A Standard Deviation Computer[1,2,3]

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ABSTRACT

The standard deviation computer is a small electronic analog device which is designed to calculate the standard deviation of a voltage signal with less than 5 per cent error. The output is a continuous voltage signal proportional to the standard deviation which is generated with a computation time equal to the sampling period. Mathematically, this instrument computes the mean absolute deviation which is related to the standard deviation by a constant factor. It accepts signals in the frequency range of from d-c to some upper limit determined by amplifier bandwidth over a wide range of amplitudes and computes with sampling times up to 3600 seconds or more. In field tests, the computer operated successfully on the output of a wind vane.

1. Introduction

Numerous field studies of atmospheric turbulence have been undertaken in the past (Scrase, 1930; Best, 1935; Hewson and Gill, 1944; Hewson, 1956; Cramer, 1953; Hewson *et al.*, 1951; Friedman, 1953; Falk *et al.*, 1954; Record and Cramer, 1958; Hay and Pasquill, 1959; Panofsky and McCormick, 1960), but deficiencies in data handling systems at the time have limited their scope and utility.

A direct method of computing an estimate of the standard deviations of the wind direction has been presented by Jones and Pasquill (1959). The experimental system reported by them was designed to compute the statistical mean of the standard deviation of wind-direction fluctuations with various averaging and sampling times.

Study of the fundamental properties of atmospheric turbulence and diffusion would be facilitated by a similar device capable of calculating the standard deviation with less than 5 per cent error and reliable enough for continuous operation. This accuracy requirement is imposed to match the accuracy of the sensor and so that the data need not be recomputed. The computer should provide several pre-set values of the sampling period up to 3600 sec. Since the computer is to perform mathematical operations on the signal, it should be possible to convert it readily to other uses, such as computing the correlation function of two signals. On-site operation demands that the computer be compact and rugged with simple but effective controls.

These requirements suggest the use of an electronic analog circuit which would consist of high-quality passive components, e.g., resistors and capacitors, and of low-cost, commercially available operational amplifiers. Using these components, precise mathematical operations can be performed and this type of computer will accept any direct analog voltage signal and will drive any conventional recorder or display device.

2. Statistics of a continuous time series

If the basic process is stationary, the statistics obtained by the finite time averaging will converge to a limit as the averaging time increases. If the process is also ergodic, then these statistics may be related to the statistical averages taken across the process. The assumptions that need to be introduced depend upon the use to which the data are put. However well these assumptions hold, one may still observe the finite time averages and attach some meaning to them as characteristic of the part of the atmosphere actually observed.

Only finite time averages will be considered in this paper and in general the averages obtained will be functions of time themselves. The mean value obtained over a period of time τ of the general continuous time series $x(t)$ is given by

$$\bar{x}(t)=\frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} x(\xi)d\xi \qquad (1)$$

and the variance by

$$\sigma_\tau^2(t)=\frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} [x(\xi)-\bar{x}(\xi)]^2 d\xi. \qquad (2)$$

Another statistic which will be useful is the mean absolute deviation

$$e(t)=\frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} |x(\xi)-\bar{x}(\xi)| d\xi \qquad (3)$$

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which is related to the standard deviation by

$$\sigma_\tau = 1.25e \qquad (4)$$

if $x(t)$ is normally distributed. This relation may also hold for some non-normal distributions but if (4) does not hold, the distribution is not normal (Brooks and Carruthers, 1953).

If one wishes to determine the contribution to the variance due to the frequencies present within some fixed range one may smooth the data before computing the variance. The operation of forming $x-\bar{x}$ provides a low frequency cut-off determined by the magnitude of the sampling time τ. Smoothing the data eliminates the high frequency components and is equivalent to averaging with a smaller sampling time s. It will be shown later that the sensing instruments do not report their inputs exactly but operate on them in such a way as to perform a near-equivalent to mathematical smoothing. Therefore meteorological data obtained with the use of wind vanes, anemometers, etc., is always smoothed to some extent. Another method for obtaining this variance, designated $\sigma_{\tau,s}^2$, is to compute directly the variance for the sampling time τ and the smaller sampling time s. Then the component variance is given by $\sigma_{\tau,s}^2=\sigma_\tau^2-\sigma_s^2$.

If the raw data are labeled $y(t)$, the smoothed data will be obtained by the operation

$$x(t)=\frac{1}{s}\int_{t-s/2}^{t+s/2} y(\xi)d\xi. \qquad (5)$$

3. Specification of realizable filters

The filters introduced in the previous section may not be realizable; this depends upon the computational system used. In the case of an electrical analog system, the filters consist primarily of resistors and capacitors and amplifiers. Such a computing system can observe only current values of the input, remember in a limited way, past values, and has no access to future values. This constitutes the first important restriction for a realizable system, the filter must not require future values of the input. Because of this restriction, Eq (1), (2), (3) and (5) must be rewritten in an equivalent form:

$$x(t-\tau/2)=\frac{1}{\tau}\int_{t-\tau}^{t} x(\xi)d\xi, \qquad (1a)$$

$$\sigma^2(t-\tau)=\frac{1}{\tau}\int_{t-\frac{3}{2}\tau}^{t-\frac{1}{2}\tau} [x(\xi-\tau/2)-\bar{x}(\xi-\tau/2)]^2 d\xi, \qquad (2a)$$

$$e(t-\tau)=\frac{1}{\tau}\int_{t-\frac{3}{2}\tau}^{t-\frac{1}{2}\tau} |x(\xi-\tau/2)-\bar{x}(\xi-\tau/2)| d\xi, \qquad (3a)$$

$$x(t)=\frac{1}{s}\int_{t-s}^{t} y(\xi)d\xi. \qquad (5a)$$

The block diagrams in Fig. 1 show a set of filters that, ideally, could be used to compute the variance or the mean absolute deviation. The term filter is used in this paper to mean a mathematical operation which may exist only as a concept or which may be a series of operations performed by hand or by some computing machine. If smoothing is not desired, the first filter in each series should be deleted. These formulae have incorporated a time shift to avoid the need for future data but they are still exact and will be used as a standard of performance.

