On Coastal Trapped Waves: Analysis and Numerical Calculation by Inverse Iteration

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Abstract

Waves of sub-inertial frequency in a continuously stratified ocean and trapped over a continental shelf and slope are considered. They form one infinite discrete sequence of modes with frequencies decreasing to zero. The mode frequencies increase with stratification. All modes progress with the coast on their right in the Northern Hemisphere. In three formal asymptotic limits the waves adopt special forms: (1) large longshore wavenumber [Rhines (1970) bottom-trapped waves]; (2) small stratification [barotropic continental shelf waves]; and (3) large stratification [baroclinic (internal) Kelvin-like waves].

These results are illustrated by numerical calculations using the method of inverse iteration, which avoids time integration. Further calculations demonstrate the strong influence of the depth and density profiles on the wave forms. In particular, a realistic context (i.e., a gently sloping shelf bounded by a steeper continental slope, together with greater stratification near to the surface) appears to concentrate the motion over the upper slope and shelf, where it tends to be barotropic.

1. Introduction

This paper considers waves of sub-inertial frequency, trapped over and propagating along a monotonous continental shelf and slope in a stratified ocean. In this context of large depth changes, we neglect the latitudinal variation of the inertial frequency which supports Rossby waves. Then all waves below the inertial frequency are trapped. Above the inertial frequency, Kelvin waves in uniform depth against a coastal wall (and a few other cases with special depth profiles) may be trapped. In general, however, waves above the inertial frequency include internal wave contributions of sufficiently fine vertical structure to radiate energy into the deep sea. Thus the conditions of trapping and sub-inertial frequency are essentially equivalent.

The two elements of the situation, namely, topography and stratification, are separately associated with continental shelf waves (first specifically studied by Robinson, 1964) and internal Kelvin waves against a coastal wall (Fjeldstad, 1933). The existence and properties of both wave types are well-documented (e.g., Clarke, 1977).

In practice, topography and stratification are both present. Thus Mysak (1967) and Gill and Clarke (1974) studied shelf models with a two-layer density structure (only the upper layer extending over the shelf). An internal Kelvin wave is found against the vertical continental slope, and continental shelf wave analogs occur with phase speeds greater than in a homogeneous ocean [agreeing with (2.9) below; phase speeds always increase with stratification]. The two-layer models of Kajiura (1974), Wang (1975) and Allen (1975) have both layers extending over a shelf. The natural modes correspond closely with an internal Kelvin wave against the coastal wall and a barotropic shelf wave. As stratification or longshore wavenumber increases, the internal Kelvin wave speed increases toward that of the shelf wave. Instead of crossing, however, the dispersion curves "kiss" and the two modes exchange identities (Allen, 1975).

In the context of uniform (small) sea floor slope $h'$ and Brunt-Väisälä frequency $N$, Rhines (1970) found bottom-trapped wave solutions. They have frequency $Nh'$ if the horizontal wavenumber vector is parallel to the depth contours and progress cyclonically relative to the deep water, as do barotropic continental shelf waves and internal Kelvin waves. In Section 7 we find that these waves form the solution of the general problem in the limit of large longshore wavenumber $k$. They are concentrated near the position of maximum $Nh'$ on the bottom, implying that realistic results for large $k$ require accurate representation of the depth and density profiles. This generally involves numerical calculation.

Wang and Mooers (1976) present numerical calculations for continuous stratification and various depth profiles, using the method of resonance iteration. They include the exponential depth profile considered by Wang (1975) and Allen (1975) in discussing "kissing modes", and the phenomenon is retained. However, the identification of modes as internal Kelvin or barotropic shelf waves is not always
clear, and the first shelf profile of Wang and Mooers (1976) does not appear to exhibit "kissing modes".

We formulate the problem in Section 2, where some general analytic results are also given. One special depth profile is treated analytically in Section 3. Section 4 describes the numerical method using inverse iteration. The following sections describe analytical and numerical investigations of the dependence of the wave modes on the various parameters, viz., horizontal divergence (Section 5), stratification (Section 6), longshore wavenumber (Section 7), depth profile (Section 8) and stratification profile (Section 9).

2. Formulation and general analysis

a. Equations of motion

We consider a rotating Boussinesq incompressible sea with uniform Coriolis or inertial frequency $f$, total density $\rho_0(z) + \rho(x,t)$ and pressure $\rho_0(z) + \rho(x,t)$. The equilibrium values $\rho_0$ and $\rho_0$ are in hydrostatic balance, i.e.,

$$0 = -\frac{\partial \rho_0}{\partial z} - \rho_0(z)g.$$

Plane horizontal coordinates $x$, $y$ (Fig. 1) are taken offshore and along the shelf (uniform in $y$) described by the monotonic bottom profile $B: z = -h(x)$; $z$ increases vertically upward to zero on the equilibrium sea surface $T$. We consider constituents of the form $\exp(i(ky + \sigma t))$, harmonic alongshore with wavenumber $k$ and in time $t$ with frequency $\sigma$. Adopting the convention $\sigma > 0$, waves traveling with the coast on their right have $k > 0$. All quantities are nondimensionalized on appropriate scales as follows:

- **horizontal** velocity $(u,v)$
  - shelf breadth $L$
  - deep-sea depth $H$

- **vertical** velocity $w$
  - $UHL/L$
  - $\rho_0(0)FUL/ho_0(0)(gH)$
  - $L^{-1}$

- $n^2$ representing $-g\rho_0^{-1}\cdot \frac{\partial \rho_0}{\partial z}dz$ $N^2 = \max(-g\rho_0^{-1}\cdot \frac{\partial \rho_0}{\partial z}).$

Nondimensional parameters $D^2 = f^2L^2/(gH)$, measuring horizontal divergence and $S = N^2H^2/(f^2L^2)$, measuring stratification, also appear.

The linearized equations of motion in the long wave approximation ($L \gg H$) are then:

$$\begin{align*}
\text{horizontal momentum} & \quad \begin{cases}
\sigma u - v = -\partial p/\partial x & (2.1a) \\
\sigma v + u = -ikp & (2.1b)
\end{cases} \\
\text{vertical momentum} & \quad \begin{cases}
0 = -\partial p/\partial z - \rho & (2.1c)
\end{cases} \\
\text{incompressibility} & \quad \begin{cases}
\frac{\partial u}{\partial x} + ikv + \frac{\partial w}{\partial z} = 0 & (2.1d)
\end{cases} \\
\text{continuity} & \quad \begin{cases}
i\sigma p - w n^2 S = 0. & (2.1e)
\end{cases}
\end{align*}$$

The appropriate boundary conditions for unforced trapped waves are 1) motion $\rightarrow 0$ in the deep sea $(x \rightarrow \infty)$; 2) no flow through the sea floor: $w = -\frac{udh}{dx}$ on $B: z = -h(x)$, interpreted as $u = 0$ at vertical segments, e.g. a coastal wall; 3) zero total pressure at the sea surface $H z \approx HUw/(L,\sigma)$, i.e.,

$$0 = -\rho_0(0)g \frac{HU}{L} \frac{w}{i\sigma} + \rho_0(0)fULp \quad \text{on} \ T: z = 0.$$

Eqs. (2.1a)-(2.1c) and (2.1e) are easily solved for $u$, $v$, $\rho$ and $w$ in terms of $p$:

$$(u,v) = \begin{cases}
\frac{1}{\sigma^2 - 1}, & (i\sigma \partial p/\partial x + ikp, -\sigma kp - \partial p/\partial x)
\end{cases}$$

$$\begin{align*}
\rho & = -\partial p/\partial z \\
w & = -\frac{i\sigma}{n^2 S} \cdot \partial p/\partial z.
\end{align*}$$

Expressing (2.1d) and the boundary conditions in terms of $p$ alone, we have

$$\begin{align*}
\frac{\partial^2 p}{\partial x^2} + \frac{1 - \sigma^2}{S} \cdot \frac{\partial (1/\partial p)}{\partial (n^2 S \partial z)} - k^2 p & = 0, \\
p & \rightarrow 0 \quad \text{as} \quad (x \rightarrow \infty),
\end{align*}$$

$$\begin{align*}
\frac{dh}{dx} \left( \frac{\partial p}{\partial x} + \frac{k}{\sigma} \right) + \frac{1 - \sigma^2}{Sn^2} \cdot \frac{\partial p}{\partial z} & = 0 \quad \text{as} \quad (z = -h),
\end{align*}$$
\[
\frac{1}{Sn^2} \frac{\partial p}{\partial z} + D^2 p = 0 \quad (z = 0). \tag{2.3c}
\]

b. Variational form

By multiplying (2.2) by \( p \), integrating over the \((x,z)\) region \( \mathcal{A} \) occupied by the sea, integrating by parts and using the boundary conditions (2.3), it is straightforward to show that

\[
k\sigma = \frac{I_1 + \frac{1 - \sigma^2}{S} I_5 + k^2 I_4 + D^2(1 - \sigma^2) I_4}{I_5}, \tag{2.4}
\]

where

\[
I_1 = \int_A \left( \frac{\partial p}{\partial x} \right)^2 dA, \quad I_2 = \int_A \frac{1}{n^2} \left( \frac{\partial p}{\partial z} \right)^2 dA
\]

\[
I_3 = \int_A p^2 dA, \quad I_4 = \int_T p^2 dx, \quad I_5 = \int_B p^2 dz
\]

This result was obtained by Clarke (1977) neglecting terms in \( \sigma^2 \) and \( k^2 \), and is also given in Wang and Mooers (1976) neglecting \( D^2 \). The right-hand side (RHS) of (2.4) is stationary with respect to small variations of \( p \) if and only if \( p \) is a solution of (2.2) and (2.3). Thus (2.4) represents a variational formulation of (2.2) and (2.3).

There is a corresponding orthogonality relation. For given \( k, G = (1 - \sigma^2)/S \) and \( E^2 = D^2(1 - \sigma^2) \),

\[
\int_B p_n p_m dz = 0 \tag{2.5}
\]

for solutions \( p_n \) and \( p_m \) corresponding to different values \( 1/c_n \) and \( 1/c_m \) of \( k\sigma \). However, \( k, S \) and \( D^2 \) are the given parameters in any one physical context. Hence the modes are only orthogonal in the case \( \sigma^2 \ll 1 \), for which (2.5) was given by Clarke (1977).

c. General analysis

We consider only sub-inertial frequencies, i.e., \( \sigma < 1 \), for which (2.2) is elliptic. All terms of RHS of (2.4) are then positive. Thus, generalizing Clarke (1977),

Any trapped mode of sub-inertial frequency \( (\sigma < 1) \) propagates cyclonically relative to the deep sea (i.e. \( k > 0 \)). \tag{2.6}

This is the same result as for barotropic continental shelf waves (Huthnance 1975), internal Kelvin waves and bottom-trapped waves (Rhines, 1970).

