On Geostrophic Adjustment in Sea Straits and Wide Estuaries: Theory and Laboratory Experiments. Part I: One-Layer System

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ABSTRACT

The dynamics of outflows from sea straits and wide estuaries are examined through a simplified frictionless model whose primary motions are not constrained to be quasi-geostrophic. The potential vorticity equation is solved by means of approximate analytical methods. Some of the model predictions are tested in the laboratory.

The mathematical model predicts that an outflow from a channel with uniform velocity distribution deflects to the right or left depending on the depth of the basin into which it debouches. There is a "critical" Rossby number below which the flow separates from one of the basin banks. When a non-uniform velocity is introduced upstream the direction of deflection may differ substantially from the upstream uniform flow case. The model shows that rotation is important whenever the ratio between the relative depth variation to the Rossby number is not negligible; rotational effects can be important even if the ratio between the channel width and the Rossby deformation radius is entirely negligible.

An experimental system consisting of a rotating channel with an abrupt cross-sectional variation was used in the laboratory to test the theory described above. Deflections resulting from "supercritical" conditions were tested qualitatively with favorable results.

1. Introduction

This study is concerned with the interaction of a current with a large body of fluid into which it debouches. Currents from straits connecting oceans with their Mediterranean basins and from estuaries belong to this category. The problem consists of an initial flow in a channel and a subsequent spreading process subject to the influence of the earth's rotation. In the channel the boundaries restrict the streamlines to be straight and parallel; removal of the boundaries, at a point, causes the current to enter into an adjustment process in which it approaches a new state of geostrophic balance. The problem is related to the classical problem of adjustment toward a geostrophic balance (Rossby, 1938). Previous investigations related to the initial flow and the spreading process are mentioned below.

a. Flow in a rotating channel

Huppert and Stern (1974a,b) studied the dynamics of low Rossby number stratified flow in a rotating channel for which the bottom height varied in the downstream direction, and the effect of side walls on the flow in a rotating channel with two-dimensional obstacles. Using the nonrotating hydraulic principle of maximum transport in flow over a weir, Whitehead et al. (1974) investigated the geometrical restriction of sea straits on the rotating flow in the lower layer.

b. Spreading of the flow

Savage and Sobey (1975) studied a turbulent jet issuing horizontally from a circular orifice into a rotating basin. They found that for a deep layer the jet path is a catenoid but is a straight line for a shallow layer as it would be in the absence of rotation. Paul and Lick (1974) studied river discharge into a lake, using a numerical model which included the Coriolis parameter. Their results are symmetrical and deflection of the flow is not noticed. However, their model does not include buoyancy near the outlet, which results in a limited rotational effect. Takano (1954, 1955) investigated slow-moving frictional river discharge. He neglected the inertia terms and showed analytically that as a result of the earth's rotation, the viscous plume deflects to the right in the Northern Hemisphere.

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2 Rossby's problem described the mass and velocity changes required to "adjust" an initially unbalanced current to a state of geostrophic balance.

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These studies are of limited significance to the problem posed in this study where accelerations and mass adjustments cause an evolution toward a new geostrophically balanced flow. The goal of this study is to discuss the dynamics of outflows and the currents associated with them. No attempt will be made to produce detailed models of particular oceanic situations. It is hoped that this study will enable one to draw some general conclusions about the dynamical behavior of rotational outflows and their induced circulation. In order to explain some of the features which occur in the process we address ourselves first to a one-layer model of steady flow where bottom topography, inertia, rotation and pressure are included but friction, buoyancy, diffusion, tidal effect and pre-existing longshore currents are excluded. The next step will be to study a two-layer model which includes buoyancy and the same physical processes (this study will be reported later). It can be shown that, for our study, the neglect of friction is justified for straits and estuaries which are relatively deep [O(10–100)m] and wide [O(1–10)km]. Eddy viscosity estimates given by Defant (1961) and Officer (1976) suggest that for such channels, the time scale required for penetration of frictional effects into the core of the outflow is much larger than the one required for completing the adjustment process. This results in a small ratio of the boundary layers thickness to both the width and the depth of the outflow.

The geometry of the theoretical model has been adopted in such a way as to simplify the boundary conditions and to ensure that the problem is well posed. The hydrostatic and the rigid lid approximation are invoked but the primary motions are not constrained to be quasigeostrophic. The potential vorticity equation yields a linear second-order partial differential equation of the elliptic type which is solved by means of conformal mapping, Fourier integral and Fourier series. The results are presented and discussed in Sections 3 and 4.

Laboratory experiments were performed on a rotating table in order to assess the validity and weaknesses of the mathematical solutions. The results are presented in Section 5 where qualitative agreement with the theory is generally noted.

2. Mathematical formulation

We consider a homogeneous layer of fluid as shown in Fig. 1. The flow is assumed to be steady and the pressure hydrostatic. For hydrostatic motions the pressure is a linear function of z and the horizontal pressure gradients depend on x and y alone. Therefore, it is reasonable to assume that the horizontal velocity components (u, v) are also independent of z. For conditions as such, the potential vorticity equation is (Gutman, 1972, Whitehead et al., 1974)

$$\nabla_H \cdot (\nabla_H \psi / h) + f = hK(\psi),$$

(2.1)

where $\nabla_H$ is the horizontal del-operator, $f$ the Coriolis parameter, $h$ the total depth of the fluid column and $\psi$ a transport function defined by

$$-uh = \partial \psi / \partial y, \quad vh = \partial \psi / \partial x.$$  

(2.2)

The function $K(\psi)$ is to be determined from the upstream conditions.