Each of these formulae require integration over a finite period of time with a sharp cut-off which is an operation that cannot be performed continuously with electrical filters. Therefore the above equations cannot be realized exactly by this method but must be approximated. If it were not necessary to use the above definitions and if the greatest emphasis were to be placed upon the immediate past then one could use exponentially-mapped-past statistics (Otterman, 1960). In this system, the mean value is defined as

$$\bar{x}(-\tau)=\frac{1}{\tau}\int_{-\infty}^{0} x(\xi)e^{-\xi\tau}d\xi, \qquad (6)$$

where the past extends from 0 to $-\infty$. A similar expression exists for the variance. This type of average gives the greatest weight to the present value and exponentially decreasing weight to past values. It has the advantage that it can be implemented exactly by electrical means. And it has the disadvantage that it is not compatible with most other systems of obtaining the mean and variance, therefore the method of exponentially-mapped-past statistical variables will not be used in this paper.

In the block diagram of Fig. 1, the first filter required is the smoothing filter given in Eq (5a). What is needed is an approximation to Eq (5a) which is sufficiently accurate but which can be implemented with a simple electrical analog. If the approximation can be stated as the ratio of two polynomials in the frequency domain, in the form

$$y(j\omega)=\frac{a_0+a_1(j\omega)+a_2(j\omega)^2+\cdots+a_n(j\omega)^n}{b_0+b_1(j\omega)+b_2(j\omega)^2+\cdots+b_m(j\omega)^m}, \qquad (7)$$

then there exists a direct electrical analog. The amount of equipment required in the filter, and therefore the cost, is proportional to the order of the expression so we try to keep this as small as possible. To find this approximation, take the Fourier transform of (5a).

$$F\left[\frac{x(t-s/2)}{y(t)}\right]=\frac{\sin\frac{1}{2}\omega s}{\frac{1}{2}\omega s}e^{-j\omega s/2}=\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega s)^n}{(n+1)!}. \qquad (8)$$

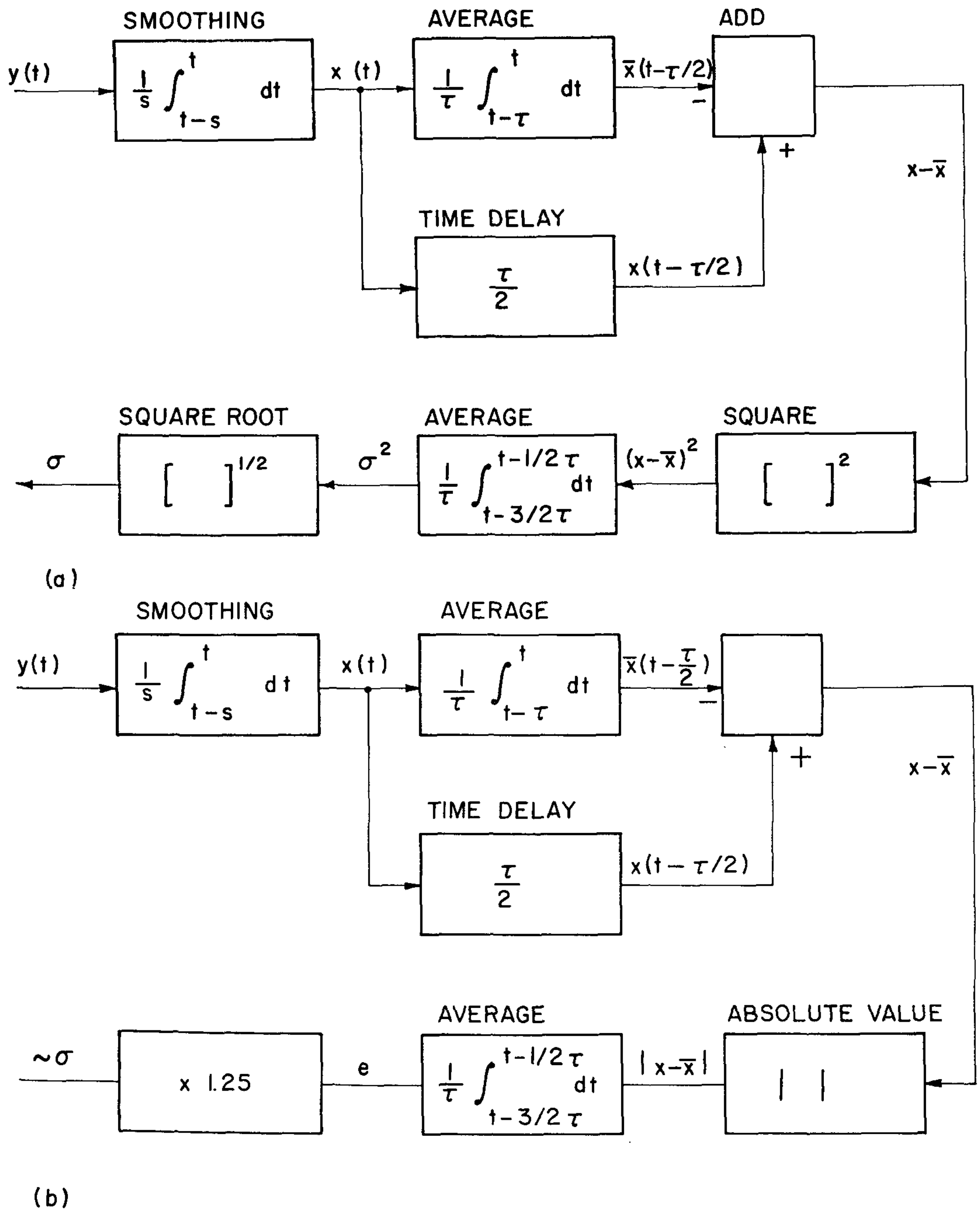

FIG. 1. Block diagram for computing the standard deviation and the mean absolute deviation showing all of the filters that are needed in the ideal system.