Since \( \sigma < 1 \), the minimization (with respect to functions \( p \)) of the RHS of (2.4) for given \( k,G > 0 \) and \( E^2 > 0 \) is a well-posed problem. It results in an increasing sequence of values for \( k\sigma \):

\[
\begin{align*}
1/c_0 &= \text{the absolute minimum of RHS (2.4)} \\
1/c_1 &= \text{the minimum (at } p_1) \text{ among functions} \\
1/c_2 &= \text{the minimum (at } p_2) \text{ among functions}
\end{align*}
\tag{2.4b}
\]

orthogonal to \( p_0 \) and \( p_1 \)

These minima of (2.4) are stationary with respect to small variations of \( p \), so that \( p_0, p_1, p_2 \ldots \) are solutions of (2.2), (2.3).

In section 6b it is found that for small \( G \), \( 1/c_m \rightarrow \infty \) as \( m \rightarrow \infty \). Hence \( 1/c_m \rightarrow \infty \) as \( m \rightarrow \infty \) in all cases, since \( 1/c_m \) increases with \( G \) by (2.4). Consequently, the sequence \( p_0, p_1, p_2, \ldots \) includes all solutions of (2.2) and (2.3) (Courant and Hilbert, 1953).

For any given \( (E^2, G, k) \) we thus deduce one infinite sequence of wave modes \( p_0, p_1, p_2, \ldots \) with natural frequencies \( \sigma = kc_0, kc_1, kc_2, \ldots \) decreasing to zero. We must discard a possible finite number \( M \) of the lowest modes for which \( kc_0, \ldots, kc_M > 1 \) contrary to the assumption \( \sigma < 1 \).

We should prefer the wave frequencies \( \sigma(E^2, G, k) \) to be reexpressed in terms of the independent parameters of the physical context, \( viz. \), \( D^2, S \) and \( k \) \((E^2 \text{ and } G \text{ themselves involve } \sigma) \). This is possible as is now shown. We regard \( G \) and \( k \) as temporarily fixed. Then the natural frequencies \( \sigma(E^2, G, k) \) may be regarded as curves "stacked" above one another in order of mode number, with \( \sigma = kc_0 \) uppermost (Fig. 2). By (2.4),

\[
-k\sigma^2 \left. \frac{\partial \sigma}{\partial E^2} \right|_{G, k} = I_4/I_5
\]

since the RHS of (2.4) is stationary with respect to variations of \( p \). Thus \( \partial \sigma/\partial E^2 < 0 \) on each curve, and the \((\sigma, E^2)\) relation is invertible to give \( E^2 \) as a func-

![Fig. 2. \( \sigma(E^2, G, k) \) for the first three modes.](image-url)
tion of $\sigma$, with the lower modes having greater $E^2$ (rotate Fig. 2 counterclockwise through 90°). Since $D^2 = E^2(1 - \sigma^2)$, $D^2$ is also a function of $\sigma$ and lower modes have greater $D^2$. Finally (Appendix A),
\[
\left. \frac{\partial \sigma}{\partial D^2} \right|_{s_{\lambda k}, k} < 0,
\]
so that the $(D^2, \sigma)$ relation is invertible (read $D^2$ for $E^2$ in Fig. 2). Thus $\sigma$ is a function of $D^2$ with lower modes having greater $\sigma$, and we obtain $\sigma(D^2, G, k)$ retaining the order of the curves from $\sigma(E^2, G, k)$. Similarly, one can transform from $\sigma(D^2, G, k)$ to $\sigma(D^2, S, k)$ using (Appendix A)
\[
\left. \frac{\partial \sigma}{\partial G} \right|_{s_{\lambda k}, k} < 0,
\left. \frac{\partial \sigma}{\partial S} \right|_{s_{\lambda k}, k} > 0.
\]
The ordering of modes is again retained. Hence

There is an infinite discrete sequence of wave modes $p_0, p_1, \ldots$ with decreasing frequencies $\sigma(D^2, S, k)$.

(2.7)

A finite number of the lowest modes may be absent, as noted above, where their frequency function $\sigma(D^2, S, k)$ inconsistently enters $\sigma > 1$. This may occur in particular for small $D^2$ or large $S$ or $k$. The number absent tends to decrease with $D^2$ and increase with $S$ since (Appendix A)

$\partial \sigma/\partial D^2 < 0$: the natural wave frequency decreases as the horizontal scale increases in comparison with the Rossby radius of deformation for barotropic motion

(2.8)

and

$\partial \sigma/\partial S > 0$: the natural wave frequency increases with the measure $S$ of stratification.

(2.9)

The result (2.8) is the same as for barotropic continental shelf waves (Huthnance, 1975). In practice, $D^2$ is small ($10^{-2}$ or less for most shelves), and by Appendix A

\[
\left| \frac{\partial \sigma}{\partial D^2} \right| \leq \sigma(4 L_s \sigma)/(3 L_s k).
\]

Hence the natural frequencies of modes $1, 2, \ldots$ are almost as if $D^2 = 0$, in agreement with the more detailed study by Buchwald (1973) for barotropic shelf waves. However, mode 0 is essentially a barotropic Kelvin wave dominated by surface wave dynamics; it depends on nonzero $D^2$ for its physical existence. Both (2.8) and (2.9) correspond to the intuitive notion that increased gravity should increase wave speeds by enhancing restoring forces. The natural modes are fastest in the rigid-lid approximation $D^2 = 0$ and with strong stratification.

The orthogonality relation (2.5) corresponds to a partition between modes of a combination of wave energy

\[
\mathcal{E} \equiv \frac{1}{2} \int_A \left[ \frac{u^2}{v^2} + \rho^2/(Sn^2) \right] dA + \frac{1}{2} D^2 \int_T \rho^2 dx
\]

and wave energy flux along the shelf

\[
\mathcal{F} = \int_A p v dA.
\]

As for barotropic shelf waves, (2.10) to (2.12) are summarized symbolically by

\[
\mathcal{E}_m \quad \mathcal{F}_{mn} \quad \mathcal{E}_{mn} = 0,
\]

where $\mathcal{E}_{mn}$ and $\mathcal{F}_{mn}$ are the interactive contributions to $\mathcal{E}$ and $\mathcal{F}$, and $\mathcal{E} = (\sigma_n - \sigma_m)(k_n - k_m).

3. One analytic example

Eq. (2.2) is separable in $x$ and $z$. For uniform stratification $n^2$ it remains separable in some other coordinate systems (Morse and Feshbach, 1953); of these only parabolic coordinates yield boundary conditions (2.3) allowing a separated form of solution. To find this we take the profiles

\[
h(x) = x^{1/2}, \quad n^2(z) = 1,
\]

and write $G = (1 - \sigma^2)/S$. We introduce parabolic coordinates $(\xi, n)$:

\[
x = \frac{1}{2} \left[ \frac{\xi^2 + 1}{2G} - \eta^2 \right], \quad z = \xi \eta G^{1/2},
\]

the sea surface is $\eta = 0$ [$\xi = 0$ in $x < 1/(4G)$] and the sea floor is $\eta = -(2G)^{-1/2}$ (see Fig. 3). Eq. (2.2) becomes

\[
\begin{align*}
\partial^2 p/\partial \xi^2 - k^2 \xi^2 p + \partial^2 p/\partial \eta^2 - k^2 \eta^2 p = 0,
\end{align*}
\]
and for $D^2 = 0$ the boundary conditions (2.3) become

$$p \to 0 \quad (\xi \to \infty)$$

$$\frac{\partial p}{\partial \eta} + \frac{k}{\sigma(2G)^{1/2}} p = 0 \quad (\eta = -(2G)^{-1/2})$$

$$\frac{\partial p}{\partial \eta} = 0 \quad (\eta = 0)$$

$$\frac{\partial p}{\partial \xi} = 0 \quad (\xi = 0).$$

We have taken $D^2 = 0$ to enable separated - form solutions; these are

$$p = e^{-\xi \sigma/2} H_{2m}(\xi \eta^{1/2}) e^{\xi \eta t/2} H_{2m}(\eta \xi^{1/2}),$$

$$m = 1, 2, \ldots$$

$$1/\sigma = 1 - H_{2m}(y) / H_{2m}(y) \bigg|_{y = \xi \eta^{1/2}}$$

where $H_{2m}$ are symmetric Hermite polynomials with $2m$ real zeros $\pm \xi_1$, $\ldots$, $\pm \xi_m$, so that mode $m$ has $m$ nodal lines

$$x = \frac{1}{2} \left( \xi_r^2 + \frac{1}{2G} - \frac{z_r^2}{\xi_r^2 G} \right), \quad r = 1, \ldots, m.$$  

Hence the nodal lines of $p$ run from the bottom to the surface and tilt away from the coast (Fig. 3). As the stratification $S$ increases ($1/G$ increases), they pass from the vertical ($S = 0$; continental shelf waves) to almost horizontal as $S \to \infty$.

For the lowest mode $m = 1$, $H_{2m}(y) = 4y^2 - 2$ and $\sigma = [-2 + (9 + 5S\xi^{1/2})/5].$ Hence $\sigma$ increases with $S$ and $k$, reaching $\sigma = 1$ at $S\xi = 8$. The model is atypical in that $\sigma$ does not decrease to zero as $k \to 0$, reflecting the absence of well-defined length or depth scales as the slope continues indefinitely.

4. Numerical calculation

The example just treated appears to be the only nontrivial case amenable to analysis, apart from some cases of extreme parameter values considered in Sections 6 and 7. Numerical calculation therefore seems to be required for quantitative results, and is of course necessary to handle realistic depth and density profiles.

