NOTES AND CORRESPONDENCE

Observational Evidence for Vertical Diffusion Driven by Internal Waves of Tidal Origin in the Oslofjord

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ABSTRACT

In an earlier paper (Stigebrandt, 1976), a model was proposed for internal wave-induced vertical mixing in a sill fjord. The observational evidence for the theory at that time was rather sparse, but the void is materially filled by observations presented in this note. Current measurements show, in accordance with the theory, that progressive internal waves radiate out from the Dröbak sill in the Oslofjord. An experiment with tracer dye below the sill depth in the same fjord showed that the coefficient of vertical diffusion of the tracer was an order of magnitude less than the overall coefficient for vertical diffusion of density during the same period. This finding, indicating large horizontal inhomogeneities in the mixing field, gives strong support to the mixing model proposed in the earlier paper referred to above.

1. Introduction

In an earlier paper (Stigebrandt, 1976; hereafter referred to as S1) the present author developed a theory for vertical diffusion driven by internal waves in a sill fjord. It was shown that an oscillating current over the fjord sill should generate internal waves when a pycnocline was present at sill level and when a densimetric Froude number based on the current velocity over the sill was less than 1. The internal waves were supposed to be progressive and the waves going into the fjord were assumed to break against the sloping bottom, thereby creating small-scale turbulence capable of working against the buoyancy forces. The breaking waves were supposed to be the dominating energy source for the turbulence below the sill level. The theory, which in particular was meant to be valid for the inner Oslofjord, certainly also gives a qualitatively correct explanation for the mixing of the deep water in several other fjords. Thus Perkin and Lewis (1978) have found that the mixing in an arctic fjord, Cambridge Bay, is significantly driven by internal waves at tidal and higher forced frequences. In the following, however, we mostly concentrate on the Oslofjord when discussing observational evidences for the correctness of the theory in S1.

We begin in Section 2 with a brief description of the theory in S1. In Section 3 we present observations of internal waves of tidal origin radiating out from the Dröbak sill in the Oslofjord. The observed waves are compared to predicted waves. Section 4 is devoted to a presentation of some measurements of vertical diffusion showing that the vertical mixing in the deep water is horizontally strongly inhomogeneous. The vertical mixing is most probably dominated by boundary mixing, presumably caused by breaking internal waves. The paper is brought to an end in Section 5 by a discussion and proposals for desired future work.

2. Elements of the theory

It was shown in S1 that when a pycnocline is situated at sill depth, a forced oscillating current over the sill can give rise to two progressive internal waves, one on each side of the sill. Waves appear when an appropriate densimetric Froude number based on the current velocity over the sill is less than 1. The stratification was described by a two-layer approximation. The orbital velocity u_{1i}, in the upper layer caused by the internal waves, is given in S1 by

$$u_{1i} = \begin{cases} \alpha \dfrac{h_0}{H_0} \sin[k_i(x - L_1) + \omega t], & \text{inside the sill} \\ -\alpha \dfrac{h_0}{H_0} \sin[k_i(x - L_1) - \omega t], & \text{outside the sill} \end{cases} \tag{1}$$

where

$$\alpha = \frac{\omega A Y_1}{B_1(H_0 + h_0)} . \tag{2}$$

The x axis is directed out of the fjord. H_0 and h_0 are depths of the upper and lower layers, respectively, k_i is the wavenumber and ω the frequency. The sill

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TABLE 1. List of symbols.

A	amplitude in water level inside the sill of oscillating tidal component with frequency ω.
a	amplitude of the internal wave
B	width of the fjord
B_1	width of the sill
c_i	phase velocity of the internal wave
g	acceleration of gravity
H_0	depth of the upper layer
h_0	depth of the lower layer
k_i	wavenumber of the internal wave
L_1	position of the sill
ΔL	distance between two current meter stations
u_{1i}	orbital velocity in the upper layer caused by the internal wave
Y_1	surface area of the fjord inside the sill
$\Delta\rho$	density difference between the two layers
α	amplitude of the barotropic tidal current
ω	frequency of the internal wave
$\mathscr{H}$	phase of internal wave current

is situated at $x = L_1$ and the width of the sill is B_1. The fjord area inside the sill is Y_1 and the amplitude of the water-level fluctuation, caused by the forcing oscillating current, inside the sill is A. The phase velocity of the internal waves is given by

$$c_i = \left(g\,\frac{\Delta\rho}{\rho}\,\frac{H_0 h_0}{H_0 + h_0}\right)^{1/2},$$

where g is acceleration of gravity and $\Delta\rho$ the density difference between the layers. For a given forced frequency ω, the wavenumber k_i is $k_i = \omega/c_i$. The notations used in this paper are summarized in Table 1.

