Analog Computing Techniques Applied to Atmospheric Diffusion: Continuous Line Source[1,2,3]

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(Manuscript received 15 January 1962)

ABSTRACT

The partial differential equation which describes steady-state diffusion from an infinite line source is replaced with a set of simultaneous ordinary differential equations solved on an electronic analog computer. One space dimension, distance downwind, is represented by computer time; the other, height, is replaced with finite differences. Solutions are obtained for constant, power law, and logarithmic wind profiles, and for diffusion of particulates which can settle out and deposit on the ground.

All solutions are obtained with one basic computing circuit. Each problem requires only a particular setting of the coefficient potentiometers in the circuit. Implementation of this circuit requires only 9 integrating amplifiers and 26 coefficient potentiometers, available in any medium sized computer.

The solution's accuracy has been measured by comparing the computer plots with the analytical solution for constant wind profile. This measures the total error due to the finite difference approximation and to computer errors. The solution's accuracy is found to be 5 per cent or better over most of the field.

1. Introduction

Diffusion due to atmospheric turbulence can be described by the parabolic diffusion equation. This is the direct physical approach in contrast to that which invokes statistical concepts. Through the use of such an equation and its boundary conditions, diffusion problems are conveniently specified, but analytical solutions are not so readily obtained. So great is the difficulty, indeed, that it is unusual to find a solution that fits problems of interest. With the aid of various computing devices, such solutions can be found. Given such a device, in this case an electronic analog computer, it is of interest to explore the form of the solutions as a function of the various equation parameters.

The physical problem considered here is that of steady-state diffusion from an infinite line source, oriented normal to the wind. The solution will specify the concentration of the diffusing substance as a function of position in the region after the source has been emitting at a steady rate for a long time. It will be obtained as a plot of concentration as a function of distance downwind for discrete height intervals. As examples of this process, consider diffusion of exhaust gases from automobiles on a busy road, or of smoke from a uniform, slowly advancing grass fire. The physical model, that is, the idealized case, is shown in Fig. 1 and consists of a line source of infinite extent along the y axis. The x axis is oriented downwind which is normal to the source and the z axis extends vertically. The wind vector is assumed to have no components in the y or z directions. Since the source has no height or width, the concentration at the source must be infinite.

A solution for one case of interest, that of the logarithmic wind profile, was given by Karplus and Allder

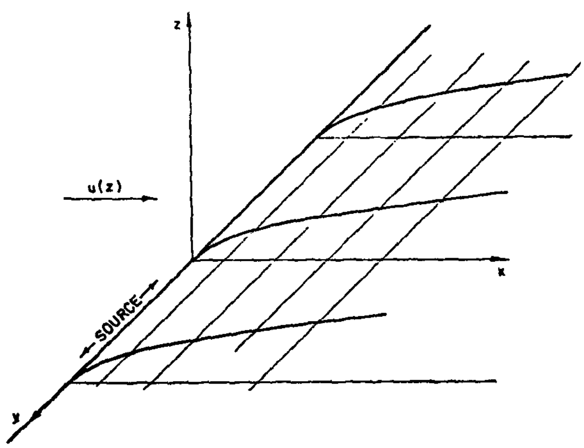

FIG. 1. Physical model of diffusion from an infinite line source showing coordinate axes and cloud outlines.

[1] Publication No. 61 from the Meteorological Laboratories, The University of Michigan.

[2] Contribution No. 18 from the Great Lakes Research Division of the Institute of Science and Technology, The University of Michigan.

[3] Research conducted under Research Grant No. G-11404 from the National Science Foundation.

(1956). They used the electric analog which is similar in principle to the technique reported here. It consisted of building a ladder network of resistors and capacitors which satisfied a set of simultaneous differential equations serving as an approximation to the partial differential equation.

The electronic analog approach is different primarily in that a standard electronic analog computer is utilized instead of building a special circuit as was done by Karplus and Allder. This has the advantage of using standard, general purpose equipment, and is much more flexible.