3. Outflow from a channel with uniform velocity distribution

Upstream in the channel the flow is assumed to be in a geostrophic balance. To simplify the upstream conditions it is assumed that the channel bottom has a transversal slope of $dz_b/dx = fV_0/g$, where $z_b$ is the bottom elevation, $g$ the gravitational acceleration and $V_0 = gf^{-2} \partial \eta / \partial x$ is the geostrophically balanced uniform velocity in the channel ($\eta$ is the surface displacement). With such conditions the column depth $H_0$ in the channel is independent of cross-stream direction x. Upstream, in the channel, we define $\psi$ such that $\psi = 0$ at $x = -b$. Substitution of the upstream condition into (2.1) yields, $K(\psi) = f/H_0$ which from (2.1) gives the governing equation

$$\nabla_H \cdot (\nabla_H \psi / h) = f(h - H_0)/H_0.$$  

(3.1)

\footnote{It will become clear later that due to the rigid lid approximation this assumption can be avoided.}
This equation holds for all streamlines which originate in the channel. In the subsequent analysis, the following nondimensional scaled variables are used:

\[ \psi^* = \psi/(2V_0H_0b), \quad x^* = x/b, \]
\[ H_1^* = H_1/H_0, \quad h^* = h/H_0, \]
\[ Ro = V_0/2fb, \quad F = V_0^2/gH_0, \]
\[ y^* = y/b, \quad \eta^* = \eta/H_0 \]
\[ v^* = v/V_0, \quad u^* = u/V_0 \]
\[ \Delta H^* = (H_1 - H_0)/H_0, \quad \nabla = b\nabla_H \]

(3.2)

The nondimensional equation in the basin \((y^* \geq 0)\) is

\[ 4\nabla \cdot [\nabla\psi^*/(1 + \Delta H^* + \eta^*)] = [\Delta H^* + \eta^*]/Ro. \]  

(3.3)

It can be shown that the rigid lid approximation is valid as long as \(F \ll |\Delta H^*|\) and \(F/Ro \ll |\Delta H^*|\).

For such cases (3.3) can be approximated by the Poisson equation

\[ \psi^* = 0, \quad x^* = -c, \quad 0 < y^* < \infty \]  
\[ \psi^* = 0, \quad -c < x^* < -1, \quad y^* = 0 \]  
\[ \psi^* = (1 + x^*)/2, \quad -1 \leq x^* \leq 1, \quad y^* = 0 \]  
\[ \psi^* = 1, \quad 1 < x^* < c, \quad y^* = 0 \]  
\[ \psi^* = 1, \quad x^* = c, \quad 0 < y^* < \infty \]  
\[ \partial\psi^*/\partial y^* = 0, \quad -c \leq x^* \leq c, \quad y^* \rightarrow \infty \]

(3.6)

where \(c (= a/b)\) is the relative widening of the channel. Conditions (a), (b), (d) and (e) state that the basin walls are streamlines; (c) is the uniform flow assumption at the mouth; and (f) reflects the completed nature of the adjustment far downstream.

The general solution of (3.4) consists of a homogeneous part which satisfies (3.6) and a particular solution which satisfies the homogeneous conditions

\[ \psi_p = 0, \quad x^* = -c, \quad 0 < y^* < \infty \]  
\[ \psi_p = 0, \quad -c < x^* < c, \quad y^* = 0 \]  
\[ \psi_p = 0, \quad x^* = c, \quad 0 < y^* < \infty \]

(3.7)

The detailed homogeneous solution is presented in the Appendix. Far downstream (at \(y^* \rightarrow \infty\)) the solution (A5) reduces to the simple form

\[ \lim_{y^* \to \infty} \psi_H = (x^* + c)/2c \]  

(3.8)

which describes a uniform flow as one expects.

Due to the problem geometry, it is reasonable to assume that far downstream the particular solution \(\psi_p\) is independent of \(y^*\). This assumption is supported by the idea of geostrophic adjustment since one expects the flow to be in geostrophic balance far downstream which requires that the flow also be one-dimensional. Therefore, one may assume that \(\psi_p\) consists of

\[ \psi_p = \tilde{\psi}(x^*, y^*) + \breve{\psi}(x^*) \]  

(3.9)

where

\[ \lim_{y^* \to \infty} \tilde{\psi}(x^*, y^*) = 0 \]
a. The solution far downstream

The general form of the asymptotic solution is

$$\psi^* = \lim_{y^* \to \infty} \psi_H(x^*, y^*) + \tilde{\psi}(x^*),$$

where the first term in the right-hand side is given by (3.8) and \( \tilde{\psi}(x^*) \) is to be determined. From (3.4) and (3.7) one obtains

$$\tilde{\psi}(x^*) = D[(x^*)^2 - c^2], \quad (3.10)$$

where

$$D = \Delta H^*(1 + \Delta H^*)/8 \text{ Ro}. \quad (3.11)$$

The velocity far downstream is

$$v^* = \frac{2}{H_1^*} \left[ \lim_{y^* \to \infty} \left( \frac{\partial \psi_H}{\partial x^*} \right) + \frac{\partial \tilde{\psi}}{\partial x^*} \right]$$