Wang and Mooers (1976) used the method of resonance iteration, in which the equations of motion are integrated in time, with the inclusion of a forcing function iteratively adjusted to induce resonance of the natural mode sought. The method follows Platzman’s (1972) calculation of ocean basin modes. Platzman (1972) also indicated the alternative of inverse iteration (see, e.g., Forsythe and Wasow, 1960) adopted here. This also uses a forcing function iteratively adjusted to excite the desired mode, but solves the eigenvalue problem [essentially Eq. (2.4)] directly without time integration.

a. Formulation

The natural mode problem with continuous variables is represented by (2.2) and (2.3) to which the variational form (2.4) is equivalent. For discrete variables, we let the system be represented by the pressure perturbation $p$ evaluated at the points of the finite-difference grid in Fig. 4. Then the integrals $I_1$ to $I_5$ in (2.4), all homogeneous of second degree in $p$, may be approximated by the finite-difference analogs specified in Appendix B. These are also homogeneous of second degree in the values of $p$ at all grid points. The continuous equations (2.2) and boundary conditions (2.3) for $p$ result from requiring (2.4) to hold for arbitrary small variations of $p$. Finite-difference analogs are similarly obtained by differentiating the finite-difference form of (2.4) $I_5$ with respect to the value of $p$ at each grid point. They take the form

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(L - \lambda I_B)p = 0. \tag{4.1}

where L is a square matrix (constant for given \( E^2 \), \( G \) and \( k \)), \( \lambda \) represents the eigenvalue \( k/\sigma \), \( I_B \) is the identity matrix on the bottom boundary values \( \tilde{p}_K \) (and zero elsewhere) and p is the vector of all grid values \( p_{1,K} \). The scheme is of second order, having centered differences in the interior.

b. Vertical structure modes

We consider finite-difference analogs of the barotropic and first, second, etc., baroclinic modes in the deep-sea region of uniform depth. They are important for representing the trapping condition \( p \to 0 \) (\( x \to \infty \)) in a finite model, and as a basis for an exact solution of (4.1).

Just as (2.2) is separable in \( x \) and \( z \), so its finite-difference analogs in (4.1) are separable in \( f \) and \( K \). In the deep sea where the bottom is flat, there are solutions of separated form

\[ p_{1,K} = f_m^{l*} \psi_K^{(m)}, \quad m = 1, \ldots , KM, \]

which also satisfy the surface and (flat) sea floor boundary conditions. The number of vertical structure modes \( \psi^{(m)} = (\psi_1^{(m)}, \ldots , \psi_{KM}^{(m)}) \) equals the number \( KM \) of grid points in the vertical. The associated factor \( f_m \) between horizontally adjacent positions corresponds to the exponential growth or decay of continuous modes at subinertial frequencies (\( \sigma < 1 \)).

All the \( \psi^{(m)} \), \( f_m \) are easily and rapidly found as solutions of an eigenvalue problem with a tridiagonal matrix of dimension \( KM \). In practice a library routine following Wilkinson and Reinsch (1971) was used. The vectors \( \psi^{(m)} \) are orthogonal and, by assumption, normalized with respect to the scalar product (Fig. 4).

\[ \{\psi^{(m)}, \psi^{(n)}\} = X \sum_{K=1}^{KS-1} \psi_1^{(m)} \psi_1^{(n)} + \frac{1}{2}(1 + X) \sum_{K=1}^{KA} \psi_1^{(m)} \psi_1^{(n)} \tag{4.2} \]

The \( KM \) modes \( \psi^{(m)} \), being orthogonal, are complete on \( K = 1, \ldots , KM \). Hence for any \( I \) there are coefficients \( a^{(m)} \) such that

\[ p_{I,K} = \sum_{m=1}^{KM} a^{(m)} \psi_K^{(m)}. \]

Then the requirement that \( p_{I,K} \) satisfies the difference equations in the deep sea implies \( a^{(m)} = a_m f_m^{l*} \), where \( f_m < 1 \) for seaward decay. Hence the general finite-difference solution in the deep sea is

\[ p_{I,K} = \sum_{m=1}^{KM} a_m f_m^{l*} \psi_K^{(m)}, \tag{4.3} \]

so that

\[ \{p_I, \psi^{(m)}\} = a_m f_m^{l*}, \]

where \( p_I \) denotes the vector \( p_{I,K} \). Thus a condition equivalent to \( p_I \to 0 \) (\( x \to \infty \)) is

\[ \{p_{IU}, \psi^{(m)}\} = f_m^{l*}\psi^{(m)}, \quad m = 1, \ldots , KM, \tag{4.4} \]

where \( IU = IM + 1 \) as in Fig. 4, and \( IM \) need be only \( I(KM) + 1 \). The use of (4.4) enables truncation of the finite-difference grid at \( IU = I(KM) + 2 \), including only three complete vertical columns of grid points. (Application of \( p_I \to 0 \) as \( x \) or \( I \to \infty \) requires in principle the inclusion of arbitrarily many columns. The truncation is particularly valuable for mode 0, a barotropic Kelvin wave, which decays very slowly offshore.) With this truncation, \( I_1 \) to \( I_4 \) of the variational form (2.4) are evaluated supposing a deep-sea boundary at \( IM + \frac{1}{2} \). The condition (4.4) is incorporated into (2.4) by substituting \( I_1 + I_8 \) for \( I_1 \); \( I_4 \) is specified in Appendix B. These modifications to (2.4) leave all difference equations (4.1) in \( I \leq IM \) unchanged, since \( I_8 \) depends only on \( p_{IU} \). Hence the only effect on the problem (4.1) is to introduce the deep-sea conditions (4.4), when (2.4) is differentiated with respect to \( p_{IU} \). Since (4.4) are an exact equivalent of \( p_I \to 0 \) (\( x \to \infty \)), we have in effect merely introduced a practical expression of the latter requirement.

The general deep-sea solution (4.3) may be extended toward the coast, since it continues to satisfy the interior and surface difference equations. The \( KM \) sea-floor boundary conditions (the equations with \( \lambda \) in (4.1)) as the floor rises through the \( KM \) grid levels form \( KM \) constraints on the coefficients \( a = (a_1, \ldots , a_{KM}) \) in (4.3): \( (A - \lambda B)a = 0 \), or \( (AB^{-1} - \lambda I)Ba = 0. \) This eigenvalue problem may be solved for the \( KM \) inverse wave speeds \( \lambda \) and the orthogonal bottom boundary values of pressure \( \tilde{p} = Ba \). A library routine following Wilkinson and Reinsch (1971) was used. Unfortunately, this simple, rapid and exact solution is feasible only with coarse grids \( (KM < 20 \) in practice), owing to the rapid coastward growth of the higher vertical structure modes. As \( KM \) increases, the matrices \( A \) and \( B \) are eventually unbalanced beyond any finite resolution. Nevertheless, the method does provide a quick initial approximation for the inverse iteration.

c. Inverse iteration

We wish to solve (4.1) for the eigenvectors \( p \) and eigenvalues \( \lambda = k/\sigma \). In principle, the equations of (4.1) without \( \lambda \) determine \( p_{I,K} \) \( I > I(K) \); see Fig. 4 in terms of the bottom boundary values \( \tilde{p} \). The remaining \( KM \) equations of (4.1) (with \( \lambda \)) then form an eigenvalue problem, yielding \( KM \) eigenvectors and eigenvalues

\[ \tilde{p}_r \] (hence \( p_r \), \( \lambda_r \), \( r = 1, \ldots , KM \)).
We may suppose that in seeking one of these, \((p_m, \lambda_m)\), we have an \(i\)th approximation \(p^{(i)}\) to \(p_m\) with small error \(e^{(i)}\). Immediately, (2.4) furnishes an \(O(e^{(i)})^2\) approximation to \(\lambda_m\):

\[
\lambda^{(i)} = (I + I_0 + Gl_2 + k^2 I_3 + E^2 I_4)_{I_5}p^{(i)} = \lambda_m + O(e^{(i)})^2
\]

since the RHS of (2.4) is stationary with respect to the \(O(e^{(i)})\) errors \(p^{(i)} - p_m\). The first approximation \(p^{(1)}\) is taken as the exact solution on a coarse grid.

Inverse iteration exploits the fact that if (4.1) are forced by \(\Sigma b_r \hat{p}_r\) including the constituent \(\hat{p}_m\), then there is a near-resonant response of the form \(p_m\), viz.,

\[
p = \sum \frac{b_r}{\lambda_r - \lambda} p_r.
\]

Thus mode \(m\) enjoys great relative amplification if \(\lambda\) is near \(\lambda_m\). Hence we seek \(p^{(i+1)}\) by solving

\[
(L - \lambda^{(i)} I_0)p^{(i+1)} = [\lambda^{(i)} - \lambda^{(i-1)}]_0 p^{(i)}, \quad (4.5)
\]

in which the RHS is the best current estimate of \(\hat{p}_m\). If \(p^{(i)} = \sum b^{(i)} \hat{p}_r\), then \(p^{(i+1)} = \sum b^{(i+1)} \hat{p}_r\), obtained thus has

\[
b^{(i+1)} = \frac{\lambda^{(i)} - \lambda^{(i-1)}}{\lambda_r - \lambda^{(i-1)}} b^{(i)}.
\]

The \(b^{(i+1)} (r \neq m)\) form the error in \(p^{(i+1)}\). Hence

\[
e^{(i+1)} = (\epsilon^{(i+1)})^0 \epsilon^{(i)}
\]

which is approximately second-order convergence. The factor \([\lambda^{(i)} - \lambda^{(i-1)}]\) on the RHS of (4.5) keeps the magnitudes of \(p^{(i)}\) and \(p^{(i+1)}\) nearly equal, so that \(p^{(i)}\) is a good initial approximation to \(p^{(i+1)}\). This step from \(p^{(0)}\) to \(p^{(0+1)}\) is itself performed iteratively (an inner iteration), (4.5) being solved by successive overrelaxation (see, e.g., Forsythe and Wasow, 1960). A difficulty arises at this stage. If \(L\) is symmetric with an eigenvalue less than \(\lambda^{(i-1)}\), then the corresponding eigenvector grows in successive iterations (Forsythe and Wasow, 1960). In the present case \(L\) is not precisely symmetric, but we do expect \(KM\) real eigenvalues and the same phenomenon occurs. That is, the contributions of all eigenvectors \(p_r\) lower than \(p_m\) grow during the iteration. Hence the lowest eigenvector \(p_1\) and eigenvalue \(\lambda_1\) must be sought first. Then \(p_1\) is repeatedly removed during the iteration for \(p_2, p_3\) and \(p_4\) are both removed while seeking \(p_5\) and so forth. In practice it is only necessary to remove the lower modes \(p_r\) from the bottom boundary values \(\hat{p}\). This is easily done via the orthogonality relation (4.2); \(\hat{p}\) is replaced by

\[
\hat{p} - \sum p_r \hat{p}_r p_r, \quad (4.6)
\]

d. Tests of the model

Two computational checks test the accuracy of programming. First, inverse iteration should give the same result as the exact solution (Section 4b) if performed on the same grid. Second, if the iterative equations based on (4.1) do not correspond with (2.4) exactly, first-order rather than second-order convergence results.