For the Oslofjord there is one dominating tidal period, the M_2, and in our analysis of observed currents we concentrate on this period. When comparing observed current amplitudes and phases with amplitudes and phases calculated from the expressions given above we must take into account the real distribution of width in the fjord. The depth variations, however, are not so important for the phase speed of the internal waves as long as the lower layer is much thicker than the upper. Under this condition the phase speed does not vary appreciably. The phase difference $\Delta\mathscr{H}$ between two observational sections can therefore be calculated from the earlier given expression for the phase speed where we take h_0 as the mean depth of the lower layer between the sections. The amplitude of the internal wave, however, is sensitive to changes in width *and* depth. Defant (1961) discussed changes in the amplitude of a barotropic long wave propagating in a canal of varying depth h and width B. He gives the variations in the amplitude of the barotropic wave as proportional to $B^{-1/2}h^{-1/4}$. This result is based on the assumption of constant energy flux. The variations in the amplitude of a long, progressive internal wave in a canal with varying width and depth is generally more complicated. The internal waves in the Oslofjord are supposed to break when meeting sloping bottoms in the *inner* parts of the fjord. In the more or less channel-formed outer part of the fjord we expect breaking against the bottom to be insignificant and in that part of the fjord the energy flux should be nearly constant. The energy flux ϵ_i in S1 [Eq. (15)] was given by

$$\epsilon_i = BEc_g,$$

where E is the energy density and $c_g = c_i$ for long internal waves. Outside the dissipation area (the inner part of the fjord) we thus expect that

$$\frac{d\epsilon_i}{dx} = \frac{\partial\epsilon_i}{\partial B}\frac{\partial B}{\partial x} + \frac{\partial\epsilon_i}{\partial h_0}\frac{\partial h_0}{\partial x} + \frac{\partial\epsilon_i}{\partial a}\frac{\partial a}{\partial x} = 0$$

or

$$\frac{\partial a}{\partial x} = -\frac{\dfrac{\partial\epsilon_i}{\partial B}\dfrac{\partial B}{\partial x} + \dfrac{\partial\epsilon_i}{\partial h_0}\dfrac{\partial h_0}{\partial x}}{\dfrac{\partial\epsilon_i}{\partial a}}, \tag{3}$$

where a is the amplitude of the wave. In our Oslofjord case $(\Delta B/\Delta x)/(\Delta h/\Delta x) \gg 1$ and $(\partial\epsilon_i/\partial B)/(\partial\epsilon_i/\partial h_0) \approx 1$ (actual parameter figures are given in Table 4) and from this it follows that

$$\frac{\partial a}{\partial x} \approx -\frac{\dfrac{\partial\epsilon_i}{\partial B}}{\dfrac{\partial\epsilon_i}{\partial a}} \cdot \frac{\partial B}{\partial x}$$

or $a \propto B^{-1/2}$. The effects on the amplitude of variations in the width dominate those at depth in the Oslofjord case. Thus, assuming a constant energy flux by the internal waves, we expect that the orbital velocity u_{1i} at fjord width B is described by

$$u_{1i} \approx u_{1i1}(B_1/B)^{1/2}, \tag{4}$$

where u_{1i1} is the orbital velocity and B_1 the width at the sill. Defant (1961) showed that the $B^{-1/2}h^{-1/4}$ law for barotropic long waves approximately satisfies the linear wave equation when the transverse dimensions of the canal vary only by small fractions of themselves over a wavelength. However, the good agreement between predicted [from Eq. (4), see Section 3] and measured internal wave orbital velocities in the Dröbak sound [which is a channel outside the sill with nearly constant width except for a rather abrupt narrowing just at the sill (see Fig. 1)] gives reason to believe that the linear theory presented by Defant is too restrictive. A future investigation of the full nonlinear problem of waves in canals with varying width and depth will give the correct limits for the $B^{-1/2}h^{-1/4}$ law (and its equivalent for internal waves).