$$= c^{-1}(1 + \Delta H^*)^{-1} + \Delta H^* x^*/2 \text{ Ro}. \quad (3.12)$$

Since both \( \Delta H^* \) and \( x^* \) are either positive or negative the velocity according to (3.12) may become negative for sufficiently small \( \text{Ro} \). However, it can be shown that such a situation is impossible. A negative \( v^* \) corresponds to parcels which have originated at \( y^* \to \infty \) and so have not passed across the step; therefore the right-hand side of (3.4) is altered in such regions and the negative \( v^* \) flow found in (3.12) does not obey the proper equation. One concludes that (3.12) is valid as long as \( v^* > 0 \). If \( v^* \) [as determined by (3.12)] is negative, separation occurs, \( \tilde{\psi} \) can no longer satisfy simultaneously the two conditions \( \tilde{\psi} = 0 \) at \( x = \pm c \), and mathematically the problem becomes poorly defined.

We define a critical Rossby number as the one which first causes separation. For larger than critical Rossby numbers the velocity is always positive in the field. The critical Rossby number is determined from the conditions

- \( v^* = 0; \ x^* = -c \) for a step down \( (\Delta H^* > 0) \)
- \( v^* = 0; \ x^* = c \) for a step up \( (\Delta H^* < 0) \).

Substitution of the above conditions into (3.12) yields the single relation

$$R_{oc} = c^2 |\Delta H^*| (1 + \Delta H^*)/2. \quad (3.13)$$

With \( \text{Ro} < R_{oc} \) one of the boundary conditions should be replaced by a new condition which will be determined from the location of the separation line (where \( \psi^* = 0 \) or unity in the interior of the basin). However, due to the nonuniqueness of the problem it is impossible to determine the location of the separation without an additional assumption. This additional information can be provided by considering the far downstream field of a case with a step down under subcritical conditions as shown in Fig. 2. If the Rossby number is decreased (e.g., rotation rate increased) to its critical value the velocity vanishes at the left bank. If the Rossby number is further decreased, the flow may detach either from the left bank or from the right bank. These two possible velocity distributions are marked by (a) and (b), respectively. At the point of detachment from the right bank [case (b)], the shear near the right bank (looking downstream) becomes infinite. Such infinite shear cannot occur but viscosity could decrease this shear and allow the flow to exist. However, it is reasonable to assume that situation (a) in which there is no velocity discontinuity at the point of detachment is more likely to occur. That is, we assume that the flow favors the situation in which the shear near the wall is minimized.

It is instructive to examine the role of the bottom Ekman layers. One can see that for a flow with a step down (Fig. 2), the Ekman layer on the bottom will flux fluid toward the left in a similar fashion to the bottom flow in the so-called "tea cup problem." Therefore, if case (b) existed the bottom Ekman flow may cause the separated current to drift toward case (a). The same logic implies, however, that for the step up case the Ekman flux may cause the current to drift from the left [corresponds to (a) in the step down case] to

![Fig. 2. The velocity distribution far downstream, for a step down.](image-url)
the right bank [corresponds to (b)]. But, it will be shown later that the laboratory experiments show that this drift does not occur. With a step up the current was found to be stable and concentrated near the left bank. Therefore, one concludes that the Ekman layers on the bottom probably do not play an important role in determining the location of the separation line. It will be shown later, however, that the Ekman layer may alter the details of the solution.

In order to determine the location of the separation line we assume that the velocity vanishes at the separation line even if the separation line is at some distance from the basin bank. This seems to be the only logical condition at the separation line. With the aid of this condition the problem is now uniquely defined.

We shall consider now the separated solution for a step down. Separation occurs if \( \text{Ro} < \text{Ro}_c \); the new boundary conditions at \( y^* \to \infty \), equivalent to a zero value of \( \psi(x^*) \) at \( x^* = \pm c \) in the nonseparating case, are

\[
\psi = 0, \quad x^* = c, \quad x^* = -\delta_{usc}, \tag{3.14}
\]

where the subscript \( \text{usc} \) denotes that the variable in reference is associated with a step down and a supercritical Rossby number, and \( \delta_{usc} \) is the location at which the particular solution vanishes. Note that the particular solution does not vanish at the separation line but at another location.

At the asymptotic separation line \( (x^* = d_{usc}) \) for a step down

\[
\psi^* = \lim_{y^* \to \infty} \psi_H + \tilde{\psi} = 0
\]

\[
v^* = \frac{2}{H_T^*} \left[ \frac{\partial}{\partial x^*} \left( \lim_{y^* \to \infty} \psi_H \right) + \frac{\partial \tilde{\psi}}{\partial x^*} \right] = 0 \tag{3.15}
\]

where \( \tilde{\psi}(x^*) \) has the new form

\[
\tilde{\psi}_{usc}(x^*) = D(x^* - c)(x^* + \delta_{usc}). \tag{3.16}
\]

Substitution of (3.16) and (3.8) into (3.15) yields a pair of algebraic equations with two unknowns, \( \delta_{usc} \) and the location of the separation line \( d_{usc} \). Two solutions exist; the physically relevant solution is

\[
\delta_{usc} = -c + 2|D|^{-1/2} - (2c|D|)^{-1} \tag{3.17}
\]

and

\[
d_{usc} = -|D|^{-1/2} + c. \tag{3.18}
\]