The second order approximation to (8) is

$$y_1(j\omega)=\frac{12}{12-(\omega s)^2+j6\omega s}. \tag{9}$$

This was found by restricting (7) to order 2, dividing the denominator into the numerator and comparing the resulting terms to the infinite series of (8). The quality and frequency range of the approximation is shown in Figs. 2 and 3. Fig. 3 shows that the phase delay is proportional to the frequency, up to some limiting value and thus the actual time delay, phase angle in radians divided by the frequency, is $\frac{1}{2}s$ just as was anticipated in equation (5a). It is characteristic of second order systems that the maximum phase shift is just 180 deg. Improvement of the phase response at high frequencies would require a higher order system.

The next step in the implementation of Eq (2a) and

(3a) is the generation of $\bar{x}(t-\tau/2)$ and the delayed input $x(t-\tau/2)$. The $\bar{x}$ term could be formed as above and then a separate circuit would be needed to accomplish the delay of $x(t)$ for $\tau/2$ sec. A more practical approach is to generate the term $x(t-\tau/2)-\bar{x}(t-\tau/2)$ in one circuit configuration which requires fewer components. The procedure, as above, is to form the Fourier transform and then find the best approximation in the form of Eq (7), again restricting ourselves to a second order approximation.

$$F\left[\frac{x(t-\tau/2)-\bar{x}(t-\tau/2)}{x(t)}\right]=\left[1-\frac{\sin\frac{1}{2}\omega\tau}{\frac{1}{2}\omega\tau}\right]e^{-j\omega\tau/2}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(j\omega\tau)^{n+2}[2^{n+2}-(n+3)]}{2^{n+2}(n+3)!}. \quad (10)$$

The second-order approximation is

$$y_2(j\omega)=\frac{-0.067\omega^2\tau^2}{1-0.067\omega^2\tau^2+j0.50\omega\tau}. \quad (11)$$

The quality and range of the approximation involved in the y_2 filter are shown in Figs. 4 and 5. The time delay of this filter is $\tau/2$ sec for most input frequencies just as required in Eq (10).

If we wish to generate the variance in accordance with Eq (2a), we must have a squaring network, and an absolute value network is required for generation of the mean absolute deviation as shown in (3a). Assuming that such networks are available and that they have adequate accuracy and frequency response, then the choice between them depends upon their relative cost. These circuits will be introduced later. For the present designate them as y_3 (square) and y_3 (abs).

The last filter required is to average over time τ the

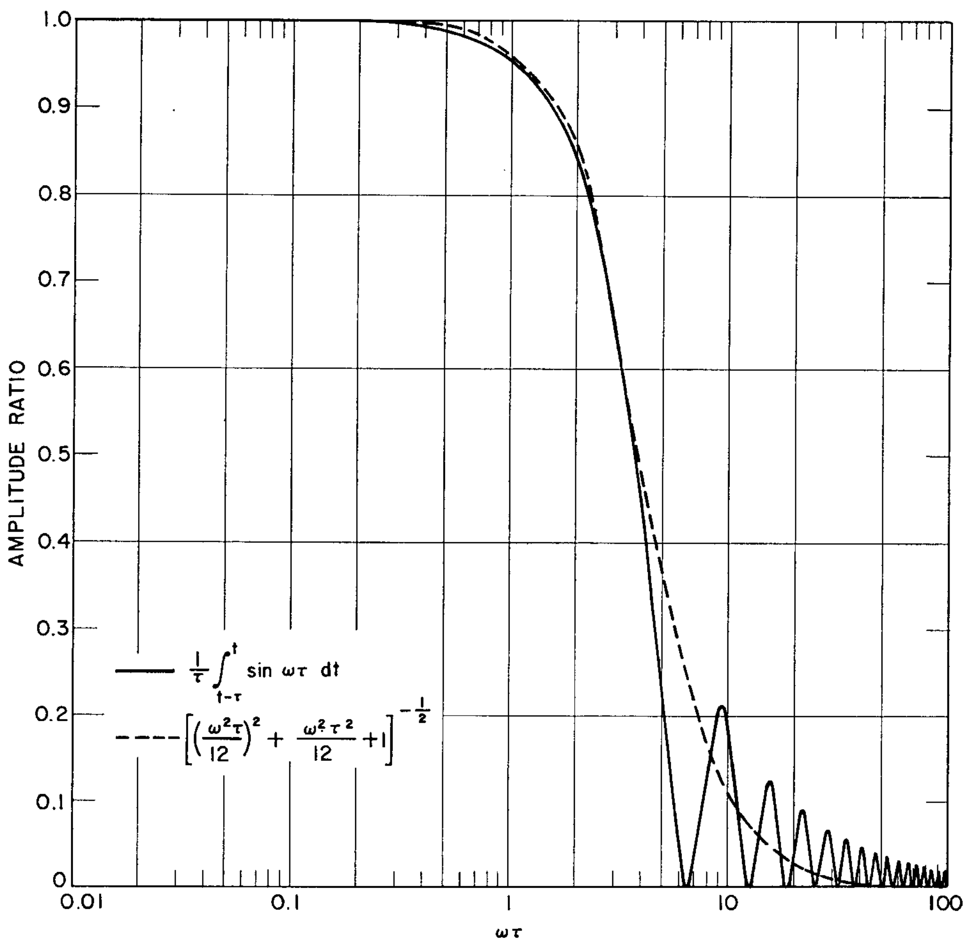

FIG. 2. Amplitude response of filters 1 and 4 compared to the ideal and showing experimental points from the computer. This shows the quality and frequency range of the approximation with respect to amplitude.

square or the absolute value of $x-\bar{x}$. This filter is just like $y_1(j\omega)$ except that we substitute τ for s, thus

$$y_4(j\omega)=\frac{12}{12-(\omega\tau)^2+j6\omega\tau}. \tag{12}$$

Figs. 2 and 3 also apply to the y_4 filter.