A limited comparison with analytic solutions is possible. Internal Kelvin waves were matched to within 1% by taking \(w = 10^{-4}\) (Fig. 4) to model a vertical coastal wall. Agreement within 2% was obtained with the results of asymptotic analysis (Section 6) for weak and large stratification \(S\) (these results are all for \(KM = 50\)). There is poorer comparison with asymptotic analysis (Section 7) for large longshore wavenumbers \(k\). The finite resolution of \(KM = 50\), together with the fine modal structure implied by large \(k\), resulted in agreement within 10% for \(k = 5\) but discrepancies of 30% for \(k = 20\). The latter are associated with the strong decay away from the sea floor by a factor of \(1/2\) per grid point. In general, the numerical results achieved the expected agreement of measure with these (approximate) analytic forms. The absence of more precise analysis is a bar to rigorous comparisons.

In addition to the case of large \(k\) above, the effects of finite resolution were also checked by varying the grid spacing \((-1/KM)\). For the control case (Section 5) \(D^2 = 0, S = 1, k = 10^{-4}\) and uniform slope and stratification, values of \(\lambda\) for mode 2 of 4.97, 5.42 and 5.67 resulted where \(KM = 10, 20\) and 50. Mode 1 converges more rapidly and higher modes less so as \(KM\) becomes large. This example is a little flattering, having no fine spatial structure (cf. large \(k\) above).

In use, as for example in the calculations of the following sections, the iteration generally converged to
\( \sigma \) depending on three given parameters: horizontal divergence \( D^2 \), stratification \( S \) and longshore wave number \( k \), together with the depth profile \( h(x) \) and the stratification profile represented by \( n^2(z) \).

The numerical model discussed in Section 4 was used to investigate and illustrate the manner in which the trapped wave modes depend on these five factors. For clarity, we adopt control values

\[
D^2 = 0, \quad S = 1, \quad k = 10^{-4},
\]

\[
h = \begin{cases} 
  x, & 0 < x < 1 \\
  1, & x \geq 1 
\end{cases}, \quad n^2 = 1 \text{ (uniform)}
\]

and vary only one parameter at a time from this set. These values are chosen so that the control retains the full structure of the problem. However, the modes should not be distorted by (say) non-uniformities of slope, although uniform deep sea depth is retained. The shoaling depth as \( x \to 0 \) gives a 45° corner in the \( x, z \) plane. This is not a singularity of either the mathematical formulation or the finite difference grid used. (By contrast, the barotropic model ordinary differential equation has a regular singularity at \( x = 0 \). The solution which is regular there must then be chosen.)

The two horizontal velocity components \( u, v \) of mode 2 in the control case are illustrated in Fig. 5, which provides a basis of comparison as the parameters are varied. Of the other flow variables (not shown), the vertical velocity \( w \) is constrained by the sea floor condition \( w = -udh/dx \) to be similar in structure to \( u \), except that \( w = 0 \) on the sea surface (if \( D^2 = 0 \)). The structure of \( v = k\sigma p + \partial p/\partial x \) is very close to \( \partial p/\partial x \) except when we vary \( k \) above the control value \( 10^{-4} \). The choice of mode 2 is a compromise between simplicity and the demonstration of features (e.g., distribution of nodes) only exhibited by higher modes. It may be remarked that in the majority of cases, including the control, \( v \) (and \( p \)) exhibit nodes equal in number to the mode number, whereas \( u \) (and \( w \)) show one less node on the sloping bottom. However, exceptions may occur (particularly in association with strongly nonuniform slopes) even for \( p \), the only variable directly constrained by orthogonality along the sloping sea floor. The familiar rule that successive one-dimensional modes have interlacing nodes does not apply to these two-dimensional solutions.

### Table 1. Dependence of wave speed \( \sigma/k \) on \( D^2 \).

<table>
<thead>
<tr>
<th>( D^2 )</th>
<th>Mode no.</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0.422</td>
<td>0.176</td>
<td>0.114</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td>1.856</td>
<td>0.374</td>
<td>0.173</td>
<td>0.113</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.761</td>
<td>0.308</td>
<td>0.165</td>
<td>0.111</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.439</td>
<td>0.211</td>
<td>0.138</td>
<td>0.101</td>
</tr>
</tbody>
</table>

---

5. Parameter variations

The natural mode problem (2.2), (2.3) or (2.4) yields a sequence of wave modes \( p \) with frequencies

\[
1 - \lambda (0)/\lambda (i+1) \ll e^{-10} \ll \lambda \times 10^{-4}
\]

after 300–400 sweeps of the grid for each mode \( (KM = 50) \). These include 100 initial iterations for choice of the relaxation factor, and typically three subsequent steps in the inverse iteration, each comprising 50–100 inner iterations. Proportionately fewer are required for coarse grids (smaller \( KM \)). The main factor retarding convergence appears to be weak stratification. This unbalances the coefficients of the interior difference equations in (4.1), horizontal differences having much smaller coefficients. The opposite imbalance with large stratification does not retard convergence, and the difficulty appears to arise from the adopted procedure of sweeping only horizontally along each row in turn during the inner iteration. Some improvement may be effected by sweeping vertically in alternate iterations.
In addition to numerical illustrations, analysis also proves to be of considerable help in investigating the limits of small or large stratification $S$ and large longshore wavenumber $k$ (in addition to providing the general results in section 2).

a. $D^2$ varies

We already know [Eq. (2.8)] that wave speeds decrease with increasing horizontal divergence $D^2$. For variations from the control model, this is illustrated by Table 1. As foreshadowed after (2.9), the effect is less for modes 1, 2, . . . ; nonzero $D^2$ modifies the phase of the pressure $p$ at the surface by up to $\pi/2$, and for higher order $(m)$ modes this is less important relative to the $O(m\pi)$ total phase change down the sloping sea floor.

Mode 0, however, is essentially a Kelvin (surface) wave, depending on nonzero $D^2$ for its physical existence with surface elevations and associated horizontal divergence. [The trivial solution $p = \exp(-kx)$, $\sigma = 1$ is retained when $D^2 = 0$.] This is the only mode for which typical oceanic values ($10^{-5}$) of $D^2$ appear to be significantly different from zero.

6. Stratification

a. Small stratification $S$: barotropic shelf waves

Clarke (1976) has discussed this case for small $\sigma$ and $k$. In general, we let

$$p(x,z) = p_0(x) + S p_1(x,z) + O(S^2).$$

By (2.2),

$$\frac{d^2 p_0}{dx^2} + \frac{\partial}{\partial z} \left( \frac{1 - \sigma^2}{n^2} \frac{\partial p_1}{\partial z} \right) - k^2 p_0 = O(S) \quad (6.1)$$

so that $n^{-2}(1 - \sigma^2)\partial p_1/\partial z$ is linear in $z$: it is then determined in terms of $p_0$ by its values at $z = -h$ and $z = 0$, as given by (2.3b) and (2.3c) at $O(S)$. Substituting in (6.1) we obtain

$$\frac{d}{dx} \left( \frac{h}{dp_0} \int \right) - \left[ - \frac{k}{\sigma} \frac{dh}{dx} + k^2 h + D^2(1 - \sigma^2) \right] p_0 = 0. \quad (6.2)$$

Either $h(x) \to 0$ as $x \to 0$, or $h = 0$ is a wall ($dh/dx = \infty$) and (2.3b) is to be interpreted there as $dp_0/dx + k p_0/\sigma = 0$. Thus

$$h(dp_0/dx + k p_0/\sigma) \to 0, \quad x \to 0, \quad (6.3a)$$

$$p_0 \to 0, \quad x \to \infty, \quad (6.3b)$$

by (2.3a). Eqs. (6.2) and (6.3) are identical with the equations [Huthnance (1975), Eqs. (2.3) to (2.5)] for the surface elevation $\xi$ in the barotropic shelf wave problem. The correspondence is completed by noting that the surface boundary condition in the present case is $p \propto \xi^2$.

As the stratification $S$ becomes small, the solutions thus approach barotropic shelf waves. The almost vertical nodes of pressure $p$ tilt slightly away from the coast, since $\partial p/\partial z$ and $\partial p/\partial x$ have opposite signs at zeroes of $p$ by (2.3b). The $O(S)$ increase in frequency from the barotropic value may be estimated (Appendix A) as

$$\frac{\sigma(1 - \sigma^2)}{k L_s/\sigma - 2\sigma^2 D^2 I_4} + O(S^2),$$

which may be calculated entirely from values for the barotropic solution.

b. Large stratification $S$: internal Kelvin-like waves

Let $\epsilon = S^{-1/2} \ll 1$. If we suppose that the vertical scale of the solution remains $O(1)$, then the trapped solution of (2.2) is

$$p = p_0(z) e^{-kx}.$$ 

However, this form can only satisfy the bottom boundary condition (2.3b) if $\sigma = 1$ (considered in Section 6c). Hence we must seek solutions varying on a reduced vertical scale, at least near the boundary $B$. Writing $\xi = x - h^{-1}(-z)$ (horizontal distance from $B$) and $\eta = z/\epsilon$, Eq. (2.2) becomes

$$\frac{\partial^2 p}{\partial \xi^2} + \frac{1 - \sigma^2}{\eta^2} \frac{\partial^2 p}{\partial \eta^2} - k^2 p = O(\epsilon), \quad (6.4a)$$

and

$$\frac{dh}{dx} \left( \frac{\partial p}{\partial \xi} + \frac{k}{\sigma} p \right) = O(\epsilon) \quad \text{on } \xi = 0, \quad (6.4b)$$

results from the sea floor boundary condition (2.3b). The trapped solutions are

$$p = \phi(\eta) e^{-k\xi/\sigma} + O(\epsilon):$$

$$\phi = \cos \left( \frac{k}{\sigma} S^{1/2} \int_{-h}^{z} n(z')dz' \right), \quad (6.5)$$

which also satisfy (2.3b): $\partial p/\partial z = 0$ on the deep sea floor $z = -1$. The surface boundary condition (2.3c) imposes the dispersion relation

$$\tan \left( \frac{k}{\sigma} S^{1/2} \int_{-h}^{0} n(z)dz \right) = D^2 \frac{S^{1/2} n(0)}{k}.$$ 

Since $D^2 S = N^2 H/\gamma$ is the fractional density change in depth, which is small in practice (and by assumption), the dispersion relation simplifies to
for the mode (6.5) with \( m \) nodes in \( z \).