Similarly, the new boundary conditions for \( \psi \) with a step up and a supercritical Rossby number are

\[
\tilde{\psi} = 0, \quad x^* = -c, \quad x^* = \delta_{usc}, \tag{3.19}
\]

where \( \text{usc} \) denotes that the variable in reference is associated with a step up and a supercritical Rossby number. At the separation line \( (x^* = d_{usc}) \),

\[
\psi^* = \lim_{y^* \to \infty} \psi_H + \tilde{\psi} = 1
\]

\[
v^* = \frac{2}{H_T^*} \left[ \frac{\partial}{\partial x^*} \left( \lim_{y^* \to \infty} \psi_H \right) + \frac{\partial \tilde{\psi}}{\partial x^*} \right] = 0 \tag{3.20}
\]

where

\[
\tilde{\psi}_{usc}(x^*) = D(x^* + c)(x^* - \delta_{usc}). \tag{3.21}
\]

By substitution of (3.8) and (3.21) into (3.20) one obtains, as before, two algebraic equations. Their solution is \( \delta_{usc} = |D|^{-1/2} - c; \delta_{usc} = \delta_{usc} \). Thus the current width is simply \( |D|^{-1/2} \) for steps up or down. The dependence of \( D \) on \( \Delta H^* \) indicates that the width will be smaller for a step down; the dependence on \( \text{Ro} \) shows the confining effect of the earth's rotation.

b. The total solution

The total solution has the general form

\[
\psi^* = \psi_H(x^*, y^*) + \tilde{\psi}(x^*, y^*) + \psi(x^*). \tag{3.22}
\]

The homogeneous solution \( \psi_H(x^*, y^*) \) was mentioned earlier and is given by (A5). \( \psi(x^*) \) is given by (3.10) for subcritical flows, and by (3.16) and (3.21) for supercritical flows with a step down or up, respectively. The function \( \tilde{\psi} \) is to be determined; it should satisfy condition (3.9) and its values on the boundaries should be such that the boundary condition (3.7) will be satisfied. We can find \( \psi \) by assuming an exponential decay in \( y^* \) and using a Fourier series representation. The detailed derivation of \( \psi \) is given in the Appendix. As with \( \psi, \tilde{\psi} \) has a different expression for each flow regime. For subcritical flows \( \psi \) is given by (A9) and for supercritical flows with a step down or up \( \psi \) is given by (A10) and (A11), respectively.

Typical total solutions are shown in Figs. 3 and 4. These figures illustrate subcritical and supercritical conditions and sensitivity to \( \text{Ro} \) and \( \Delta H^* \). It should be noted that the dependence on \( \text{Ro} \) and \( \Delta H^* \) is linked since for a given \( c \) the solution depends only on \( D \). Fig. 3 shows non-separating flows for steps up and down. The center streamline deflects to the right (left) for parcels which have experienced a cyclonic (anticyclonic) vorticity in crossing the step. Fig. 4 shows corresponding separating flows for the same basin width as shown in Fig. 3.

A somewhat disturbing matter is the difficulty of finding analytically whether the functions which constitute the total solution, produce a continuous separation line which intersects with the proper boundary. About 40 numerical calculations of \( \psi^* \) were performed with \( 3 \leq c \leq 18, 0.05 \leq \text{Ro} \leq 0.2 \) and \(-0.3 \leq \Delta H^* \leq 0.3 \); they showed a continuous separation line which intersects with the proper
boundary in all cases. However, for the range \( c = 1, \Delta H^* = 0.3 \) and \( \text{Ro} < 0.1 \) the separation line did not intersect with the proper boundary near the outlet, and for this range of parameters the total solution presented in this section is not valid. The reason for this behavior may be related to the assumption that upstream the streamlines remain unaltered until reaching the outlet, or to the assumption that \( u^* = 0 \) on the asymptotic separation line. The first of these two possibilities seems more reasonable since in reality there will be an upstream influence a distance of \( O(b) \) upstream from the step. This results in alteration of the solution also in the downstream direction [within a radius of \( O(b) \)]. Unfortunately, it is not simple to avoid this assumption since it requires solving the governing equations in two different domains (channel and basin) and matching the solutions at the mouth. The governing equations in each of these two regions differ from each other since upstream the parcels did
not pass across the step. This makes the solution even more complex.

However, since the range of parameters for which the total solution is not valid is fairly small in comparison to the range which is of interest the limitation does not seem to be of severe restriction.

c. The exterior flow

In the exterior region (i.e., across the separation line) the fluid again obeys the basic law (2.1) but \( \Delta H^* = 0 \) since these parcels have not passed the step. The domain is long and, except near the mouth, the flow is nearly one dimensional; therefore, it is reasonable to assume that it can be approximately described by the quasi-geostrophic theory. For this layer of fluid confined between a flat rigid bottom and a free surface the quasi-geostrophic potential vorticity equation in the basin is (Stern, 1975)

\[
(D/Dt)[\nabla_*^2 \psi - \psi/\lambda^2] = 0, \tag{3.23}
\]
where $\lambda$ is the Rossby deformation radius. For steady flow (3.23) can be written in the form

$$\nabla^2 \psi - \psi/\lambda^2 = K(\psi),$$

(3.24)

where $K(\psi)$ is to be determined. Here $\psi$ may be interpreted as a geostrophic streamfunction $g\eta/fH_1$. The velocity distribution far downstream is unknown and one is unable to determine $K(\psi)$ without additional information. The latter can be provided by assuming that, since there is no mechanism by which potential vorticity can be transferred from the interior to the exterior, there is no potential vorticity in the isolated exterior region. This condition yields $K(\psi) = 0$, which implies

$$\nabla^2 \psi - \psi/\lambda^2 = 0.$$  

(3.25)