As shown in Fig. 6, these filters operate in series because their design is such that they do not interact with each other.

The computational form used for the variance is the most direct form and it can be shown that it is not made easier if we expand the term in brackets in Eq (2a). Expansion gives

$$\sigma^2(t-\tau)=\frac{1}{\tau}\int_{t-\frac{3}{2}\tau}^{t-\frac{1}{2}\tau} x^2(\xi-\tau/2)d\xi$$
$$-\frac{2}{\tau}\int_{t-\frac{3}{2}\tau}^{t-\frac{1}{2}\tau} x(\xi-\tau/2)\bar{x}(\xi-\tau/2)d\xi$$
$$+\frac{1}{\tau}\int_{t-\frac{3}{2}\tau}^{t-\frac{1}{2}\tau} \bar{x}^2(\xi-\tau/2)d\xi. \tag{13}$$

The time delays involved prevent cancellation of terms in Eq (13) so that no simplification occurs.

All of the development in this section was done in the frequency domain for ease of manipulation. It may be helpful to show the results in the time domain. In Fig. 6, the filter y_1 is designed to approximate the integral of Eq (5a). This integral requires that we integrate over a precise interval of time with weight $1/\tau$. Fig. 7(a) shows this schematically as a weighting function which is just the weight applied as a function of time. This weighting function along with the weighting function of the filter used applies to filters y_1 and y_4. The analytical expression for the theoretical weighting function is

$$W(t)=\frac{1}{\tau}[h(t)-h(t-\tau)],$$

where $h(t)$ is the unit step function which is zero when the argument is less than zero and equal to unity when the argument is greater than zero. The actual weighting function appropriate to filters y_1 and y_4 is just their inverse Fourier transform.

$$W(t)=F^{-1}[y(j\omega)]$$
$$=h(t)\frac{83.14}{\tau}e^{-3t/\tau}\sin(1.732t/\tau).$$

This is the curve plotted in Fig. 7(a).

The weighting function for the generation of $x-\bar{x}$ should be

$$W(t)=\delta\left(t-\frac{\tau}{2}\right)-\frac{1}{\tau}[h(t)-h(t-\tau)],$$

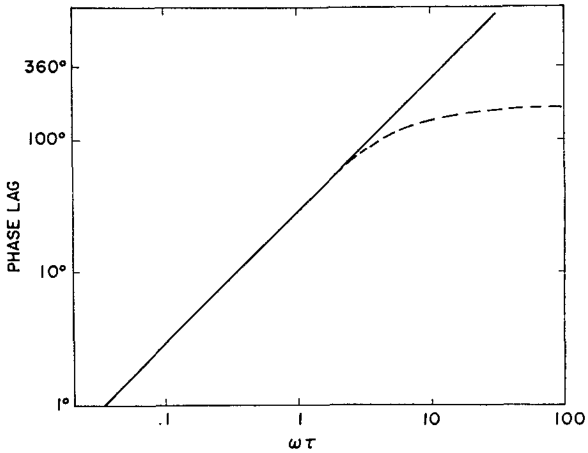

FIG. 3. Phase response of filters 1 and 4 compared to the ideal. This illustrates the quality and the frequency range of the approximation with respect to phase shift.

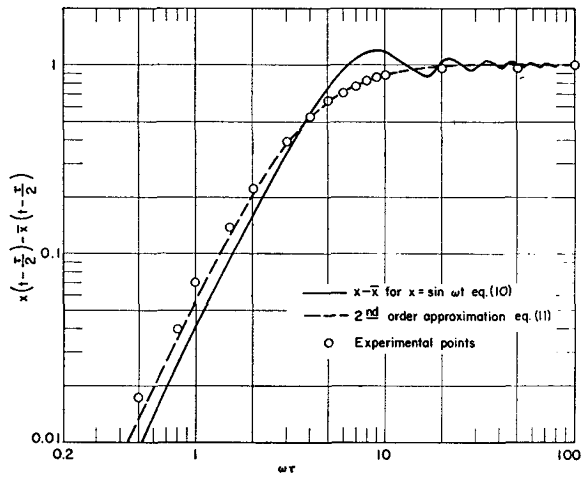

FIG. 4. Amplitude response of filter 2 compared to the ideal and showing experimental points from the computer. This shows the quality and the frequency range of the approximation with respect to amplitude.