We identify (6.5) as an internal Kelvin wave form; the offshore decay factor \( \exp(-k\xi/\sigma) \) implies \( O(\varepsilon) \) offshore velocities \( u \) compared with the longshore currents \( v \). The vertical wavenumber \( S^{1/2}n\pi/\sigma \) is also appropriate at each depth, the wave speed (6.6) is correct, and the separated form of (6.5) implies horizontal nodal lines. These internal Kelvin wave characteristics are suggested by the calculations of Wang and Mooers (1976) for large \( S = 2 \), and the identification of an internal Kelvin wave form is reinforced by the following intuitive notion. As the stratification \( S \) increases, vertical displacements in the interior are suppressed so that the motion becomes more nearly horizontal. Relative to this, the sloping sea floor appears more nearly vertical. Of course, the sloping bottom \( B \) is not really vertical and retains a purely geometric effect; \( \xi \) (horizontal distance from \( B \)) rather than \( x \) (offshore distance) appears in the internal Kelvin wave form (6.5).

c. Frequency \( \sigma \) near 1

The solutions for large \( S \) in Section 6b assumed \( \sigma < 1 \), which by (6.6) implies a large mode number \( m \). For lower modes, we expect from (2.9) that their (greater) frequencies increase with \( S \) to \( \sigma = 1 \) and pass out of consideration. This may be investigated analytically. We let \( \sigma = 1 - \gamma (\gamma \ll 1) \) and

\[
p = [p_0(z) + \gamma p_1(x,z)]e^{-k\xi},
\]

where, to satisfy (2.2) and (2.3b) at \( O(\gamma) \),

\[
\begin{align*}
\frac{d}{dz} \left( \frac{1}{S} \frac{dp_0}{dz} \right) & = 0,
\frac{d}{dz} \left( \frac{1}{S} \frac{dp_0}{dz} \right) + k^2S e^{-2k\xi} p_0 = 0,
\end{align*}
\]

Here \( X(z) \) is defined by \( z = -h(X) \). Boundary conditions for (6.7) are furnished by (2.3b) on the deep sea floor \( z = -1 \), and by (2.3c):

\[
\begin{align*}
\frac{dp_0}{dz} = 0, \quad z = -1, \\
\frac{1}{Sn^2} \frac{dp_0}{dz} + D^2p_0 = 0, \quad z = 0.
\end{align*}
\]

Eqs. (6.7) and (6.8) form a conventional eigenvalue problem for \( p_0 \). Given \( k \), successive eigenmodes \( p_0 \) correspond to increasing stratification values \( S \). Thus, as \( S \) increases, the lowest modes in turn suffer an increase of frequency through \( \sigma = 1 \) and cease to be trapped.

The control case with varying \( S \) is a simple example: \( n^2 = 1, X = -z, D^2 = 0 \) and hence (6.7) and (6.8) give

\[
p \approx \cos(m\pi z)e^{-k(z+x)},
\]

when \( S = 1 + m^2\pi^2/k^2 \). At this value of \( S \), therefore, mode \( m \) passes through \( \sigma = 1 \). For this particular example, all modes \( 1, 2, \ldots \), remain in \( \sigma < 1 \) while \( S < 1 \). If \( S > 1 \) (taking a slightly different view), the frequency of any mode increases to \( \sigma = 1 \) when the wavenumber \( k \) increases to \( m\pi/(S - 1)^{1/2} \).

d. \( S \) varies

We now summarize the influence of stratification on the modes. For weak stratification, the modes take the form of barotropic continental shelf waves [Section 6a], as illustrated in the numerical model by Fig. 6. As the stratification \( S \) increases, the mode frequencies always increase by (2.9). Variations of \( S \) from the control model are illustrated in Table 2,

![Fig. 6. Velocity components \( u \) and \( v \) for mode 2 (\( S = 0.2 \)).](image)

\( k\Delta u/\Delta u = 20.57 \).
Table 2. Dependence of wave speed \(\sigma/k\) on \(S\).

<table>
<thead>
<tr>
<th>(S)</th>
<th>Mode no. 1</th>
<th>Mode no. 2</th>
<th>Mode no. 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.310</td>
<td>0.108</td>
<td>0.062</td>
</tr>
<tr>
<td>1</td>
<td>0.422</td>
<td>0.176</td>
<td>0.114</td>
</tr>
<tr>
<td>10</td>
<td>1.045</td>
<td>0.509</td>
<td>0.339</td>
</tr>
</tbody>
</table>

which suggests that the effect is proportionately greater for higher modes. This accords with the slower decrease of frequency with mode number \(m\) for large \(S\) than for small \(S\) [\(\sigma \sim m^{-1}\) for internal Kelvin waves (6.6) but \(\sigma \sim m^{-2}\) for barotropic shelf waves].

The other principal effect as \(S\) increases is a transition from a barotropic to a horizontally layered structure. For example, the nodal lines of pressure tilt steadily away from the coast, passing from vertical to horizontal. When \(S\) (and \(\sigma\)) increases from the control model, the cross-sectional velocity components \(u\) and \(w\) increase relative to the longshore component \(v\), but all decrease strongly relative to the pressure perturbation \(p\). The strong coastal cell of longshore velocity \(v\) in the barotropic shelf wave reduces toward equality with the other cells. Eventually the frequencies of the lowest modes increase to \(\sigma = 1\) in turn, by Section 6c. The higher modes with frequency still below \(\sigma = 1\) take the form of internal Kelvin waves. Figure 7 shows mode 2 approaching an internal Kelvin wave for \(S = 10\).

7. Longshore wavenumber

a. Large wavenumber \(k\): bottom-trapped waves

For large \(k\), Eq. (2.2) implies an exponential growth rate of order \(k\) in some direction. Bounded solutions are therefore trapped against some part of the \((x,z)\) boundary other than \(T:z = 0\), where (2.3c) prevents strong downward decay. Hence we seek a solution trapped against the sloping bottom \(B: z = -h(x)\), with

\[\eta = k[z + h(x)]\]

as the appropriately scaled variable for distance above \(B\). A reduced horizontal scale also proves appropriate: for \(x_0\) and \(\epsilon\) yet to be chosen let

\[\xi = \frac{x - x_0}{\epsilon} (k^{-1} \ll \epsilon \ll 1), \quad \varphi = \frac{1 - \sigma^2}{Sn^2},\]

\[h' = \frac{dh}{dx}, \quad \varphi' = \frac{d\varphi}{dz}, \text{ etc.}\]

Then (2.2) and (2.3b) are, respectively,

\[0 = k^2 \left[ (\varphi + h'^2) \frac{\partial^2 p}{\partial \eta^2} - p \right] + k \left[ 2h' \frac{\partial^2 p}{\partial \xi \partial \eta} \right] + k \left[ (\varphi' + h'') \frac{\partial p}{\partial \eta} \right] + \epsilon^2 \frac{\partial^2 p}{\partial \xi^2}, \quad (7.1)\]

\[0 = k \left[ (\varphi + h'^2) \frac{\partial p}{\partial \eta} + \frac{h'}{\sigma} p \right] + \frac{1}{\epsilon} \left[ h' \frac{\partial p}{\partial \xi} \right] \quad (\eta = 0). \quad (7.2)\]

The largest terms give

\[p = p_0(\xi) \exp[-\eta/(\varphi + h'^2)^{1/2}],\]

\[\sigma = h'/(\varphi + h'^2)^{1/2}.\]

However, \(\sigma\) is a constant, whereas \(\varphi\) and \(h'\) vary with position on the sea floor \(z = -h(x)\). We can take \(\sigma = h'/(\varphi + h'^2)^{1/2}\) evaluated at \([x_0, z_0 = -h(x_0)]\). Discrepancies of order \(\epsilon\) then arise in (7.1) or (7.2) when \(\xi = O(1)\), at distances of order \(\epsilon\) from \((x_0, z_0)\). They may be balanced by the following terms if \(\epsilon = 1/(k\epsilon)\). We therefore take

\[\epsilon = k^{-1/2},\]

\[p = (p_0(\xi) + \epsilon p_1(\xi, \eta) + \epsilon^2 p_2(\xi, \eta) + \ldots) e^{-\eta/\sigma},\]

\[1/\sigma = (1 + \varphi/h'^2)^{1/2} x_0, z_0 + \epsilon(1/\sigma)_1 + \epsilon^2(1/\sigma)_2 + \ldots,\]

Fig. 7. Velocity components \(u\) and \(v\) for mode 2 (\(S = 10\); \(k\Delta v/\Delta u = 7.32\).
where
\[ h' = h'(x_0 + \varepsilon \xi), \quad \varphi = \varphi[-h(x_0 + \varepsilon \xi) + \varepsilon^2 \eta], \]
\[ a(\xi, \eta) = (\varphi + h'^2)^{1/2}. \]

Eqs. (7.1) and (7.2) are formally expanded to order \( \varepsilon^2 \) in Appendix C. The above form of solution balances the \( \varepsilon^4 \) terms. At order \( \varepsilon^1 \), Eq. (7.1) determines \( p_1(\xi, \eta) \) in terms of \( p_0 \) and \( \eta \); Eq. (7.2) requires that \( a/h' \), evaluated on \( z = -h(x) \), is stationary at \( (x_0, z_0) \). Moreover, \( 1/\alpha_1 = 0 \). At order \( \varepsilon^2 \), (7.1) determines \( p_2(\xi, \eta) \) in terms of \( p_0 \) and \( \eta \); Eq. (7.2) yields an ordinary differential equation for the form of \( p_0(\xi) \). The requirement of bounded solutions for large \( \xi \) then implies that \( a/h' \) must be a minimum at \( (x_0, z_0) \), and \( 1/\alpha_2 \) is determined. Hence
\[ 1/\alpha = a/h' + k^{-1}(m + 1/2)b^2/a = h''/(2h'^2)\varphi/h' \]
\[ + O(k^{-3/2}) \]
for the mode with \( m \) nodes \( (m = 0, 1, 2, \ldots) \), viz.,
\[ p_0(\xi) = H_m(b\xi) \exp[-(b\xi)^2/2] \]
\[ \times b = (h''\varphi'/2\varphi - h''/h')^{14}. \]

Thus the motion is trapped close \( (k^{-1}) \) to the bottom, near \( (k^{-1/2}) \) to a position \( (x_0, z_0) \) of maximum \( nh' \), i.e., of maximum bottom slope relative to the internal wave particle displacement slope for any given frequency. The frequency \( \sigma = h'nS^{1/2} \) and the direction of particle displacements (parallel to the bottom and normal to depth contours) correspond to a bottom-trapped wave (Rhines, 1970) progressing along the depth contours. The analysis makes the generally valid assumption that the greatest \( nh' \) occurs neither at \( z = 0 \) or \( z = -1 \). If \( nh' \) has several maxima, the lowest modes will be confined to the neighborhood of the greatest.

\[ \text{FIG. 9. Velocity components } u \text{ and } v \text{ for mode } k = 6; \]
\[ k\Delta u/\Delta u = 10.25. \]

\( b. \ k \text{ varies} \)

Dispersion relations \( \sigma(k) \) as \( k \) varies from the control model are sketched in Fig. 8 for modes 1 and 2. The three values \( S = 0, 1/2, 2 \) illustrate barotropic, weakly stratified and strongly stratified cases, respectively, noting from Section 6c that \( S = 1 \) divides weak and strong stratification in this model. If \( S < 1 \), then \( \sigma < 1 \) for all \( k \) as with barotropic shelf waves [for which \( \sigma \) increases to \( (2m + 1)^{-1} \) for \( m \) as \( k \to \infty \)]. If \( S > 1 \), then \( \sigma(k) \) increases to \( \sigma = 1 \) at \( k = m\pi(S - 1)^{-1/2} \) in keeping with the large \( S \) internal Kelvin wave dispersion relation (6.6).