Another factor we must also take into account is that observed currents have a barotropic contribution. The barotropic current u_s in S1 was written as

$$u_s = \alpha \sin\omega t. \tag{5}$$

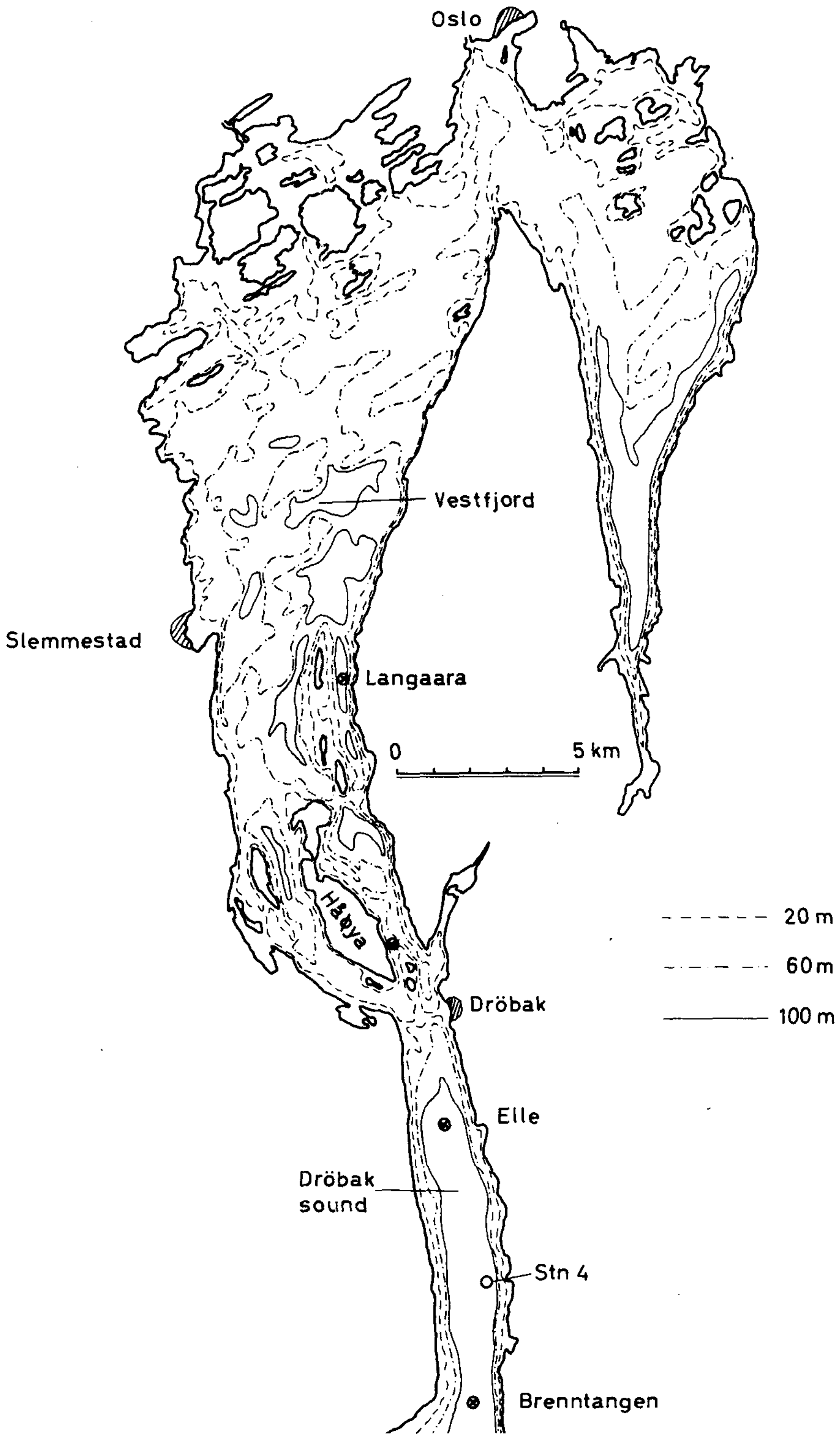

Fig. 1. Map over the Oslofjord. Current meter stations are indicated by the symbol ⊗. Hydrographic measurements in 1972 were taken at station 4.