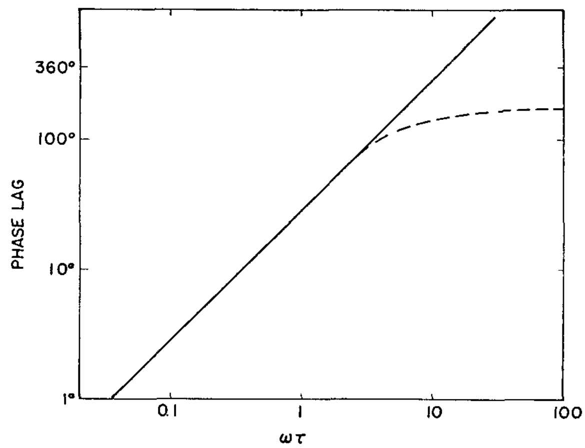

FIG. 5. Phase response of filter 2 compared to the ideal. This illustrates the quality and the frequency range of the approximation with respect to phase shift.

where $\delta(t)$ is the unit impulse function such that

$$\int_{-\epsilon}^{+\epsilon} \delta(t)dt = 1$$

for $\epsilon > 0$ and $\delta(t) = 0$ when $t \neq 0$. This is plotted in Fig. 7(b) along with the weighting function for filter y_3 which is

$$W(t) = F^{-1}[y_3(j\omega)]$$

$$= \delta(t) + h(t)\frac{14.9}{\tau}e^{-3.23t/\tau}$$

$$\times[0.50 \cos t/\tau - 1.73 \sin t/\tau].$$

4. Realization of filters

The filters of Fig. 6 can be made using resistors, capacitors and high gain, low cost, d-c amplifiers plus some special components for generating the square and the absolute value. So far, the development has been quite general with regard to the magnitude of the sampling time τ and it will continue to be except only that we will make provision for τ up to 3600 sec. The format of this section will be to show the general computing circuits and then to show how these may be implemented with hardware to emphasize accuracy, reliability or economy as the occasion may demand.

The smoothing circuit shown as filter y_1 in Fig. 6 must satisfy the transfer function of Eq (9). Either the three amplifier or the one amplifier circuit of Fig. 8 will do. The one amplifier circuit is suitable only for very small values of s, say about 10 sec or less. In either case, the only parameter which must be specified is the value of s in seconds. In the three amplifier circuit, R_1, R_2, C, α and β are determined by s according to the following relation:

$$s = \frac{3.47R_1C}{\alpha} = \frac{6R_2C}{\beta}.$$

The symbols α and β denote voltage dividers, i.e., adjustable resistors. For the one amplifier circuit,

$$s = 2C_2(2R_2 + R_1)$$
$$= [12R_1R_2C_1C_2]^{\frac{1}{2}}.$$

These circuits are presented separately from the rest of the computing circuits since they will not always be used. If they are to be used, the output voltage of the smoothing filter should be fed directly to the input of the next circuit.

The rest of the circuit, corresponding to filters y_2, y_3 and y_4, is shown in Figs. 9 and 10 for the absolute value method and the squaring method, respectively. Generation of the absolute value required by filter y_3 is accomplished with only two diodes and an extra wire while generation of the square requires a diode multiplier. The multiplier is a network of resistors and diodes such that the output current is proportional to the square of the input voltage. Another multiplier has been added to the end of the circuit and hooked up to perform the square root necessary to obtain the standard deviation if desired. This would correspond to another filter y_5, as previously shown.

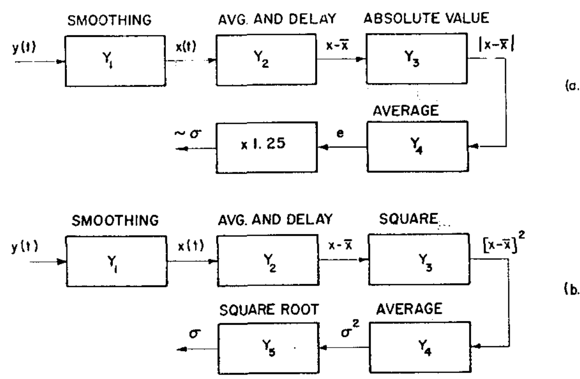

FIG. 6. Block diagram of the scheme actually used for computing (a) the mean absolute deviation and (b) the standard deviation showing filters 1, 2, 3 and 4. The (b) part has an optional filter 5 for obtaining the standard deviation from the variance.

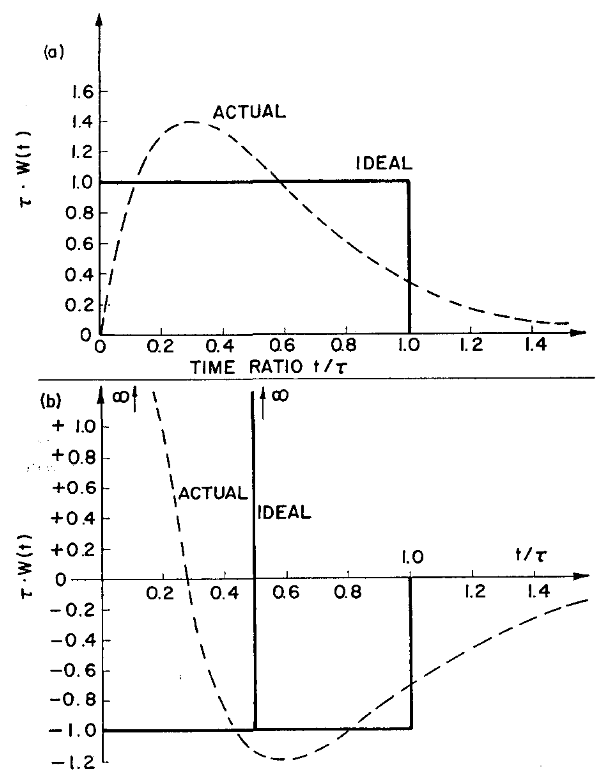

FIG. 7. Weighting function for (a) filters 1 and 4 and (b) filter 2. This shows the weight each filter gives to its input as a function of time as compared to the ideal weighting functions (solid line).