2. The parabolic diffusion equation

The diffusion equation appropriate to the physical model used here is

$$u(z)\frac{\partial \chi}{\partial x}=\frac{\partial}{\partial z}\left[K(z)\frac{\partial \chi}{\partial z}\right]+f\frac{\partial \chi}{\partial z} \qquad (1)$$

where χ = time mean concentration of the diffusing substance
x = distance downwind
z = height above ground
$u(z)$ = mean wind speed, a function of height
$K(z)$ = eddy diffusivity, a function of height
f = fall speed of particulate matter, constant over the region.

Following Calder (1961), the boundary conditions are

$$\lim_{x\to\infty}\chi=0 \qquad (2a)$$

$$\lim_{z\to\infty}\chi=0 \qquad (2b)$$

$$\lim_{x\to 0}\chi=\frac{Q}{u(h)}\delta(z-h) \qquad (2c)$$

$$\lim_{z\to 0}\left[K(z)\frac{\partial \chi}{\partial z}+(f-p)\chi\right]=0. \qquad (2d)$$

In (2c), $\delta(z-h)$ is the unit impulse function which is zero except when the argument is zero and then it is effectively infinite. Eq (2d) states that the flux at the ground is equal to the material lost to the ground. The ground is assumed to act as a sink such that the flux into the ground is proportional to the concentration near the ground with proportionality constant p.

The continuity equation for this problem is

$$\int_0^\infty u(z)\chi dz+p\int_0^x \chi(x,0)dx=Q \qquad (3)$$

where Q is the source strength.

To facilitate handling the equations and to increase the utility of the solutions, nondimensional terms will be used throughout. In these terms, Eq (1) becomes

$$\frac{\partial S}{\partial X}=\frac{C}{G_1(Z)}\frac{\partial}{\partial Z}\left[G_2(Z)\frac{\partial S}{\partial Z}\right]+\frac{CF}{G_1(Z)}\cdot\frac{\partial S}{\partial Z} \qquad (4)$$

where the terms which are functions of height are expressed as follows

$$u(z)=u_1g_1(z)=AG_1(Z)$$
$$K(z)=K_1g_2(z)=BG_2(Z). \qquad (5)$$

These are the defining equations for the constants A and B. The other nondimensional parameters used are stated in terms of the constants z_0, χ_0 and C in addition to A and B. These relations are $Z=z/z_0$, $S=\chi/\chi_0$, $X=Bx/ACz_0^2$, $F=z_0f/B$, $P=z_0p/B$, $H=h/z_0$. The unit of height, z_0, can be chosen arbitrarily and the unit of concentration is related to the source strength as defined below. The constant C appears on both sides of Eq (4); on the left side, it is incorporated in the parameter X. This constant does not affect the equation or its solution and is not related to any property of the diffusion problem. It is inserted for convenience in analog computation as explained below.

The boundary and continuity conditions become

$$\lim_{X\to\infty}S=0 \qquad (6a)$$

$$\lim_{Z\to\infty}S=0 \qquad (6b)$$

$$\lim_{X\to 0}S=\frac{Q}{\chi_0 u(h)}\delta(Z-H) \qquad (6c)$$

$$\lim_{Z\to 0}\left[G(Z)\frac{\partial S}{\partial Z}+(F-P)S\right]=0 \qquad (6d)$$

$$\int_0^\infty G_1(Z)SdZ+\frac{Bx_0}{Az_0^2}P\int_0^x S(x,0)dX=\frac{Q}{A\chi_0 z_0}. \qquad (7)$$