In view of the geometry of the exterior domain one may approximate the latter by $\psi_{yz} - \psi/\lambda^2 = 0$ for large $y$. The boundary conditions are that $\psi$ vanishes at two parallel lines. It follows that in the exterior $\psi(x, y) = 0$ at large $y$ and by virtue of the known property of the elliptic equation (3.25) (see, e.g., Forsythe and Wasow, 1960) $\psi(x, y) = 0$ in the entire exterior domain. We conclude that there is no motion in the whole exterior domain.

The above discussion has the following weakness. A basic assumption was that the exterior can be described by quasi-geostrophic theory. The latter requires that a line of constant $\psi$ will also be a line of constant $\eta$, while numerical calculations of $\eta$ based on the solution of the interior show that $\eta$ is constant along the separation line far downstream but is not necessarily constant in the vicinity of the outlet. That is, near the outlet the quasi-geostrophic theory satisfies only one of the two matching conditions. Therefore, it can be applied to the exterior only if the motions which are caused by height variations along the separation line are negligible in comparison to motions in the interior. It is reasonable to assume that height variations along the separation line can cause motions in the exterior which are comparable to the velocities in the interior along the separation line. Numerical calculations (for the cases mentioned in Section 3b) showed that the velocities along the separation line are small in comparison to the main interior motion only for the cases where the width of the interior is not very small in comparison to the basin width.

In the cases where the width of the separation current is small in comparison to the basin width one expects to find a stagnant region far downstream but a cyclonic or anticyclonic circulation near the outlet. This can be demonstrated by the following example. Fig. 4B shows that the width of the interior at say $y^* = 10$ is about two-thirds of the interior width at $y^* \to \infty$. Since the shear in the two locations must be almost the same one concludes that at $y^* = 10$ the velocity in the interior at the separation line is about one-third of the velocity near the left wall. Therefore, in such an extreme case one expects to find in the exterior near the channel outlet, a flow (which cannot be described by the quasi-geostrophic theory) with velocities of about a third of the velocities in the interior. A detailed solution of this field is beyond the scope of this study.

4. Outflow from a channel with a linear velocity distribution

In this section we shall consider a different version of the step problem. Assume that upstream in the channel the velocity field is $\psi = V_n + Ax$, where $A$ is a positive or negative constant shear parameter. Using the rigid lid approximation, the function $K(\psi)$ is found from the upstream boundary condition and (2.1) to be $K(\psi) = (A + f)/H_0$, which implies

$$\nabla^2 \psi = (AH_1 + f\Delta H)H_1/H_0.$$  

(4.1)

This equation is a generalization of (3.1) for the interior flow in the basin. In nondimensional form (4.1) is

$$\nabla^2 \psi^* = (1 + \Delta H^*)$$

$$\times [\text{Ro}^{-1}\Delta H^* + 4S^{-1}(1 + \Delta H^*)]/4.$$  

(4.2)

where the nondimensional shear parameter $S = 2V_n/bA$ can be either positive or negative. Eq. (4.2) is identical in structure to (3.4), but the boundary condition at the outlet (i.e., $-1 \leq x^* \leq 1$, $y^* = 0$) is somewhat different than the one considered in the previous section. At the outlet, $\psi^*$ is no longer given by $(1 + x^*^2)/2$ only; it has the additional term $[(x^*)^2 - 1]/2$. However, this outlet condition enters the problem through the homogeneous (potential) solution only and any changes in the distribution of $\psi^*$ at the outlet can be considered as imaginary sources and sinks (located at the mouth) whose net transport is zero. Therefore, the influence of such changes is confined to the immediate vicinity of the outlet, and one concludes that downstream, solutions of (4.2) are identical in structure to solutions of (3.4). That is, the current ultimately deflects to the right if the right-hand side of (4.2) is positive and to the left if it is negative. The separation conditions and width of the final current are now dependent on both $\Delta H^*$ and $S$.

To demonstrate the above we shall consider a special case where $S^{-1} = 0.5$ and $\text{Ro}^{-1}\Delta H^* + 4S^{-1}(1 + \Delta H^*) = 0$, corresponding to cancellation of the initial vorticity by the step. For this case, the governing equation (4.2) reduces to the Laplace equation. The boundary conditions are the same as (3.6) except at the outlet where

$$\psi^* = (1 + x^*^2)/4.$$  

(4.3)

The detailed solution is presented in the Appendix and is given by (A13). Fig. 5 shows the
streamlines for this case. Without the initial shear the current would have deflected to the left, but due to the initial channel vorticity the flow has a slight deformation near the outlet, but has no net deflection downstream.