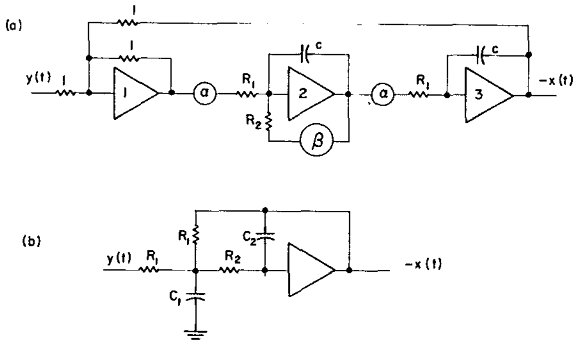

FIG. 8. Realization of filter 1 used for smoothing the data when the variance due to a certain range of frequencies is desired. Both the three amplifier and the one amplifier circuits are shown.

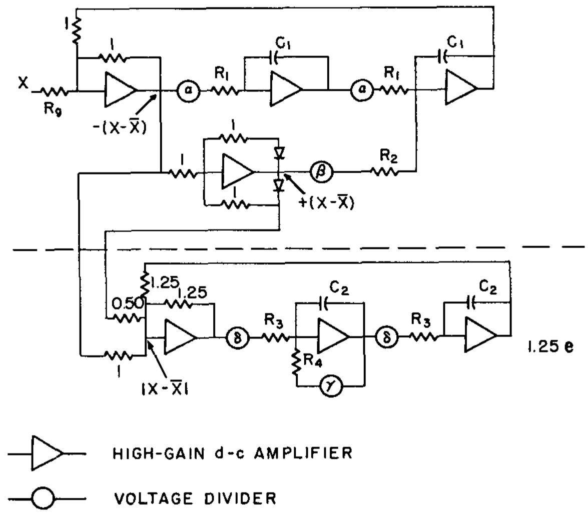

FIG. 9. Circuit for generating the approximate standard deviation, 1.25 e. The circuit includes filters 2, 3 and 4. The absolute value circuit for filter 3 is a full wave rectifier. This is the realization of the block diagram in Fig. 6(a). All resistors and capacitors are designated in units of megohms and microfarads respectively.

All the indicated resistors and capacitors are 1 per cent quality and the multipliers are the diode quarter-square type. The circuit diagram of Fig. 9 has been divided into two major mathematical operations by the dashed lines. The operations are the generation of $|x(t-\tau/2)-\bar{x}(t-\tau/2)|$ in the top half of Fig. 9, and the subsequent integration required for computation of e at the bottom. The coefficients are determined by three circuit components, a voltage divider α, β, δ, or γ followed by an RC combination as follows:

$$\tau=\frac{3.86R_1C_1}{\alpha}=\frac{7.46R_2C_1}{\beta} \tag{14}$$

from Eq (11), and

$$\tau=\frac{3.47R_3C_2}{\delta}=\frac{6R_4C_2}{\gamma} \tag{15}$$

from Eq (12). Eq (14) and (15) must be satisfied for the concept of sampling time to have meaning.

Given that the computer can handle a voltage range from −100 to +100 volts, then, for optimum use of the computer, the voltage input $x(t)$ should have extreme values of ±100 volts. If this is not practical, the input resistor R_g shown in Figs. 9 and 10 provides an adjustable gain. If the voltage extremes of $x(t)$ are, say, ±10 volts, then R_g should be 0.1 megohm which makes the effective voltage range ±100 volts since the input gain is $1/R_g$.

The relation between the voltage at any point in the circuit and the magnitude of the corresponding physical variable is determined by the amplitude scaling factor A which is defined by the relation

$$X(t)=Ax(t), \tag{16}$$

where $x(t)$ is the physical quantity and $X(t)$ is the corresponding voltage. This relation holds at every point in the circuit in Figs. 9 and 10.

To compute the standard deviation, 8 amplifiers and

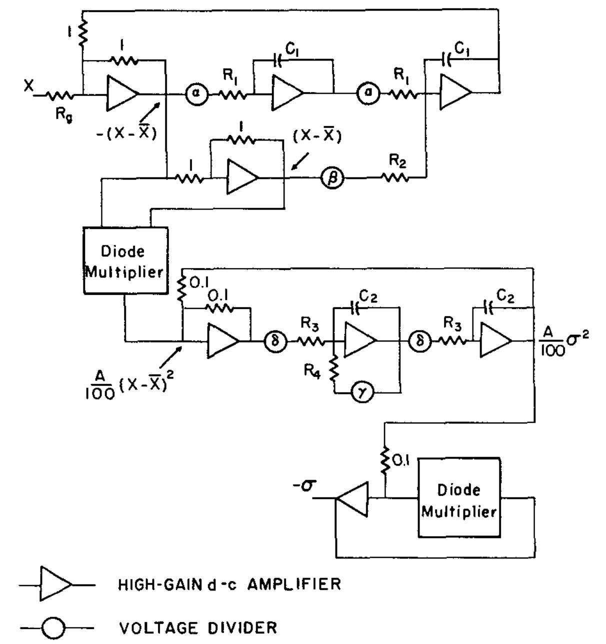

FIG. 10. Circuit for generating the variance and the standard deviation. This circuit includes filters 2, 3, 4 and 5. Filters 3 and 5 are the square and square root filters respectively. This is the realization of the block diagram in Fig. 6(b).

2 multipliers are needed but only 7 amplifiers are required for the mean absolute deviation. Simply on the basis of the amount of equipment required, it is least expensive to compute the mean absolute deviation.

It is possible to design a circuit to perform the same functions with fewer amplifiers. A 4-amplifier circuit for computing the mean absolute deviation is shown in Fig.