3. Preparation for analog computation

Since this problem is being prepared for solution on an analog computer, we must observe the restriction that the analog computer can integrate only with respect to one independent variable, namely, computer time. As stated in Eq (4), the model has two independent variables X and Z. The variable X is represented as computer time and the variable Z must be eliminated. This is done by replacing derivatives with respect to Z by finite differences as follows

$$\left.\frac{\partial S}{\partial Z}\right|_n=\frac{S(n+1)-S(n-1)}{2\Delta Z(n)}$$

$$\left.\frac{\partial}{\partial Z}\left[\frac{\partial S}{\partial Z}\right]\right|_n=\frac{1}{\Delta Z(n)}\left[\frac{S(n+1)-S(n)}{\Delta Z(n+\frac{1}{2})}-\frac{S(n)-S(n-1)}{\Delta Z(n-\frac{1}{2})}\right].$$

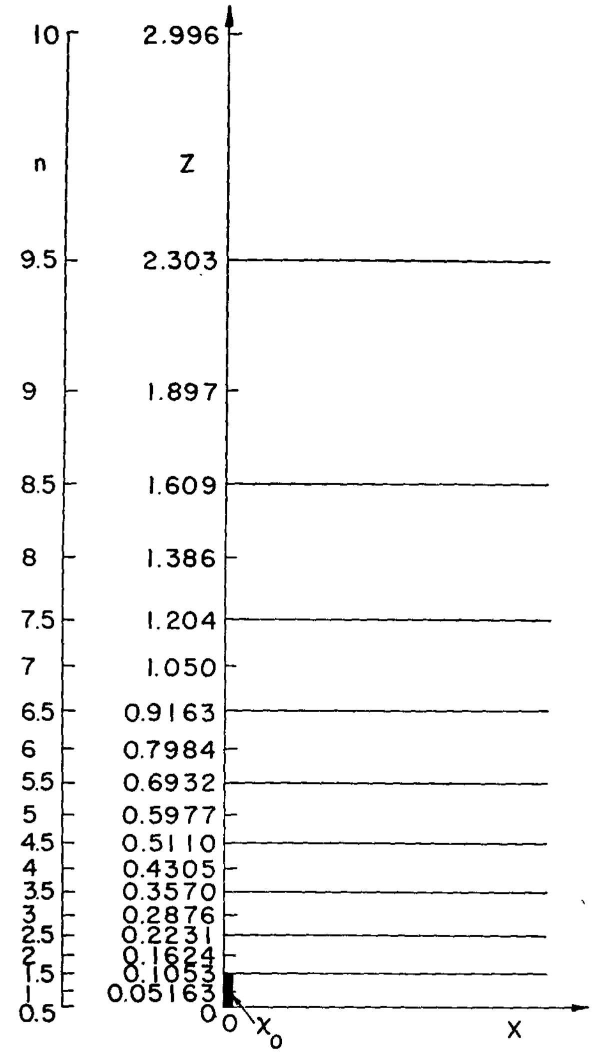

FIG. 2. Finite difference model showing the vertical coordinate axis relations.

Implementation of finite differences requires a new model to clarify the meaning of the position n and the increment ΔZ. Such a model is presented in Fig. 2 which shows the atmosphere divided into 10 layers in the vertical. The thickness of the layers increased logarithmically with height so that the thickness of the top layer is infinite. The height of the station n where N is the total number of stations is given by

$$Z(n)=\ln\left[\frac{N-\frac{1}{2}}{N-n}\right].$$

This distribution tends to decrease the error inherent in a finite difference approximation by providing a dense grouping of stations where the derivatives of concentration with respect to height are greatest, near the source.

Each layer has a thickness $\Delta Z(n)$, is infinite in the Y direction and semi-infinite in the X direction. All properties of each layer are lumped in the center of the layer so that the model may be thought of as a stack of planes spaced logarithmically. Each plane incorporates the properties of the layer of the atmosphere which it replaces. The planes are labeled 1 through 10 and represent the computing stations. Thus the notation $S(n)$ represents the concentration in the nth layer which is a function of the distance downstream.

Now the reference parameters χ_0 and z_0 can be defined in terms of the finite difference model. The height scale and the layer thickness is determined by the number of stations used and by the magnitude of z_0.