Thus the phase is considered as constant over the fjord, an approximation which is discussed in S1. Near the sill the barotropic and baroclinic velocity components in the upper layer are in phase. Further away from the sill the barotropic current leads the baroclinic current. The measured phase difference between two current meter stations a distance ΔL from each other, with both stations outside or

TABLE 2. Results of the harmonic analysis of the current records from Elle and Brenntangen using the north-south component of the M_2 during the period 7 June–6 July 1972.

Location	Depth (m)	u_{1i} (cm s^{-1})	$\mathcal{H}$ (deg)	$\Delta\mathcal{H}$ (deg)
Brenntangen	5	9.6	125.6	
Elle	5	12.5	84.0	
Brenntangen	12.5	15.1	119.1	
Elle	12.5	9.9	55.0	
Brenntangen	Mean upper layer	12.4	122.4	52.9
Elle	Mean upper layer	11.2	69.5	

inside the sill, will then be less than the phase difference caused by a pure progressive internal wave. The discrepancy between calculated [from Eq. (1)] and measured phase differences depends primarily on the distance ΔL and the ratio H_0/h_0, which is the ratio between the barotropic and baroclinic velocity amplitudes in the upper layer disregarding variations in the width. The predicted internal waves, radiating out from the Dröbak sill in the Oslofjord, are compared with observations in Section 3.

In S1 it was conjectured that the internal waves propagating into the fjord should break when meeting a sloping bottom. One reason for this belief was that this happened in a laboratory experiment where internal waves in a sill fjord were generated by the same mechanism as in nature. The laboratory fjord model had a sloping bottom and the waves were completely destroyed against this. (Unfortunately, the shadowgraphs in Fig. 5 in S1, showing two different stages of breaking waves, were printed upside down.) In the laboratory experiments it was found that when the lower layer was stratified, breaking occurred at all depths in the lower layer. Based on these findings a mixing model for a confined, stratified water mass was proposed in S1. The model was supposed to be valid for the region below the sill level and the mixing was thought to be driven entirely by internal waves breaking on sloping bottoms. The vertical mixing in such a model takes place at the sloping bottoms and the result of the mixing process is communicated to the interior by density currents. The vertical mixing is thus highly horizontally inhomogeneous. A mean value of the vertical exchange coefficient can be obtained from the time development of the vertical density structure. However, if one determines the vertical exchange coefficient from a tracer in such a system, one will at first find other and, as a rule, smaller exchange coefficients. This is because of the horizontal inhomogeneity of the mixing process. If, for example, one puts the tracer far from the mixing zones, one will find very low vertical diffusion coefficients. When the tracer eventually comes into a mixing zone the time mean of the vertical exchange coefficient increases dramatically. Only for time scales long compared to the time needed for a water parcel to reach a mixing zone from the interior, will the coefficients for vertical diffusion calculated from the time developments of the density field and the tracer, respectively, be equal. However, if the vertical mixing was horizontally homogeneous the vertical exchange coefficients for the density and tracer should be equal on even small time scales. In Section 4 some tracer measurements in the Oslofjord are briefly discussed.

3. Observations of internal waves with an M_2 Period in the Oslofjord

Two sets of observations are used in order to confirm the existence of internal waves radiating out from the sill. The first set was collected by the River and Harbour Laboratory (VHL) in Trondheim, Norway, during an investigation of possible effects of thermal pollution on the outer part of the Oslofjord. Several current records from different depths and locations in the Dröbak sound, outside the sill, were obtained with Aanderaa current meters. Here two records are used (at 5 and 12.5 m) from each of the stations Elle and Brenntangen (see Fig. 1). The analyzed period starts at 0000 LT 7 June 1972 and stops at 2300 LT 6 July 1972. The distance between the stations ΔL, is about 7.5 km.