The principal change in the form of the modes as \( k \) increases is their closer confinement to the neighborhood of the sloping bottom, illustrated in Fig. 9. This increase of pressure gradients as compared with pressure perturbations implies an increase of currents relative to the latter. Otherwise dependence on \( k \) is slight.

In an oceanic context modes 1 and perhaps 2 are most important, with periods of some days reflecting meteorological forcing. The dispersion relation then
implies moderate longshore wavenumbers $k \leq 1$, which are also emphasized by the smaller effects of dissipation expected for longer waves. The influence of $k$ on the form of the modes then appears to be small, and (Fig. 8) the wave speed $\sigma/k$ is constant for small $k$. Hence there is support for the approximation $\sigma \to 0, k \to 0$ used by Gill and Clarke (1974) in their discussion of the possible role of these waves in sea level changes and upwelling.

8. Depth profile

Fig. 10 illustrates mode 2 for the depth profile

$$h = \begin{cases} e^{b(x-1)}, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

when $b = 4$. All other parameters retain their control values. The motion is strongly affected by the introduction of the shelf. Over the shallow shelf, the motion is substantially barotropic, as might be expected from the values of

$$\frac{N^2}{f^2} \left[ \frac{d(Hh)}{d(Lx)} \right]^2 = Sh^2 e^{b(x-1)} ,$$

which is perhaps a better measure of the local stratification than $S = \frac{N^2H^2}{f^2L^2}$ and may be quite small over the shelf (e.g., $0.0054S$ at $x = 0$ if $b = 4$). The motion is also concentrated onto the shallower shelf area, with the magnitude of currents near the coast increased relative to those elsewhere.

These trends are confirmed by calculations with a sequence of profiles

$$0.1x/s, \quad 0 < x < s$$

$$h(x) = 0.1 + 0.8(x - s)/(1 - 2s), \quad s < x < 1 - s$$

$$0.1 - 0.1(1 - x)/s, \quad 1 - s < x < 1$$

developing from the control model ($s = 0.1$) to a shelf and vertical continental slope ($s = 0.5$). Calculations for a real shelf also show the same trends (figs. 12, 13; see Section 10). Since the stratification over the shelf is effectively reduced, explanation may be sought from barotropic shelf wave theory. This suggests that currents vary in magnitude over the profile roughly as $h^{-1/2}$ [exactly so for the exponential profile; see Buchwald and Adams (1968)] and so are stronger over the shallower shelf. Moreover, the modal structure is concentrated (in offshore distance $x$) where $h^{-1} dh/dx$ is greatest, i.e., in shallow water owing to the factor $h^{-1}$ and the large offshore ($x$) extent of the shallow shelf.

Special effects may attach to particular depth profiles. For example, Wang and Mooers (1976) investigated the exponential profile with $b = 1$. The depth $e^{-1}$ of the coastal wall is then sufficient to intercept the highest node of mode 2 (and of modes 3, . . .), which accordingly takes the form of an internal Kelvin wave near to the coast. If this wave is sufficiently confined against the wall to effectively “see” only uniform depth (typically, if the stratification is weak), then the identification of an internal Kelvin wave is sufficiently precise for “kissing modes” to arise (Allen, 1975). Typical continental shelves are better represented by $b = 4$; for example, Buchwald and Adams (1968) suggest $b = 5.4$ to model the East Australian shelf and Cartwright (1969) used $b = 4.4$ to model the Scottish Continental shelf. More strictly, coastal walls as such are rare although the bottom slope may be greater close to the coast.

The numerical model assumes a uniform deep sea depth beyond some offshore distance $L$. In practice the depth may continue to increase gradually beyond the point at which a finite difference grid is arbitrarily terminated, and it is desirable to estimate the effects of this change of profile. In the case of uniform slope, such truncation is equivalent to a change of the depth and length scales $H, L$ by the same factor $\alpha$ (say). Then $D^2 = f^2L^2/\bar{g}H$ and $k = (longshore wavenumber)$ $L$ also gain the factor $\alpha$; $S$ is unchanged. Assuming $D^2$ is negligibly small,
the effects reduce to those of varying \( k \), and may be expected to be small (Section 7b) if \( k \) is moderate (<1, say), apart from a proportional change of frequency \( \sigma \). Of course the calculated modal structures should in this case be re-scaled to occupy the entire slope. In the presence of a shelf, concentration of the motion there suggests that truncation effects should be reduced.

9. Stratification profile

The control model stratification \( n^2 \), representing \( dp_0/dz \), is uniform \( (n^2 = 1) \). We now consider effects of redistributing \( n^2 \) between the various (\( KM \)) grid levels while retaining the same overall density difference, i.e., keeping \( \Sigma n^2 = KM \).

A pycnocline at mid-depth, with uniform (weaker) stratification elsewhere, increases the speed of mode 1 but decreases that of mode 2. These trends appear to reflect the changes of modal structure. Mode 1 tends to a first baroclinic mode with two barotropic layers separated by the pycnocline, making best “use” of the pycnocline to increase the wave speed. However, mode 2 has 3 cells (2 nodes) of pressure perturbation and longshore current. The lowest tends to grow up to the mid-depth pycnocline while the upper two (with the largest currents) are above the pycnocline in a region of weak stratification, reducing the wave speed. These trends clearly depend on pycnocline depth. They appear to become pronounced when the density step is comparable with the total density difference elsewhere.

A weakly stratified layer \( (n^2 = 0.1) \) centered at mid-depth, with increased uniform stratification elsewhere, decreases the mode 1 speed as the layer thickens. This is essentially the opposite of the pycnocline case. The mode 2 phase speed at first increases as the modal structure migrates away from mid-depth, following the regions of increased stratification above and below. Eventually, the mode 2 speed also decreases; in the limit the effects of a vanishingly thin light surface layer and heavy bottom layer are negligible, and the reduced stratification in the central layer predominates.

Overall, stratification in the ocean decreases with depth. Exponential \( n^2 \) profiles were tested with a ratio \( e^b \) of surface to sea floor stratification. Wave speeds were found to decrease very slightly with \( b \) (about 2% for \( b = 3 \)). There is a tendency toward a horizontally layered structure in the greater near-surface stratification, and the modal structure tends to follow the stratification upward (as do the vertical structure modes of ordinary internal waves over a flat sea floor). The near-surface intensification leaves \( n^2 dp/dz \) almost unchanged from its form with uniform \( n^2 \). Hence the vertical velocity \( w \) and the offshore component \( u \) (related to \( w \) on the sloping bottom) are hardly affected. The longshore current increases slightly near the coast, but all these changes are remarkably small recalling that \( b = 3 \) represents a surface: bottom stratification ratio of 20.

Microstructure was simulated by setting \( n^2 \) to \( 1 - q \) and \( 1 + q \) at alternate grid levels. For \( q \) up to 0.99, there was no effect (to three significant figures) on the wave speeds of modes \( l - 3 \), except that the iteration for modes 2 and 3 failed for \( q \geq 0.9 \). The microstructure leaves the vertical velocity \( w \propto n^2 dp/dz \) smooth, and is reflected by sharp gradients (but no discontinuities) in the pressure perturbation and both horizontal velocity components, as may be deduced from consideration of the field equation (2.2) for bounded but discontinuous \( n^2 \). In the more extreme case of density steps, (2.2) implies that \( p, u \) and \( v \) are bounded but discontinuous and \( w \) is continuous but not smooth. The numerical results suggest negligible macroscopic effects. An order-of-magnitude estimate of these can be obtained by analytic treatment of the finite-difference equations over a flat sea floor if \( D = 0 \) and \( n^2 = 1 \pm q \) alternately. The eigenvalues associated with vertical structure modes suffer an approximate fractional reduction \( \frac{1}{2}q^2(1 - \cos2\pi r/\ K M) \) (mode \( r \)). For mode 1 with 50 grid levels and \( q = \frac{1}{2} \), this estimate is \( 2.5 	imes 10^{-4} \).

10. Discussion

We have considered the trapped waves of subinertial frequency (i.e., \( \sigma < 1 \)) which may occur in a continuously stratified ocean over a straight continental shelf and slope. There is a single infinite discrete sequence of orthogonal [Eq. (2.5)] modes with frequencies decreasing to zero. In the Northern Hemisphere, they progress with the coast on their right. Apart from the fundamental Kelvin wave mode, they are barotropic continental shelf waves if the stratification is weak.

The natural frequencies of the modes always increase with increasing stratification \( S \); indeed, essentially the same proof (Appendix A) implies frequency increases for any increase in the stratification profile \( S n^2 \). As \( S \) increases, the nodal lines (vertical for barotropic waves) tilt away from the coast, thus maintaining the oscillatory nature of the pressure perturbation along the sloping sea floor. Correspondingly, the motion decays away from the sloping sea floor; Wang and Mooers (1976) refer to this as “bottom-trapping” although internal Kelvin waves are trapped in the same sense. For large \( S \), the nodal lines are almost horizontal, corresponding to the inhibition of vertical motion and the almost vertical wavenumber imposed on internal waves. The sloping sea floor is then seen approximately as a side wall, so that the trapped modes approximate internal Kelvin waves. By contrast, for weak
stratification the sloping floor is seen as a varying depth implying potential vorticity changes.

For large longshore wavenumber \( k \) the modes take the form of Rhines (1970) bottom-trapped waves. The frequencies then decrease with mode number only slowly \((k^{-1})\) from

\[
\text{max } S^{1/2} n(z) dh/dx
\]

and the motion is concentrated on the sloping sea floor near \( k^{-1/2} \) where this maximum occurs. Clearly, therefore, both the density and depth profiles must be modelled accurately in any study of short waves.