The line source becomes a small uniform, vertical area source in the finite difference model with a minimum height equal to one layer thickness. The concentration becomes finite over the area instead of infinite. The source can be at any layer; a ground level source would be represented by the first layer. The continuity equation at $X=0$ evaluated over any one layer is

$$\int_{\Delta Z(n)} G_1(n)S(n)dZ=\frac{Q}{A\chi_0 z_0}$$

and if we choose the value of the concentration ratio in the source layer at $X=0$ to be unity, then

$$S(n)=1 \quad \text{at} \quad X=0$$

and therefore

$$\chi_0=\frac{Q}{\Delta Z(n)AG_1(n)z_0}.$$

Using finite difference expressions for the derivatives with respect to Z, Eq (4) will be replaced with a set of 10 simultaneous differential equations, one for each layer. The form of these equations is

$$\left.\frac{dS}{dX}\right|_n=\frac{C}{G_1(n)\Delta Z(n)}\left[\frac{G_2(n+\frac{1}{2})}{\Delta Z(n+\frac{1}{2})}\{S(n+1)-S(n)\}-\frac{G_2(n-\frac{1}{2})}{\Delta Z(n-\frac{1}{2})}\{S(n)-S(n-1)\}\right]+\frac{CF}{2G_1(n)\Delta Z(n)}\{S(n+1)-S(n-1)\}. \quad (8)$$

Notice that the constant C changes the rate at which the solution is generated with respect to X but not with respect to x. Since X is represented as computer time, the rate of generation with respect to time is controlled by C. The solution speed must be within limits set by the frequency response of the recording equipment, at one extreme, and by the patience of the operator at the other.

The boundary conditions and the continuity equation require a special interpretation in this model. The conditions given in (6a) and (6b) do not appear explicitly but they are implicit in the choice of this finite difference model. This model works because these conditions hold. Condition (6c) is applied only approximately as follows

$$\lim_{X\to 0} S(n)=1 \qquad \text{if } n^{\text{th}} \text{ cell is source cell}$$
$$=0 \qquad \text{otherwise.}$$

The concentration at the source is represented as finite over some height increment instead of being infinite. Condition (6d) in this model is

$$\frac{G_2(1)}{\Delta Z(1)}[S(1)-S(0)]+(F-P)S(1)=0 \qquad (9)$$

where $S(0)$ is a virtual station below ground used to evaluate the flux across the ground. When Eq (8) is evaluated at station 1, $S(0)$ will be required and is given by Eq (9).

In this paper the continuity equation is invoked only to define the concentration ratio as shown above.

Six problems were chosen for solution as stated below.

(a) *Constant velocity profile with variable source height.* The diffusing material is assumed to be a gas and the ground is a reflecting boundary. The source height can be increased in increments from one layer to the next. Each of the lowest three cells was used individually as the source cell. Use of the first cell as the source cell corresponds to a ground level line source. Each of the others corresponds to an elevated line source. Since the cell thickness increases with height the value of the initial concentration ratio in the source cell must be set inversely as the cell thickness to keep the source strength constant.

The model used here is not optimum for large elevations of the source since the cell spacing is for a ground source. With the small elevations used here, the accuracy of computation will not be much affected.

(b) *Constant velocity profile with gravitational settling.* In this case, a particulate material is diffusing from a ground level source but ground absorption or deposition is not permitted. This requires reflotation of the particles as soon as they come into contact with the ground.

(c) *Constant velocity profile with ground absorption.* In this, as in all succeeding cases, the source is on the ground. The diffusing material is a gas which is subject to ground absorption.

(d) *Constant velocity profile with gravitational settling and ground absorption.* A particulate material is emitted which is subject to deposition on the ground. It is not necessary to assume that the rate of deposition P is equal to the fall speed F. Accordingly, two cases will be considered, one with $P=F$ and one where $P-F=1$, $F=1$ in both cases.

(e) *Power law wind profile.* The diffusing material is gaseous and the ground is a reflecting boundary. The wind speed used is proportional to the seventh root of the height and the eddy diffusivity is proportional to the six-seventh root.