The second set of observations (also collected by VHL) was designed for a study of the internal waves inside the sill. Just one current meter (at 7m) was used at each of the two stations, Håöya and Langaara. The distance between the stations is about 7 km (see Fig. 1). The analyzed period starts at 1200 LT 28 June 1976 and stops at 1100 LT 28 July 1976.

The records have been analyzed for the 10 greatest tidal components (harmonic analysis) using a standard machine program at our disposal at VHL. The results of the analyses are given in Table 2 for the Elle and Brenntangen records and in Table 3 for the Håöya and Langaara records. The current reported in these tables is the north-south component of the dominating M_2 tidal constituent. The fjord is approximately oriented in that direction and the east-west current component is very weak. The phase of the north-south current is related to the moon's passage of the local meridian and the phase is related to the maximal northward current.

TABLE 3. As in Table 2 except from Håöya and Langaara for the north-south component of M_2 during the period 28 June–28 July 1976.

Location	Depth (m)	u_{1i} (cm s^{-1})	$\mathcal{H}$ (deg)	$\Delta\mathcal{H}$ (deg)
Langaara	7	6.7	102.6	40.9
Håöya	7	16.4	61.7	

From Tables 2 and 3 it can be seen that the phase of the current is smaller for the stations near the sill than it is for the stations further away from the sill. Thus, there definitely exist internal tidal waves of M_2 period that radiate away from the sill. According to the theory the phase difference $\Delta\mathscr{H}$ is

$$\Delta\mathscr{H} = k_i \Delta L,$$

where ΔL is the distance between the stations. We also have $k_i = \omega/c_i$ and

$$\Delta\mathscr{H} = \frac{\omega \Delta L}{c_i} \quad \text{or} \quad \Delta\mathscr{H} = \frac{\omega \Delta L}{c_i}\frac{360}{2\pi}\ [\text{deg}].$$

The phase velocity of the internal waves is determined by the density difference between the two layers in a two-layer approximation and the depths of the layers, as shown by the expression given in the last section. The stratification for the two periods is given in Figs. 2 and 3, respectively. The relevant parameters for the two cases are given in Table 4, where E-B and H-L denote the situations outside and inside the sill, respectively.

From Table 4 it can be seen that the predicted phase difference outside the sill, between Elle and Brenntangen, is very near the observed one. The predicted phase difference inside the sill is greater than the measured. However, there is just one observational series at each station here and the uncertainty in the phase estimates from the records can probably explain the main part of the discrepancy. In order to obtain better phase estimates one has to use more instruments over longer periods of time. The measured amplitudes are approximately equal to those predicted. For the waves outside the sill the measured amplitude is 80% of the predicted one. Inside the sill the measured amplitude is about 115% of that predicted. In the above analysis of the currents we have not subtracted the effect of the barotropic current. This current should have a small effect on the phases on both sides of the sill because the local amplitude α between Brenntangen and Elle and Langaara and Håöya is about 2 cm s^{-1} or less (near Langaara). The effect on the measured amplitudes is less than 20%. The current meter stations at Håöya and Langaara were not ideally located from an oceanographers point of view (see Fig. 1) because of heavy ship traffic to and from Oslo. We should also comment on the absolute phases for the internal waves. According to the linear theory in S1 the barotropic current and the baroclinic current in the upper layer should be in phase at the sill. The expressions for the tidal-induced internal wave currents given in Eqs. (1) are related to a barotropic current with phase 0° at the sill [see Eq. (5)]. From harmonic analysis of water level data from Oslo Harbour we have found that the phase for high water for the M_2 constituent is 135°. The barotropic current should then have a phase of approximately 45° (135° − 90°) and this should also be the phase for the internal wave currents at the sill. The Håöya current meter is situated about 2500 m from the sill and this gives a phase of 40° for the inner internal wave at the sill. The Elle current meter station is situated about 3000 m from the sill giving a phase for the outer internal wave of ~48°. The absolute phases for the internal waves are thus close to those expected from the theory in S1.