Hence some means of numerical calculation of modes is required. This is also necessary for quantitative results, the treatment of moderate stratification and wavenumbers and the handling of realistic depth and density profiles. The method of inverse iteration used here exploits the property of the variational formulation (2.4) that an approximation \( p(\varepsilon) = p + O(\varepsilon) \), \( \varepsilon \ll 1 \), to a natural mode \( p \) yields an estimate of \( k/\sigma \) with error \( O(\varepsilon^2) \). An improved approximation \( p(\varepsilon^2) \) to \( p \) may then be obtained. Inverse iteration appears to provide a rapid and fairly reliable means of calculating coastal trapped waves. It requires that the modes are found in order beginning with the lowest, a snag mitigated in the coastal trapped waves problem by the expected greatest importance of the lowest modes and the need to store only their bottom boundary values for subsequent use. However, the calculations are performed for given

\[
E^2 = D^2 (1 - \sigma^2), \quad G = (1 - \sigma^2)/S, \quad k,
\]

rather than for given values of \( D^2, S \) and \( k \) which are specified by the context. Most of the calculations in Sections 5 to 9 have avoided this problem by taking \( k \) so small \((10^{-4})\) that the resulting small values of \( \sigma \) render \( E^2 \) and \( G \) indistinguishable from \( D^2 \) and \( 1/S \). This is a good approximation for long trapped waves; variations of \( k \) within moderate values \( (\ll 1, \text{say}) \) may be expected to have little effect on wave speeds or forms (Section 7).

Other numerical methods are also possible. Wang and Mooers (1976) used resonance iteration, which bears some similarity to inverse iteration in using an iteratively adjusted forcing function to excite the desired mode. Time integration is probably less efficient (Platzman, 1972), although the above difficulty with the interdependence of parameters is avoided. For a coarse grid, Section 4c describes an exact solution formed as the appropriate combination of the \( KM \) vertical structure modes associated with \( KM \) grid levels. One can also envisage a Rayleigh-Ritz procedure using the variational statement (2.4) with the first few vertical structure modes as trial functions (they already satisfy the interior equation and seaward and surface boundary conditions).

Good estimates of the wave speed but poor representation of smaller scale structure (e.g., over a shallow shelf) might be expected.

One motive for this work is the wish to assess the role coastal trapped waves may play in upwelling, e.g., off northwest Africa, following the suggestion of Gill and Clarke (1974) that such waves may import effects of winds elsewhere along the coast. Fig. 11 shows a mean stratification profile along a section near 23°N off northwest Africa during the joint Meteor/Discovery cruise "Auftrieb '75—Upwelling '75." The mode 1 velocity components \( u, v \) are illustrated in Fig. 12; the much stronger longshore component shows a marked intensification over the shelf. A more direct comparison with the other figures is furnished by mode 2 (Fig. 13), for which the intensification over the shelf is exhibited even more strongly by both velocity components. We should like to associate this with some particular feature(s) of the situation.

Variation of \( D^2, S \) or \( k \) gave no intensification of this nature, and the calculations of Section 8 suggest that the depth profile is primarily responsible. The shallow shelf and maximum of \( h^{-1} dh/dx \) near the shelf edge intensify the motion over the shelf and upper slope. The general increase of stratification towards the surface may also assist in raising the modal structure. The lack of vertical structure in the shelf currents also accords with Section 8.

The depth profile and the stratification magnitude and form \( Sn^2 \) seem to be the most significant factors influencing the modes in practice. An example of this is the absence of "kissing modes" in the model with uniform slope and stratification. One expects this phenomenon when the dispersion curves of two modes would otherwise cross, the two modes being

---

**Fig. 11.** Mean stratification profile \( Sn^2 \) near 23°N, northwest Africa.
clearly identifiable as a barotropic shelf wave and an internal Kelvin wave. Clear identification of the latter, however, requires that it "sees" water of almost constant depth. Hence Allen (1975) assumed that the internal Rossby radius of deformation was much less than the topographic length scale $h(h/dx)^{-1}$, so that the internal Kelvin wave was confined in almost constant depth against a coastal wall. This is not possible if a beach replaces the coastal wall and $h(dh/dx)^{-1}$ tends to zero at the coast. Thus in general the wave modes involve both the topographic and stratified elements of the situation and are characterized simply by their degree of oscillation along the sloping sea floor (where the orthogonality relation applies), rather than as "barotropic shelf waves" or "internal Kelvin waves". Certainly, internal Kelvin waves cannot be identified in the constant slope model (except in the strong stratification limit when the whole of the sloping floor ($0 < x < 1$) is seen as a side wall; then they form all the trapped modes and there are still no "kissing modes").

Although many cases have been considered, the present study is restricted even in terms of its five independent parameters: horizontal divergence $D^2$, stratification $S$, wavenumber $k$, depth profile $h$ and stratification profile $n^2$. Varying one at a time is illustrative rather than comprehensive. Another potentially important factor not considered is a steady "background" longshore current and associated horizontal density gradient. This might be expected to become important if the steady current speed is a substantial fraction of the wave speed. The calculated speeds of modes 1 and 2 at the $23^\circ$N northwest Africa section are approximately 2.3 and 1.3 m s$^{-1}$, respectively, so that mean currents above 0.5 m s$^{-1}$ may be important. Mean current shears contributing potential vorticity changes comparable with those due to the depth variations will also be significant. Higher modes may be sensitive to weaker currents. The influence of a steady current may be regarded as a special nonlinear interaction; this subject, of course, is beyond the scope of the linear wave equations which form the basis of the calculations here.

Fig. 12. Velocity components $u$ and $v$ for mode 1 (northwest Africa: $D^2 = 0.0013$, $k = 0.1$); $k\Delta v/\Delta u = 7.72$. Depth profile truncated at 2300m.

Fig. 13. Velocity components $u$ and $v$ for mode 2, northwest Africa; $k\Delta v/\Delta u = 9.76$. 
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APPENDIX A

Implications of Variational Formulation (2.4)

1. We wish to show that Eq. (2.4) implies

\[ \frac{\partial \sigma}{\partial D^2} \bigg|_{\sigma,k} < 0, \quad \frac{\partial \sigma}{\partial G} \bigg|_{\sigma,k} < 0, \]

\[ \frac{\partial \sigma}{\partial S} \bigg|_{\sigma,k} > 0, \quad \frac{\partial \sigma}{\partial D^2} \bigg|_{S,k} < 0. \]

Let (2.4) be written as

\[ F(\sigma,t) = \frac{1}{\sigma} - (At + B)(1 - \sigma^2) \]

\[ - (Ct + D) = 0, \quad (A1) \]

where

(i) \( t = D^2, \quad A = I_4/(kI_5), \)

\( B = 0, \quad C = 0, \quad D = (I_1 + G I_2 + k^2 I_3)/(kI_5) \)

(ii) \( t = G, \quad A = 0, \)

\( B = D^2 I_4/(kI_5), \quad C = I_2/(kI_5), \)

\( D = (I_1 + k^2 I_3)/(kI_5) \)

(iii) \( t = 1/S, \quad A = I_2/(kI_5), \)

\( B = D^2 I_4/(kI_5), \quad C = 0, \)

\( D = (I_1 + k^2 I_3)/(kI_5) \)

(iv) \( t = D^2, \quad A = I_4/(kI_5), \)

\( B = I_2/(kI_5), \quad C = 0, \)

\( D = (I_1 + k^2 I_3)/(kI_5). \)

Since the RHS of (2.4) is stationary with respect to variations in \( \sigma, \) \( \partial \sigma/\partial D^2 \bigg|_{\sigma,k} \) is obtained by differentiating (2.4) [written appropriately in terms of \( \sigma, \) \( D^2, \) \( G \) and \( k \)] with respect to explicit appearances of \( D^2 \) holding \( G \) and \( k \) fixed. This is represented by \( d(A1)/dt, \) i.e.,

\[ \frac{\partial F}{\partial \sigma} \frac{d\sigma}{dt} = - \frac{\partial F}{\partial t} = A(1 - \sigma^2) + C. \quad (A2) \]

Similarly, (A2) represents the other cases [(ii)–(iv)] after the corresponding substitutions; we need only show that \( d\sigma/dt < 0. \)

Now

\[ D \geq \frac{I_1 + k^2 I_3}{kI_5} \]

\[ = \frac{\int_1^\infty \left[ (\partial p/\partial x)^2 + k^2 p^2 \right] dx dz}{k \int_{-1}^0 p^2(h^{-1}(-z),z) dz} \geq 1, \]

since

\[ 0 \leq \int_{-1}^0 \int_{h^{-1}}^\infty (\partial p/\partial x + kp) p^2 dx dz \]

\[ = \int_{-1}^0 \int_{h^{-1}}^\infty (\partial p/\partial x)^2 + k^2 p^2 dx dz \]

\[ - k \int_{-1}^0 p^2(h^{-1}(-z),z) dz. \]

Hence \( F(\sigma,t) \) depends on \( \sigma \) as sketched in Fig. 14, with two zeros in \( \sigma > 0, \) one on each side of the minimum at \( \delta F/\delta \sigma = 0. \) Now \( F(1,t) = 1 - Ct - D \leq 0 \) so that \( \sigma = 1 \) lies between the two zeros; the mode frequency \( \sigma < 1 \) must be the lower where \( \delta F/\delta \sigma < 0. \) Hence by (A2), \( d\sigma/dt < 0 \) as required.

2. For case (iv), we have from (A2)

\[ \frac{\partial \sigma}{\partial D^2} \bigg|_{\sigma,k} = \frac{A(1 - \sigma^2)}{1/\sigma^2 - 2(At + B)\sigma} \]

\[ = \frac{A(1 - \sigma^2)^2\sigma^2}{1 - \sigma^2 - 2\sigma^2(1/\sigma - D)} \]

\[ \leq \frac{A(1 - \sigma^2)^2\sigma^2}{1 - 3\sigma^2 + 2\sigma^3} \quad (D \geq 1) \]

\[ = \frac{A\sigma^2(1 + 2\sigma + \sigma^2)}{1 + 2\sigma} \leq \frac{4I_4\sigma^2}{3I_5k}. \]

3. For case (iii), with small \( S, \)

\[ \frac{\partial \sigma}{\partial S} = - \frac{1}{S^2} \frac{d\sigma}{dt} \]

\[ = \frac{A(1 - \sigma^2)/S^2}{1/\sigma^2 - 2\sigma(A/S + B)} \quad \text{by (A.2).} \]
Since

\[ A = I_2(kI_3) \sim S^2 \int n^{-2}(\partial p_i/\partial z)^2 \, dA(k \int \rho_2 \, dz) \]

for small \( S \), by Section 6a,

\[ \sigma(1 - \sigma^2) \int n^{-2}(\partial p_i/\partial z)^2 \, dA \]

\[ \sigma(1 - \sigma^2) kI_3/\sigma - 2\sigma^2 D^2 I_4 \]

as \( S \to 0 \).