(f) *Logarithmic wind profile.* Again the diffusing material is gaseous and the ground is reflecting. The wind speed is proportional to the logarithm of height and eddy diffusivity is proportional to the height.

4. Analog Model

The physical model now consists of a stack of planes spaced exponentially in the z direction. It is supposed that each plane has a capacity for containing some of the diffusing substance just equal to that of the layer of atmosphere that it replaces. At a given point in x, a traverse in the z direction shows the properties of the atmosphere lumped at discrete intervals. This model can be simulated with an electric analog, where each of the circuit components is analogous to some property of the atmosphere. In general, one may relate electrical resistance to atmospheric resistance to diffusion and electrical capacitance to the capacity of a layer of the atmosphere to hold the diffusing substance. The complete network analog is shown in Fig. 3 along with the electronic analog and a representation of the physical model.

The passive circuit analog was not used in this study, but it is helpful to construct it on paper because of the insight it provides. This is particularly useful in determining the method of applying the boundary conditions.

The electronic analog circuit could be designed directly from the passive network or from the set of simultaneous differential equations. The latter method is much the easier course after Eq (8) is reduced to the form

$$\left.\frac{dS}{dX}\right|_n = a_nS(n+1)-b_nS(n)+c_nS(n-1). \qquad (10)$$

The principal element in the electronic analog is the integrating amplifier which can be used to represent one station as shown in Fig. 4. In the passive network analog, there was a direct correspondence between the circuit elements and the physical problem, while in the electronic analog there exists a one-to-one correspondence between the circuit elements and the defining Eq (10). The coefficients a, b, and c are set on the correspondingly designated coefficient potentiometers. The integrating amplifier sums the three inputs, integrates the sum, and inverts the sign. Ordinarily, the gain of the integrating amplifier is unity, set by the 1 megohm input resistor and 1 microfarad feedback capacitor. If one of the coefficients is greater than unity, the corresponding input resistor used is 0.1 megohm which provides a gain of 10. This is necessary because the maximum setting of the coefficient potentiometers is 1.

The electronic circuit is much more flexible than the passive network permitting almost any conceivable