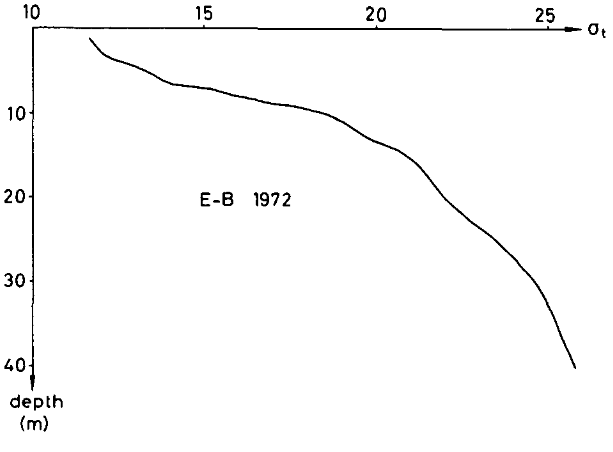

FIG. 2. The mean vertical distribution of density σ_t in the Dröbak Sound (Station 4) during the period 12 June 1972–7 July 1972.

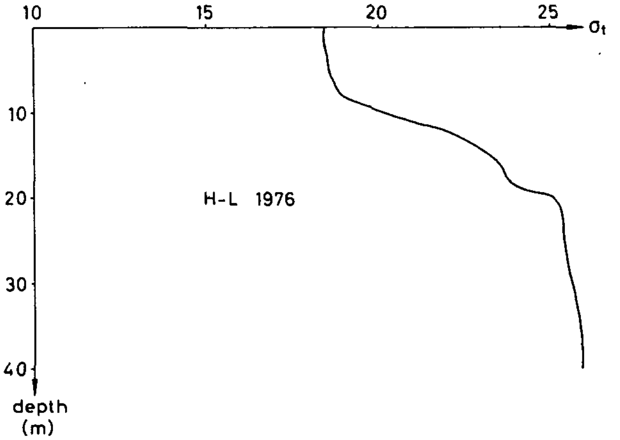

FIG. 3. The mean vertical distribution of density σ_t in the Vestfjord during the period 4 July 1976–4 August 1976.

In this connection we will also mention that Perkin and Lewis (1978) have measured the vertical oscillations of the isotherms in the pycnocline in Cambridge Bay (an arctic fjord). They found very satisfactory agreement between the amplitude predicted from the theory in S1 and the measured amplitude.

TABLE 4. Observed and predicted phase differences and orbital velocities for the two internal waves generated by the M_2 tide at the sill. The outer wave, measured at Elle and Brenntangen, is denoted by the symbol E-B. The inner wave, measured at Håöya and Langaara, is denoted by the symbol H-L.

$g = 10\ \mathrm{m\ s^{-2}}$, $\omega = 1.41 \times 10^{-4}\ \mathrm{s^{-1}}$, $B_1 = 600\ \mathrm{m}$ $Y_1 = 2 \times 10^8\ \mathrm{m^2}$, $A = 0.13\ \mathrm{m}$	common parameters

		E − B	H − L
$\Delta\rho$	(kg m^{-3})	7.5	5.5
H_0	(m)	20	20
h_0	(m)	100	70
c_i	(m s^{-1})	1.11	0.92
ΔL	(m)	7500	7000
B	(m)	1800	1800 (Håöya)
u_{1i1}	(m s^{-1})	0.26	0.24
$\Delta\mathscr{H}_{\mathrm{pred}}$	(deg)	55	61
$\Delta\mathscr{H}_{\mathrm{obs}}$	(deg)	53	41
$u_{1i,\mathrm{pred}}$	(m s^{-1})	0.15	0.14
$u_{1i,\mathrm{obs}}$	(m s^{-1})	0.12 (Brennt)	0.16 (Håöya)