11 where the transfer function of the first amplifier is

$$\frac{e_0}{e_i}=\frac{C_1C_2R_1R_2p^2}{R_1R_2C_2C_3p^2+R_1(C_1+C_2+C_3)p+1} \quad (17)$$

and $C_1=C_3$ for unity gain. To determine parameter values, Eq (17) must be matched with Eq (11). The transfer function of the last amplifier is

$$\frac{e_0}{e_i}=\frac{R_5}{R_3}\frac{1}{R_4R_5C_4C_5p^2+C_5/R_3(R_4R_5+R_3R_4+R_3R_5)p+1} \quad (18)$$

and $R_5=1.25R_3$ for a gain of 1.25. Eq (18) must be matched with Eq (12). This circuit has the disadvantage that it does not work well for values of the sampling period greater than about 10 sec. An amplifier with an output capacity of 20 milliamp at ± 100 volts could not drive the complex feedback circuits for τ greater than 12 sec. Also, the magnitude of the resistors and capacitors required is much greater than for the 7-amplifier circuit. For comparison, the 7-amplifier circuit, using the same amplifiers, can operate with a sampling period of 3600 sec and with reasonable component values. For example, a τ of 3600 sec can be obtained with 10-megohm resistors, 10-microfarad capacitors, and voltage divider settings between 0.09 and 0.17.

It is considered that there are three general situations in which the standard deviation computer would be used. These are: (1) analysis of recorded data, (2) on-line operation, i.e., accepting a voltage directly from the sensing instruments and (3) on-line operation as an element in a control system. The first alternative is the most attractive provided that some means of recording the data and later reproducing it as an analog voltage is available. This affords the greatest flexibility and the possibility of the greatest accuracy especially if a small, general purpose analog computer is available. In that event, the circuits shown could be set up on the computer.

Perhaps the opposite extreme is case (2) where economy would be stressed so as to provide a number of computers for field use. In this case, one should undertake to build a small, special purpose computer providing the necessary power supplies, etc. It would be feasible to purchase the amplifiers since adequate amplifiers can be obtained for less than $90 per pair.

The last case (3) would occur when one wished to control, for example, smoke emission from a plant depending upon the turbulence of the air stream of which the standard deviation of the wind speed or direction would be one measure. In this case reliability would be stressed and the best solution would be to install a small analog computer of about 10 amplifier capacity. Many such computers have individual trouble lights to indicate when an amplifier is out of balance or saturated, thus trouble spotting would be facilitated. Also since these computers utilize plug-in amplifiers, a spare could be quickly substituted for a faulty one. In both cases (2) and (3), one would probably wish to use the absolute value type of circuit for economy. In case (1) the use of multipliers would simply depend upon their availability in the installation.

5. Evaluation of the computer

In testing the performance of the computer it is most helpful to have an analytical solution to Eq (3a), which would give the theoretical output of the computer, exclusive of the smoothing filter, for an input sine wave of arbitrary frequency and amplitude. The Fourier transform method shows that the output of y_2 for an input of $x(t)=\sin\omega\tau$ is according to Eq (10),

$$x(t-\tau/2)-\bar{x}(t-\tau/2)=\left[1-\frac{\sin\frac{1}{2}\omega\tau}{\frac{1}{2}\omega\tau}\right]\sin\omega(t-\tau/2).$$

The term in brackets is the amplitude factor which is constant for a given ω and τ. Therefore

$$e(t-\tau)=\left|1-\frac{\sin\frac{1}{2}\omega\tau}{\omega\tau}\right|\int_{t-\frac{3}{2}\tau}^{t-\frac{1}{2}\tau}|\sin\omega(\xi-\tau/2)|\,d\xi$$

and the absolute value term can be expressed as a Fourier series

$$|\sin\omega(t-\tau/2)|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\omega(t-\tau/2)}{4n^2-1},$$

therefore,

$$e(t-\tau)=\left|1-\frac{\sin\frac{1}{2}\omega\tau}{\frac{1}{2}\omega\tau}\right| \times\left[\frac{2}{\pi}-\frac{4}{\pi\omega\tau}\sum_{n=1}^{\infty}\frac{\sin n\omega\tau\cdot\cos 2n\omega(t-\tau)}{n(4n^2-1)}\right]. \quad (19)$$

This expression converges rapidly so that only a few terms are needed to evaluate it.

For testing purposes, the computing circuit for the mean absolute deviation method was set up on a small analog computer. The computer performance was evaluated with respect to Eq (3a) and an attempt was made to evaluate Eq (4). The procedure involved both laboratory and field tests.

(a) *Laboratory tests.* Since any input can be made up of a combination of sine waves of various frequencies, it is valid to study the frequency response of the computer. A sine wave generator was used as the input, and to facilitate the work, a sampling period of 2 sec was used. The result was plotted in Fig. 12 in terms of $\omega\tau$

so that the data apply to any sampling period. The output of the computer was compared with the theoretical response given by Eq (19).

The mean absolute deviation of a sine wave consists of a steady component plus a component which oscillates in time. This is indicated in Fig. 11 by plotting the mean or steady value and the bounds of the oscillating components.

For $0.1 \leqq \omega\tau \leqq 100$, the computer error relative to Eq (11) and (12) depended upon the amplitude of the output so that the error was less than 5 per cent at the 5-volt level, less than 3 per cent at the 10-volt level, and less than 2 per cent at the 50-volt level. Error depending on the amplitude level is a phenomenon common to all such electrical computing devices.

(b) *Field test.* As a further demonstration of the computer performance, it was given a field test. A potentiometer-type wind vane was mounted on the roof of a building. Using the circuits shown in Figs. 9 and 10, the standard deviation and the mean absolute deviation were computed. The sampling times used were 24 and 60 sec. The wind vane was in a region of pronounced mechanical turbulence; the standard deviation was in the range of from 15 to 20 deg.

Some of the results were recorded with a chart speed of 10 millimeters per second, which is fast enough to permit good resolution of the input signal. From these, e was hand-calculated to check the computer.