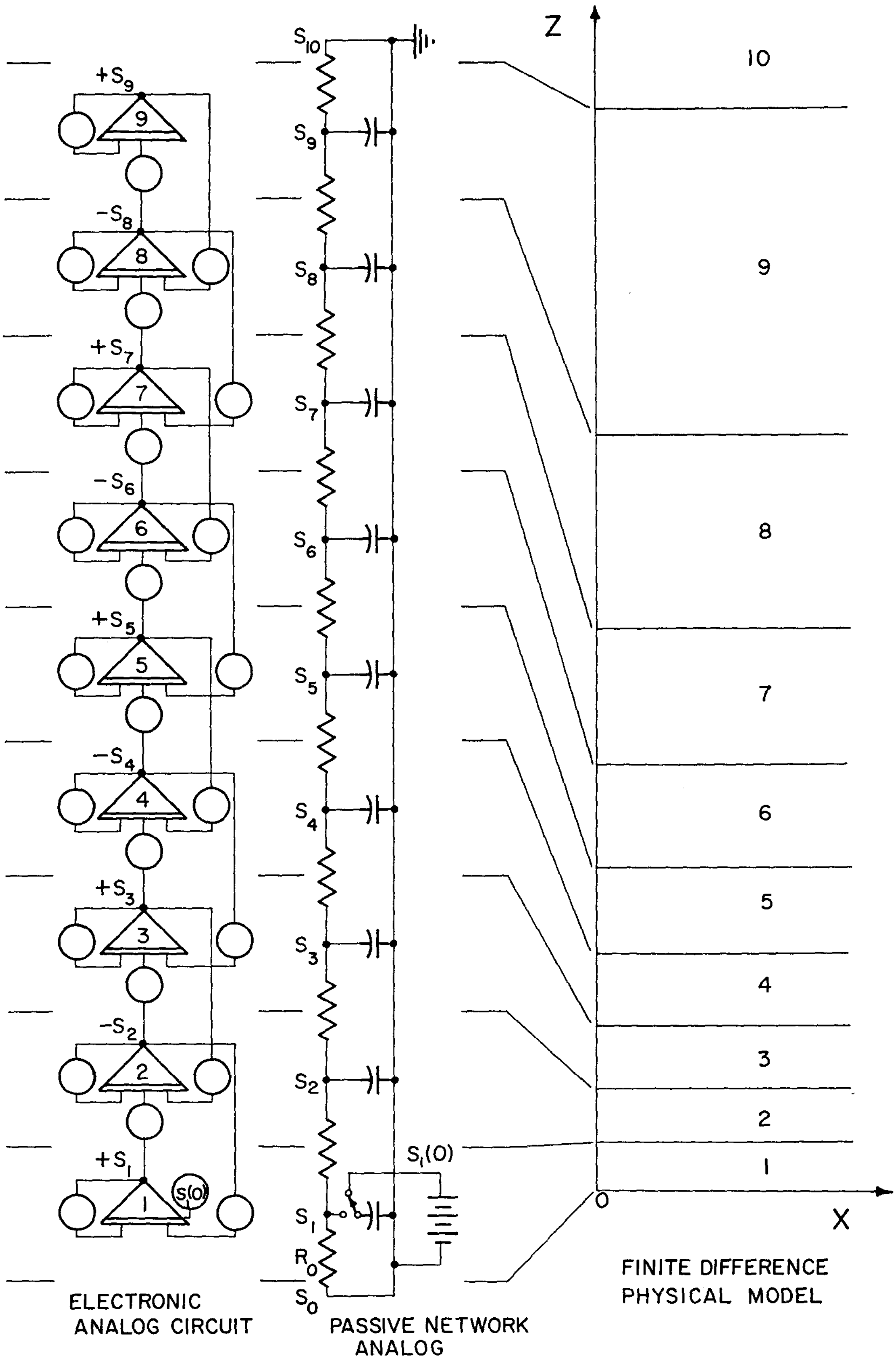

FIG. 3. Electronic analog, passive network analog, and finite difference model of the atmosphere. Note that each group of components represents a layer of the atmosphere.

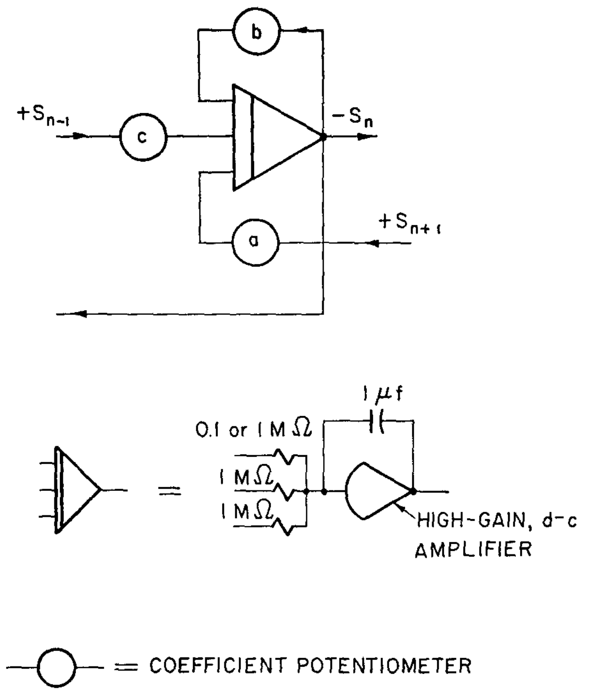

FIG. 4. Typical station in the active or electronic analog circuit. The concentration is represented as a voltage, and equation coefficients as potentiometer settings.

form of boundary conditions to be set up readily. The circuit shown in Fig. 3 was used for all of the problems discussed in this paper. To change the source height, the initial condition voltage was applied to the appropriate integrating amplifier. All other changes were implemented using the coefficient potentiometers in the circuit.