4. Observations of vertical mixing in the Oslofjord.

In connection with the design of a submerged waste water outlet at Slemmestad, on the western side of the Oslofjord (see Fig. 1), a tracer (rhodamine B) was introduced at the level which waste water is planned to reach after initial dilution (~30 m below the sea surface). The horizontal and vertical extensions of the tracer sky was carefully determined several times during the period 19 July 1977–18 August 1977 and the coefficient of vertical diffusion of tracer was determined by standard procedure. The measurements were conducted by the Norwegian Institute for Water Research (NIVA) and are reported by Bjerkeng, *et al.* (1978). The tracer moved slowly southward during the experiment. The most interesting result obtained from the experiment, from our point of view, was that the coefficient for vertical diffusion of tracer was found to be only about 10% as high as the coefficient for vertical diffusion of density determined from changes in the vertical density structure during the same period [see Eq. (22) in S1]. Gade (1970) gives a detailed discussion of the method used by Bjerkeng *et al.* The only probable explanation for the anomalously low value of the coefficient of vertical diffusion of tracer is that the vertical mixing is highly inhomogeneous horizontally. The most probable reason for a horizontally inhomogeneous vertical mixing in this contained fluid volume is that some sort of boundary mixing is dominating the vertical mixing. In the case of the Oslofjord the energy source for boundary mixing should be the internal waves whose existence was proved in Section 3. It was shown already in S1 that the internal waves generated at the Dröbak sill radiate an effect (~1000 kW) to the Vestfjord (the fjord inside the sill) which is about 20 times greater than the effect used by working against the buoyancy forces in the volume below the pycnocline. This means, if most of the mixing energy below the pycnocline emanates from the internal waves, that the efficiency of turbulence in working against the buoyancy forces is 0.05. Thus 95% of the turbulent energy goes into heat and 5% is used for work against the buoyancy forces. This partition seems to be quite normal under stable stratification. In concluding this section, we can state that the results of the above-mentioned tracer experiment further support the theory in S1.

5. Discussion

Observations have shown that internal waves radiate out from the Dröbak sill. The measured phases and amplitudes of the waves have been reasonably predicted by the theory in S1. Further improvements in agreement between observations and predictions should be possible by using more and longer observation series. Slight improvements in the theory could possibly be obtained by using the whole spectrum of the modes of internal waves of a given frequency instead of just one mode in a two-layer approximation. Perkin and Lewis (1978) also found very good agreement between the measured amplitude of internal tides in Cambridge Bay and the predicted value from the theory in S1.

The model for vertical diffusion in the water volume below the sill level given in S1 predicts that the vertical diffusion is horizontally highly inhomogeneous. All the vertical mixing should, according to that theory, occur in a boundary layer at sloping bottoms where internal waves break. A tracer study, reported by Bjerkeng *et al.* (1978), showed coefficients of vertical diffusion an order of magnitude smaller than average values determined from the time evolution of the density field in the fjord. Such results are possible only in systems with large horizontal inhomogeneities with respect to vertical mixing, thus strengthening the validity of the diffusion model in S1.

Direct measurements of breaking internal waves have not yet been undertaken in the Oslofjord because of financial problems but there is some hope of conducting such measurements in a near future. Perkin and Lewis (1978), however, have recorded breaking internal waves in Cambridge Bay, using an Aanderaa current meter placed near the bottom. The breaking process which they described showed the same characteristic features as in the laboratory experiment described in S1. The downward surges were much stronger than the upward ones.

It has already been noted in S1 that there often exists possibilities of tuning fjord mouths in order to generate a maximal energy flux by internal waves into the fjord. A correct tuning of the fjord mouth should then give a maximal vertical diffusion and also a minimal residence time for the deep water. This subject was further investigated in Stigebrandt (1977) where attention was especially directed towards the exchange of deep water over the sill and the effect of barotropic current fluctuations on such an exchange.

Although the internal waves generated by the tides in the Oslofjord seem to follow the linear theory given in S1 very well it should be emphasized that the situation can be more complex in other places with different conditions. If, for example, the tidal current over a sill or bank is supercritical, i.e., the densimetric Froude number as defined in S1 is greater than 1, the kind of waves described in S1 and in this work will not be generated. Instead fronts will develop on the lee sides of the obstacles and from these fronts internal waves with higher frequencies (than the forcing frequency) will develop. These nonlinear phenomena were described and analyzed by Lee and Beardsley (1974) later by Maxworthy (1978). Farmer and Smith (1978) described and analyzed similar, tidally generated, high-frequency, nonlinear internal waves in Knight inlet; there the waves were thought to be generated by the breakdown of an internal hydraulic jump over the sill.

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