5. Laboratory experiments

a. Basic design considerations

The experimental apparatus was designed to qualitatively test the theory described previously. A schematic diagram of the experimental apparatus is shown in Fig. 6. The test section consists of a channel 7.7 cm wide and a basin 30.5 cm wide, 30.5 cm long and 12.7 cm high. The apparatus was designed such that frictional forces which arise due to vertical walls have a limited effect on the flow. This has been achieved by choosing the apparatus characteristics such that the minimum channel Reynolds number is of the order of several hundred, and whenever possible by constructing the channel width and length so that the horizontal boundary layer thickness is small in comparison with the channel width. However, this boundary layer is not necessarily thin compared to the separated current width. For example, Schlichting (1968) gives $5(\nu/l/\nu)^{1/3}$ for the former, which for $l = 15$ cm, $\nu \approx 0.5$ cm s$^{-1}$, and $\nu = 10^{-2}$ cm$^2$s$^{-1}$ is $\approx 2.5$ cm—a thickness comparable to the separated current. Thus the details of the final current will be influenced by side boundary friction.

It is expected that frictional forces on the bottom will establish an Ekman layer of finite thickness $\pi(\nu/\Omega)^{1/3}$ which for $\Omega = 2$ rad s$^{-1}$ gives a thickness of $\approx 2$ mm, much smaller than the working layer thickness of $\approx 7$ cm. The associated spin down time scale $t = O(f^{-1}H(\Omega/\nu)^{1/3})$, (Greenspan, 1968) for the influence of secondary motion upon the main layer is of $O(10)$ s). The advection transport time scale for the basin is of $O(30)$ s. Therefore, bottom friction and secondary circulations can also significantly influence the laboratory results.

The channel was designed such that the flow at the outlet is close to a state of geostrophic balance. This was accomplished by extending the channel length downstream from the filter so that the corresponding Rossby number based on the distance from the filter to the outlet was small.

b. Apparatus and method of observation

The walls and bottom were constructed from 1.25 cm plexiglass plates. The bottom height could be changed by removing or adding 2.5 cm thick plexiglass plates in either the channel or the basin. This enables one to provide steps up or down as desired.\(^4\) Two filters were constructed (in the inlet and outlet) to produce a desired velocity distribution across the channel. Filters with a constant thickness (measured downstream along the $y$ axis) produce a uniform velocity distribution due to the viscosity of the water and the porosity of the filter. A triangle-shaped filter (the thickness varies linearly in $x$ from almost zero at $x = b$ to 9 cm at $x = -b$) was used to produce a linear velocity distribution across the channel. An impeller 12 V pump was used to maintain a flow of $\approx 3$ l min$^{-1}$ through the channel. Typical flow speeds in the channel were 0.5 cm s$^{-1}$. To eliminate the influence of air

\(^4\) The height of the step was chosen to be large in comparison to the surface variations caused by the rotation of the table ($\approx 1$ cm in the basin).
flow on the fluid, the test container was covered with a clear flat plexiglass plate. The system was mounted on an 80 cm diameter turntable driven by a variable speed ac motor with a single reduction gear drive. Before the experiments were performed the basin was leveled to within 30° of arc and centered to less than ±0.1 cm of the rotation axis of the table. Short-term stability of the rotation rate (in few revolutions) was checked by a strobe light and found to be within a fractional deviation of 0.1% with a careful balance of the apparatus on the table. The rotation rate was very stable over long periods of time and the deviation was not larger than 0.5% in the long runs (1 h). A typical rotation rate was \( \Omega = 2 \text{ rad s}^{-1} \). The Rossby number varied between 0.01 and 0.05.

Water was used as the working fluid and for flow visualization a Dupont rodamine dye was injected into the fluid after a steady state had been reached. A 35 mm camera was mounted on a stationary frame vertically above the rotating test container. A picture was photographed every few seconds as the dye was advected by the fluid.

c. Experimental procedure and results

Experiments were performed only for separating flows (supercritical Rossby numbers) since nonseparating flows require a parabolic bottom with deviation of ~1 cm over the basin. The depth of the resting fluid varied between 7.5 and 10 cm. Experiments with an abrupt step down (\( \Delta H = +2.5 \text{ cm} \)) showed a deflection to the right and with a sudden step up (\( \Delta H = -2.5 \text{ cm} \)) a deflection to the left as the theory predicts. These experimental results are shown in Fig. 7; one may compare them with the theoretical prediction for approximately the same width ratio and step size shown earlier in Fig. 4. Detailed comparison is impossible due to (i) the influence of friction, (ii) the fact that the dye width does not represent the current width and (iii) the depth variation in the \( x \) and \( y \) directions. However, the photographs show that the high velocity is near the basin wall as the theory shows. About 30 experiments were performed with \( \Omega = 1.9 \text{ rad s}^{-1} \) and 0.01 \( \approx \text{Ro} \approx 0.05 \) (where \( \text{Ro} < \text{Ro}_c \)) and all showed the result mentioned above.

With a completely flat bottom (no step) the current deflected to the left (Fig. 8) due to the parabolic shape of the surface which acts as a gradual step up on the inlet side of the basin. When a linear velocity distribution with a cycloidal shear of about 0.13 s\(^{-1}\) was introduced upstream, the deflection was neutralized (Figs. 8c and 8d) as the theory predicts (Fig. 5). The proper rotation rate for this experiment was determined by assuming that the gradual step up [\( \Delta H(r) = \Omega r^2 / 2g \)] can be considered as an abrupt step located at the outlet. For such conditions (4.1) yields \( \Omega = (g \text{Ho} A / r^2)^{1/3} \). For \( A = 0.13 \text{ s}^{-1} \), \( r = 15 \text{ cm} \) and \( H_0 = 5 \text{ cm} \), this relation gives \( \Omega = 1.4 \text{ rad s}^{-1} \). A number of preliminary experiments and adjustments of the volume flux were needed in order to achieve the flow pattern shown in Fig. 8. Photographs (c) and (d) in Fig. 7 and (a) and (b) in Fig. 8 show a similar deflection while the step in the first two was about four times higher than the equivalent step in the last two. The theory predicts a stronger deflection in the first case and this disagreement between the theory and the experiment may be a result of bottom friction. The ratio