5. Comparison of analog and analytical solutions

The form of the solution is a voltage at the output of each amplifier which is proportional to the concentration at the appropriate level in Z and which is a continuous function of X. When recorded on a two-coordinate plotter, the result is a family of curves representing the concentration at discrete levels of Z.

The evaluation of errors in this type of computation is especially important since there are two very different possible sources. One is the approximation involved in using finite differences and the other is due to the computer. The procedure here will be to compare the computer solution of case (a) constant wind profile with the ground level source, with the analytical solution which may be obtained from Sutton (1953).

$$\chi(x,z)=\frac{Q}{u_1\Gamma(\frac{1}{2})}\left[\frac{u_1}{4K_1x}\right]^{\frac{1}{2}}\exp\left[\frac{u_1z^2}{4K_1x}\right]. \qquad (11)$$

In terms of the nondimensional parameters S, X and Z, (11) is

$$S=0.5941X^{-\frac{1}{2}}\exp\left[-\frac{25Z^2}{X}\right] \qquad (11a)$$

where $K_1=u_1=z_0=1$. We can determine whether the finite difference solution is a good approximation to (11a) by comparing the form of the solutions with respect to X and Z and by comparing values of S at selected points in the X, Z space. A simple relation between S and X exists at $Z=0$ such that

$$S\Big|_{Z=0}=0.5941X^{-\frac{1}{2}}. \qquad (12)$$

Then we can observe the relation of S to the exponential term by finding the slope of logarithm of S with respect to Z^2.

$$\text{Slope}=\frac{\log S_1-\log S_2}{Z_1^2-Z_2^2}=-\frac{25}{X}. \qquad (13)$$

Table 1 shows how well (12) and (13) are satisfied. The exponential term seems to fit well out to $X>100$ where the slope was very small. At $X\geqq 200$, all the concentrations are small, $S\leqq 0.04$ so that a constant error which is small when S is large becomes a large percentage error when S is small. This is shown in Table 2 where computer values of concentration are compared with analytical values.

The most significant feature of Table 2 is that the magnitude of the error is fairly constant. The percentage error increases as the concentration decreases indicating that there is a limit to useful computation. When the concentration ratio S falls to less than 0.001, computer error in the form of noise becomes significant. Virtually all the error shown in Table 1 is due to the limitations of the finite difference model. The error in this computation is less than 5 per cent provided that $S>0.001$ and except in layer 9. Greater errors should be expected in this layer since it is the largest active one and is next to the infinite layer for which zero concentration is assumed.

TABLE 1. Determination of the fit of the finite difference solution to the analytical solution.

x	$x^{\frac{1}{2}}S\|_{Z=0}$	$-X$ [slope]
1	0.549	24.8
5	0.586	26.3
10	0.573	25.6
25	0.572	25.3
50	0.566	25.5
100	0.573	25.4
200	0.572	32.1
300	0.532	43.2
Analytical values	0.5941	25.0

TABLE 2. Comparison of analytical and finite difference solutions at selected points.

n	x	Concentration, analytical	Magnitude of error	Percentage error
1	25	0.119	−0.0050	4.2
2		0.116	−0.0040	3.5
3		0.109	−0.0040	3.7
4		0.0987	−0.0045	4.6
5		0.0831	−0.0038	4.6
6		0.0628	−0.0027	4.3
7		0.0394	−0.0019	4.8
8		0.0174	−0.0011	6.3
9		0.00325	−0.00025	7.7
1	100	0.0594	−0.0021	3.5
2		0.0590	−0.0020	3.4
3		0.0582	−0.0021	3.6
4		0.0567	−0.0020	3.5
5		0.0543	−0.0020	3.7
6		0.0502	−0.0015	3.0
7		0.0447	−0.0013	2.9
8		0.0368	−0.0016	4.3
9		0.0242	−0.0022	9.1