APPENDIX B

The Finite-Difference Model

\[ wI_1 = \sum_{K=1}^{K_S} \left[ C_{IK}^1(p_{IK}+1, K) - \hat{p}_K \right]^2 \]

\[ + \sum_{IK=1}^{IS+1} (p_{IK+1} - p_{IK})^2 \]

\[ + X \sum_{IK=1}^{IM} (p_{IK} - \hat{p}_K)^2 \]

\[ + \frac{1}{2}(1 + X)[C_{IK}^1(p_{IK}+1, K) - \hat{p}_K] \]

\[ + \sum_{IK=1}^{IM} (p_{IK} - \hat{p}_K)^2 \]

\[ \int \frac{K_S-1}{K_S} \sum_{I=1}^{IS} (p_{IK} - \hat{p}_K)^2 \]

\[ + \sum_{IK=1}^{IM} (p_{IK} - \hat{p}_K)^2 \]

\[ I_2/w = \sum_{K=2}^{K_S} n_{K-1/2}^2 (C_{2K}(D_Kp))^2 \]

\[ + \sum_{IK=1}^{IS+1} (p_{IK} - \hat{p}_K)^2 \]

\[ + X^{-1} \sum_{IK=1}^{IM} (p_{IK} - \hat{p}_K)^2 \]

\[ \frac{1 - X}{2X} \sum_{K=2}^{K_S-1} n_{K-1/2}^2 (p_{IK} - p_{IK-1})^2 \]

\[ + \sum_{IK=1}^{IM} (p_{IK} - \hat{p}_K)^2 \]

where

\[ D_Kp = C_{IK}^{-1} \left[ \hat{p}_K - (1 - C_{IK}p_{IK+1}, K) \right] \]

\[ - C_{IK}^{-1} \left[ (\Delta K - 1 + C_{IK-1})p_{IK-1}, K-1 \right] \]

\[ + (1 - \Delta K - 1)\hat{p}_{K-1} \]

\[ \Delta K = I(K) - I(K - 1) \]

\[ I_4/wd^2 = \sum_{K=1}^{K_S-1} \left[ X^2 \sum_{IK=1}^{IS} p_{IK}^2 + X(1 + X) (\sum_{IK=1}^{IS} p_{IK}^2) \right] \]

\[ + X \sum_{IK=1}^{IM} p_{IK}^2 \]

\[ + \sum_{K=K+S+1}^{K_S} \sum_{IK=1}^{IM} p_{IK}^2. \]

\[ I_4 = wd(1 + \frac{1}{2}sn_{1/2}D^2Xd)^{-1} \times \left[ \mu XC_{1/2}^2(\hat{p}_K - (1 - C_{IK})p_{IK+1})^2 \right] \]

\[ + X \sum_{IK=1}^{IM} p_{IK}^2 + \frac{1}{2}(1 + X)p_{IK}^2 + \sum_{IK=1}^{IM} p_{IK}^2. \]

\[ I_6 = d[X \sum_{IK=1}^{IS} \hat{p}_K^2 + \frac{1}{2}(1 + X)\hat{p}_K^2 + \sum_{IK=1}^{IM} \hat{p}_K. \]

\[ wI_6 = \sum_{m=1}^{KM} (\dot{p}_m, \dot{p}_m)^2 (1/f_m - 1) \]

(see Section 4b).

For evaluation, but not for differentiation,

\[ wI_6 = (\dot{p}_m, \dot{p}_m) \]

\[ KS, KM, IS, IM, IU, X, w \text{ and } d \] are shown on Fig. 4. For a given level \( K, I(K) \) is the least \( I \) for which a grid point is included, the criterion being that the horizontal separation from the sloping bottom is at most \( 1/2wd \).

\( \hat{p}_K \) represents \( C_{IK}p_{IK}, (1 - C_{IK})p_{IK+1}, K \) and is used as a replacement for \( p_{IK}, K \) in order to retain an orthogonality relation between different modes \( \dot{p}_m, \dot{p}_m^m \), viz,

\[ C_{IK}, C_{2K} \text{ and } \mu \text{ are constants whose values are not immediately apparent in the translation to finite differences; they are chosen so that the bottom boundary conditions, obtained by considering variations of } \hat{p}_K, \text{ are first-order representations of the continuous forms. If we write (see Fig. 4) } \]

\[ X_K = I(K) - 1 + \nu_K \]

\[ C_{11} = 1 + \Delta_1 \left( \frac{3}{2x_2} - 1 \right) \]

\[ C_{1K} = 1 - \Delta_K(\frac{1}{2} + x_{KS+1} - IS) \]

\[ C_{1K} = 1 - \Delta_K(x_{KS+1} - IS + X(IS - 1 - x_{KS}), \]

\[ C_{1K} = 1, \]

\[ C_{1K} = 1 - \Delta_K(\frac{3}{2} + x_{KS} - I(K + 1))/(x_{KS} - x_K) \]
otherwise,
\[ C_{2K} = \frac{1}{2X} + IS - 1 - x_{KS}, \]
\[ C_{2K} = I(K) - \frac{1}{2} - x_K \]
otherwise,
\[ \mu = \frac{1}{2}. \]

For definiteness, the grid is arranged (by choice of \( w \) as necessary) so that \( I(K + 1) \leq I(K) + 1 \) and \( IS = I(KS - 1) + 1 \).

**APPENDIX C**

**Large Wavenumber \( k \)**

With the indicated expansion in powers of \( \epsilon \), Eqs. (7.1) and (7.2) become, respectively,
\[ 0 = \epsilon^2 \hat{a} \hat{a} p_0 / \hat{a} \eta^2 - 2 a \hat{a} p_0 / \hat{a} \eta - 2 h' / a \hat{a} \eta / \hat{a} \eta \]
\[ + \epsilon^2 \hat{a} \hat{a} p_0 / \hat{a} \eta^2 - 2 a \hat{a} p_0 / \hat{a} \eta \]
\[ + 2 h' / a \hat{a} \eta + a \hat{a} p_0 / \hat{a} \eta + \hat{a}^2 p_0 / \hat{a} x^2 \]
\[ - (\phi' + h') \hat{p}_0 / \hat{a} \eta \]
\[ + 2(1 - \eta) a \hat{h' a'}(a^2 + a_3 p_0) + O(\epsilon^3) \]
(C1)

where
\[ a_1 = \epsilon^{-1} \hat{a} a / \hat{a} \xi = 0 \]
\[ a_2 = \epsilon^{-2} \hat{a} a / \hat{a} \eta = 0 \]
and
\[ 0 = \epsilon^2 \hat{a} \hat{a} p_0 / \hat{a} \eta + \hat{h'} \hat{p}_0 / \hat{a} x^2 \]
\[ + \left( \frac{1}{1 - \eta} \hat{a} / \hat{a} \eta + \hat{a} \hat{a} p_0 / \hat{a} \eta \right) \]
\[ + \epsilon^2 \hat{a} \hat{a} p_0 / \hat{a} \eta + \hat{h'} \hat{p}_0 / \hat{a} \eta + (1/\sigma) h' \hat{p}_0 / \hat{a} \eta \]
\[ + (1/\sigma) h' \hat{p}_0 / \hat{a} \eta = O(\epsilon^3), \eta = 0. \]
(C2)

Order \( \epsilon \). Eq. (C2) requires
\[ p_1 = -h' \eta a^{-2} \hat{p}_0 / \hat{a} x^2 + A(\xi) + B(\xi) \epsilon^{2n/3}, \]
where \( A(\xi) = 0 \) for boundedness at large \( \eta \) and we may take \( A(\xi) = 0 \) by inclusion of \( \epsilon A \) in \( p_0(\xi) \). Thus let \( p_1 = -h' \eta a^{-2} \hat{p}_0 / \hat{a} x^2 \); the terms of (C1) and (C2) in \( \epsilon \) now sum to \( O(\epsilon^3) \). Then (C2) also requires
\[ h' \hat{p}_0 / \hat{a} \eta + (1/\sigma) h' \hat{p}_0 / \hat{a} \eta = O(\epsilon^3), \quad (\eta = 0) \]
so that \( a'h' = \text{constant} + O(\epsilon^3) \) within \( O(\epsilon) \) of \( x_0 \) on \( z = -h(x) \); i.e., \( a'h' \) is stationary at \( (x_0,z_0) \). We choose \( (1/\sigma) p_0 \) to be the stationary value of \( a'h' \) so that \( (1/\sigma) p_0 \) = 0.

Order \( \epsilon^2 \). Substituting for \( p_1 \), the terms of (C1) in \( \epsilon^2 \) sum to \( O(\epsilon^3) \) if
\[ p_2 = A(\xi) + B(\eta) \]
\[ + 2 a^2 h' / (3/2) a^{-2} \hat{p}_0 / \hat{a} x^2 + B(\xi) + C(\xi) x^2, \]
where
\[ A = \frac{1}{2} (h' / a^4) \hat{a} p_0 / \hat{a} x^2 - \frac{1}{2} a^{-2} (h' a / a^2 + a_3) p_0, \]
\[ B = \frac{1}{2} a^{-2} (1 - h'' / a^2) \hat{p}_0 / \hat{a} x^2 - (\phi' + h') p_0 / \hat{a} x^2 \]
\[ + \frac{1}{2} a^{-2} (h' / a^2 + a_3) p_0. \]

From (C2) we then have
\[ 0 = a^2 B + (1/\sigma) h' p_0 + \epsilon^2 [h'(1/\sigma) - a] p_0, \quad \text{i.e.,} \]
\[ 0 = d^2 p_0 / \hat{a} x^2 + p_0 [h' a / h'^2 + 2 a h' / (\sigma - 1)] \]
\[ - p_0 \epsilon^2 [h'^{2/2} (2 \phi' - h'' / h')] \]
This is the parabolic cylinder equation (Abramowitz and Stegun, 1965) for \( p_0(\xi) \). There is a bounded solution with bounded derivative \( d p_0 / \hat{a} x^2 \) for large negative and positive \( \xi \) if and only if
\[ b^4 = [h'^{2/2} (2 \phi' - h'' / h')]_{x=x_0} > 0 \]
[i.e., \( (x_0,z_0) \) is a minimum of \( a'h' \)] and
\[ (1/\sigma) = \phi / (2 a h' / [2 m + 1] h^2 - h' a / h'^2), \]
\[ m = 0, 1, 2, \ldots, \]
\[ p_0(\xi) = H_m(b \xi) \exp \left[ -(b \xi)^2 / 2 \right], \]
where \( H_m \) is the Hermite polynomial of degree \( m \) and has \( m \) zeros.

**REFERENCES**