![Fig. 7. The flow pattern of the uniform flow experiment with the basin's bottom deeper than the channel [(a) and (b), taken at increasing times from dye injection] and the basin's bottom higher than the channel [(c) and (d) taken at consecutive times]. Bottom's elevation variations are much larger than the free-surface height variations (caused by rotation). Physical constants: \( c = 4, F = 10^{-3}, \Omega = 1.9 \text{ rad s}^{-1}, \text{Ro} = 0.018, \Delta H = 2.54 \text{ cm} \).](image-url)
between the thickness of the bottom boundary layer and the fluid depth was, in the first case, about twice as large as the ratio for the second case. Therefore, bottom friction was more important in the first case and perhaps retarded the deflection more effectively.

6. Summary

The results of the theory can be summarized as follows:

1) An outflow from a channel with a uniform velocity distribution deflects to the right (in the Northern Hemisphere) if the basin is deeper than the channel and to the left if it is shallower.

2) There is a "critical" Rossby number below which the current separates from one of the basin banks, forms a longshore current with a linear velocity distribution, and downstream, produces a stagnant domain across the separation line. The critical Rossby number is a function of the step size and the ratio between the width of the basin to the width of the channel.

3) When a nonuniform velocity distribution is introduced upstream in the channel the direction of deflection may differ from the one mentioned above. The current deflects to the right if the sum of the initial relative vorticity and the vorticity created by the step is positive, and to the left if the sum is negative. If the sum of the two is zero, there will be a slight deformation in the vicinity of the outlet, but there will be no-deflection downstream.

4) Rotation is important if the ratio between the relative step size to the Rossby number is not negligible.

The basic results mentioned above can also be obtained by quasi-geostrophic theory. However, this theory is more restrictive and requires that both the relative step size and the Rossby number be small, in addition to the assumptions which have been made throughout the study. Hence, the full theory allows for flows which may be far from local states of geostrophic balance. Prediction 1) and the last part of prediction 3) were tested in the laboratory, the direction of deflection as well as the qualitative velocity distributions agreeing with the theory in both cases.

APPENDIX

The Detailed Solution

a. The homogeneous solution $\psi_h(x^*, y^*)$

We shall find the homogeneous solution by transforming the interior of the basin into the upper half of the complex plane. Such mapping is done by a straightforward Schwartz-Christoffel transformation of the form

$$\Phi = \alpha + \beta i = \sin[\pi(x^* + iy^*)/2c],$$  \hspace{1cm} (A1)

where $\Phi$ denotes the complex plane and $\alpha$ and $\beta$ are

$$\alpha = \sin\left(\frac{\pi x^*}{2c}\right) \cosh\left(\frac{\pi y^*}{2c}\right),$$  \hspace{1cm} (A2)

$$\beta = \cos\left(\frac{\pi x^*}{2c}\right) \sinh\left(\frac{\pi y^*}{2c}\right).$$

The transformed boundary conditions are...
\[\psi_H(\alpha,0) = 0,\]
\[\psi_H(\alpha,0) = \frac{1}{2}[1 + (2c/\pi) \sin^{-1} \alpha],\]
\[\psi_H(\alpha,0) = 1,\]

and the solution is given by the Fourier integral

\[\psi_H = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta E(\varphi) d\varphi}{\beta^2 + (\alpha - \varphi)^2},\]  

(A4)

where \(E(\varphi) = \psi_H(\alpha,0)\) and is given by (A3). The above integral yields (see, e.g., Gradshteyn and Ryzhik, 1968)

\[\psi_H = \frac{\alpha}{2} + \left(\frac{c\alpha}{\pi^2} + \frac{1}{2\pi}\right) \tan^{-1}\left(\frac{\alpha + \pi/2c}{\beta}\right)\]
\[\quad - \left(\frac{c\alpha}{\pi^2} - \frac{1}{2\pi}\right) \tan^{-1}\left(\frac{\alpha - \pi/2c}{\beta}\right)\]
\[+ \frac{\beta c}{\pi^2} \ln\left[\frac{\beta^2 + (\alpha + \pi/2c)^2}{\beta^2 + (\alpha - \pi/2c)^2}\right]\]
\[+ O(\pi/24\alpha^2),\]  

(A5)

where \(\alpha(x^*,y^*)\) and \(\beta(x^*,y^*)\) are given by (A2) and we have used the approximation \(\sin \alpha = \alpha\) for \(\alpha \ll 1\) in calculating the integral from \(-\sin(\pi/2c)\) to \(\sin(\pi/2c)\). Due to this approximation (A5) holds, with reasonable accuracy, only for basins with \(c > 2 \sim 3\). However, it is easy to show that for \(c = 1\) the exact solution is \(\psi_H = (1 + x^*)/2\) in the whole field.

b. The function \(\tilde{\psi}(x^*,y^*)\)

The relation between \(\tilde{\psi}\) and \(\tilde{\psi}\) is given by (3.9). It can be easily verified that the function

\[\tilde{\psi} = \sum_{n=1}^{\infty} A_n \exp(-n\pi y^*/2c) \sin[n\pi(x^* + c)/2c]\]  