This calculation, over a 45-sec portion of the record where $\tau = 24$ sec, showed an average 4 per cent error in the mean absolute deviation. Instantaneous values of the error were as great as 8 per cent, but this is at least partly attributable to abstracting error.

It was stated above, Eq (4), that e and σ should be related by the form factor 1.25 when the input is normally distributed. Having computed both e and σ, one can readily check this. To do so, e and σ/e were determined at a number of points in an interval of the record. The form factor and the mean absolute deviation were averaged for the points to eliminate random measurement errors as shown in Table 1.

6. Comparison with other systems

The standard of comparison in this area is certainly the excellent paper by Jones and Pasquill (1959). In principle our system is identical with theirs although the method used for obtaining the circuits and the circuits themselves are different.

They obtain $\sigma(\tau,s)$ by computing $\sigma(\tau)$ and $\sigma(s)$ and then taking the difference after the outputs have been recorded instead of using a smoothing filter as we prefer. Jones and Pasquill claim that the largest τ which their instrument can achieve is 180 sec whereas this computer can operate to 3600 sec while using the same magnitude resistors and capacitors, 10 megohms and 10 microfarads. Both systems use an absolute value circuit but the greatest difference occurs in the filter which follows the absolute value circuit. They use a single storage filter with a longer sampling time of up to 500 sec which is intended to at least form a partial average of the standard deviation. Thus it may be said that the two instruments are very much alike in theory and in design, differing only in detail.

TABLE 1. Form factor for several values of the sampling time

Sampling time, sec	Number of points	Length of record, sec*	$\bar{e}$, deg	Form factor
24	20	20	14.4	1.21
24	20	20	14.3	1.21
60	20	40	12.7	1.23
60	20	40	14.6	1.24

* The length of record given was a sample from a much longer record.

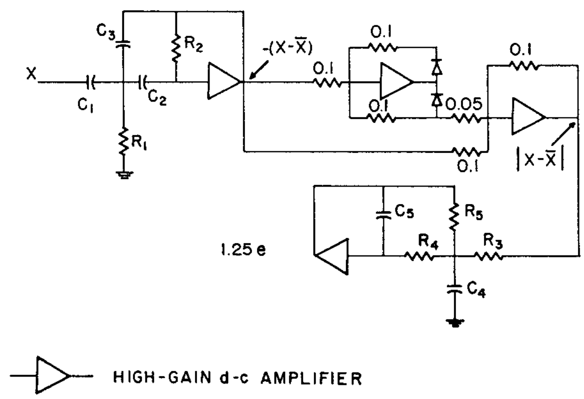

FIG. 11. Alternative circuit for computing the mean absolute deviation. This circuit performs exactly the same function as the circuit in Fig. 9 but is good only for small values of the sampling time τ.

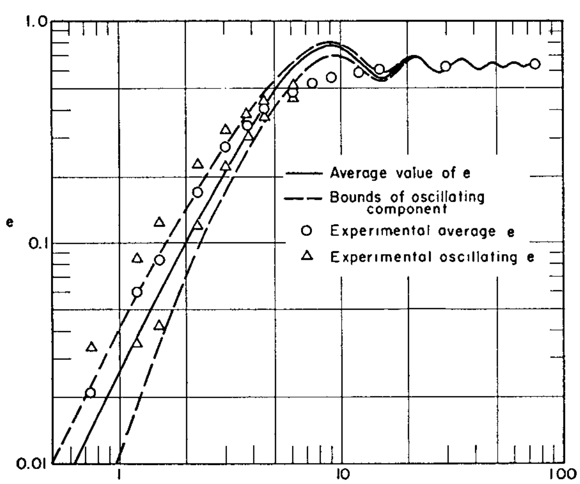

FIG. 12. Plot of the theoretical versus the actual output of the computer using the circuit of Fig. 9. The theoretical curve was obtained from Eq (19).

7. Summary and conclusions

There were two key steps involved in this method of computing the standard deviation. The first was to find a mathematical model which is an adequate approximation to the defining model and which could be readily implemented as an electrical analog. The second was to find this electrical analog which has all the desired properties, e.g., accuracy, simplicity, and reliability.

The defining model was expressed in Eq (4) and (3a) and the approximate model was partially stated in Eq (11) and (12). The complete process is shown in Fig. 6.

Computer circuits have been shown (Figs. 8, 9, and 10) for obtaining the standard deviation by two methods. The first used the concept of the mean absolute deviation and its relation to the standard deviation, while the second circuit obtained the standard deviation directly with the variance as a by-product. The second circuit has the advantage of being somewhat more exact, but it is also more expensive to implement. The decision to use the first circuit was made solely on the basis of initial cost. As another measure of economy, the circuit of Fig. 11 can be used, provided that only small values of the sampling time are required.

This computer can accept any direct analog voltage signal at any amplitude level. It can attenuate or amplify that signal to its own optimum range. The upper limit in frequency of input is the frequency at which the amplifier band width interferes, so that the upper limit should be at above one kilocycle. The maximum sampling time that can be used is ultimately set by the capacity of the amplifiers to drive the computing networks and by the difficulty in avoiding leakage paths around very large resistors. With amplifiers having a forward gain of 200,000 and an output capacity of 20 milliamp at ± 100 volts, the accuracy will be unimpaired with sampling periods up to 3600 sec but will begin to fall off for larger sampling periods.

The primary limit on accuracy is the quality of the mathematical approximations used. The computing circuits were implemented with 1 per cent quality resistors and capacitors, but if more accuracy in the computer were needed, one could easily use 0.05 per cent components.

Stability of operation is an inherent feature, even though the d-c amplifiers used are not chopper-stabilized. The computer can operate unattended, with its accuracy unimpaired by amplifier drift, for periods of up to one week.

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