(A6)

satisfies the Laplace equation and the homogeneous boundary conditions at all boundaries except at \(y^* = 0\). However, one may express the coefficients \(A_n\) such that at \(y^* = 0\) the total particular solution (the sum of \(\tilde{\psi}\) and \(\tilde{\psi}\)) will satisfy the complete homogeneous boundary conditions

\[\sum_{n=1}^{\infty} A_n \sin[n\pi(x^* + c)/2c] = -\tilde{\psi}(x^*),\]  

(A7)

\[\psi_H(\alpha,0) = 0,\]
\[\psi_H(\alpha,0) = \frac{1}{2}[1 + (2c/\pi) \sin^{-1} \alpha^2],\]
\[\psi_H(\alpha,0) = 1,\]

\[-\infty < \alpha < -\sin(\pi/2c)\]
\[-\sin(\pi/2c) \leq \alpha \leq \sin(\pi/2c)\]
\[\sin(\pi/2c) < \alpha < \infty\]

which yields

\[A_n = -(1/c) \int_{-\infty}^{\infty} \tilde{\psi} \sin[n\pi(x^* + c)/2c].\]  

(A8)

We shall first construct the subcritical flow (Ro \(> Ro_c\)) \(\psi_{bc}\) (the subscript \(bc\) denotes that the variable in reference is associated with subcritical conditions). Substitution of (3.10) into (A8) and consideration of (A6) give

\[\tilde{\psi}_{bc} = -16c^2D \sum_{n=1}^{\infty} [(\cos n\pi - 1)/n^3\pi^3]\]
\[\times \exp(-n\pi y^*/2c) \sin[n\pi(x^* + c)/2c].\]  

(A9)

When supercritical (Ro \(< Ro_c\)) conditions hold a further connection is necessary. For a step down one seeks an additional function that at \(y^* = 0\) cancels the contribution of the difference between \(\psi_{\text{asc}}\) given by (3.16) and \(\tilde{\psi}\) given by (3.10). This contribution is linear in \(x^*\) and vanishes at \(x^* = c\). In view of these considerations one finds

\[\tilde{\psi}_{\text{asc}} = \tilde{\psi}_{bc} + 16cD(\delta_{\text{asc}} - c) \sum_{n=1}^{\infty} [(\sin(n\pi/2)/n^2\pi^2]\]
\[\times \exp(-n\pi y^*/4c) \sin[n\pi(x^* - c)/4c].\]  

(A10)

where \(\delta_{\text{asc}}\) is given by (3.17) and \(D\) by (3.11). For a step up with Ro < \(Ro_c\), one seeks an additional function which at \(y^* = 0\) cancels the contribution of the difference between \(\psi_{\text{asc}}\) given by (3.21) and \(\tilde{\psi}\) given by (3.10). With the above considerations one finds

\[\tilde{\psi}_{\text{asc}} = \tilde{\psi}_{bc} + 16cD(\delta_{\text{asc}} - c) \sum_{n=1}^{\infty} [(\sin(n\pi/2)/n^2\pi^2]\]
\[\times \exp(-n\pi y^*/4c) \sin[n\pi(x^* + c)/4c].\]  

(A11)

c. Outflow from a channel with initial vorticity and a cancelling step

to find out the solution for this case we shall apply the Schwartz-Christoffel transformation which was used earlier in calculating the homogeneous solution.

The transformed boundary conditions are

\[\tilde{\psi}(x^*) = \psi_{bc} + 16cD(\delta_{bc} - c) \sum_{n=1}^{\infty} [(\sin(n\pi/2)/n^2\pi^2]\]
\[\times \exp(-n\pi y^*/4c) \sin[n\pi(x^* + c)/4c].\]  

(A12)
Substitution of (A12) and \(\sin \alpha \approx \alpha\) into the Fourier integral (A4) yields

\[
\psi^* = \frac{\beta}{2\pi} \int_{-\pi/2c}^{\pi/2c} \frac{2c^2 \varphi^2/\pi^2 + 2c \varphi/\pi + 1/2}{\beta^2 + (\alpha - \varphi)^2} d\varphi
\]

\[
+ \frac{\beta}{\pi} \int_{\pi/2c}^{\infty} \frac{d\varphi}{\beta^2 + (\alpha - \varphi)^2} + O(\pi/24c^2)
\]

which gives

\[
\psi^* = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{\alpha - \pi/2c}{\beta}\right)
\]

\[
+ \frac{1}{\pi}\left[\frac{1}{4} + \frac{c\alpha}{\pi} + \frac{c^2(\alpha^2 - \beta^2)}{\pi^2}\right]
\]

\[
\times \left[\tan^{-1}\left(\frac{\alpha + \pi/2c}{\beta}\right) - \tan^{-1}\left(\frac{\alpha - \pi/2c}{\beta}\right)\right]
\]

\[
+ \frac{c\beta}{\pi^2} + \frac{\beta c}{\pi^2}\left(\frac{1}{2} + \frac{\alpha c}{\pi}\right)
\]

\[
\times \ln\left[\frac{\beta^2 + (\alpha - \pi/2c)^2}{\beta^2 + (\alpha + \pi/2c)^2}\right]
\]

\[+ O(\pi/24c^2). \quad (A13)\]

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