6. Additional results of computation

(a) *Constant wind profile with variable source height.* The pertinent equation coefficients are $u(z)=u_1$, $G_1(Z)=1$, $K(z)=K_1$, $G_2(Z)=1$, $C=1/100$, $F=P=0$. The last statement reduces Eq (9) to $S(0)=S(1)$. The source was placed in cells 1, 2 and 3, respectively, to show the effect of a small increase in source height. Since the cell thickness increases with height the value of the initial concentration in the source cell must be set inversely as the cell thickness to keep the source strength constant. The following table gives the height, cell thickness and initial concentration ratio for the cells used as source cells.

TABLE 3. Values of the initial concentration ratio as a function of the height of the source cell. Source strength is maintained constant.

n	$Z(n)$	$\Delta Z(n)$	$S(n)\vert_0$	Fig. No.
1	0.05163	0.1053	1.000	5,6
2	0.1624	0.1178	0.8939	7
3	0.2876	0.1339	0.7864	8

The solution, as generated by the computer, is shown in Fig. 5 for the source at ground level. This shows the nondimensional concentration for each cell as a function of distance downwind. The same solution is shown in Fig. 6 as lines of constant concentration in the X, Z space. Figs. 7 and 8 show the results with the source in cells 2 and 3, respectively. In both of these cases, the plume axis heads toward the ground and concentrations are generally lower since the material has a greater immediate volume to diffuse into.

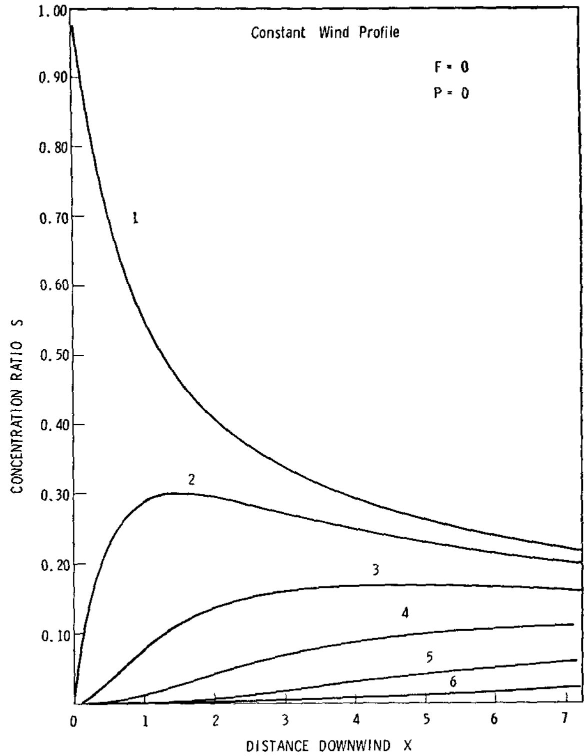

FIG. 5. Solutions as plotted by the computer for the constant wind profile case with fall speed, $F=0$, and ground absorption coefficient, $P=0$. The solutions are obtained as continuous functions of distance downwind for each station. The height of the stations involved, one through six in this case, is indicated in Fig. 2.

To find the physical parameters χ, x, and z from the ratios S, X, and Z, the following relations may be used

$$\chi=\frac{Q}{\Delta Z(n)G_1(n)Az_0}S$$
$$x=\frac{ACz_0^2}{B}X \tag{14}$$

and $z=z_0Z$, where n is the number of the source cell. Interpretation of the solutions exhibited requires evaluation of these expressions. The constants A and B are determined from the profiles using the defining Eq (4). Q is the source strength in units of M/LT since this is a line source. The choice of height scale determines z_0. $G_1(n)$ is the nondimensional wind speed at the center of the source cell. The scale factor C is given for each solution.

In this case, with a ground level source, the concentration and downwind distance become

$$\chi=\frac{9.497Q}{u_1z_0}S$$
$$x=\frac{u_1z_0^2}{100K_1}X.$$

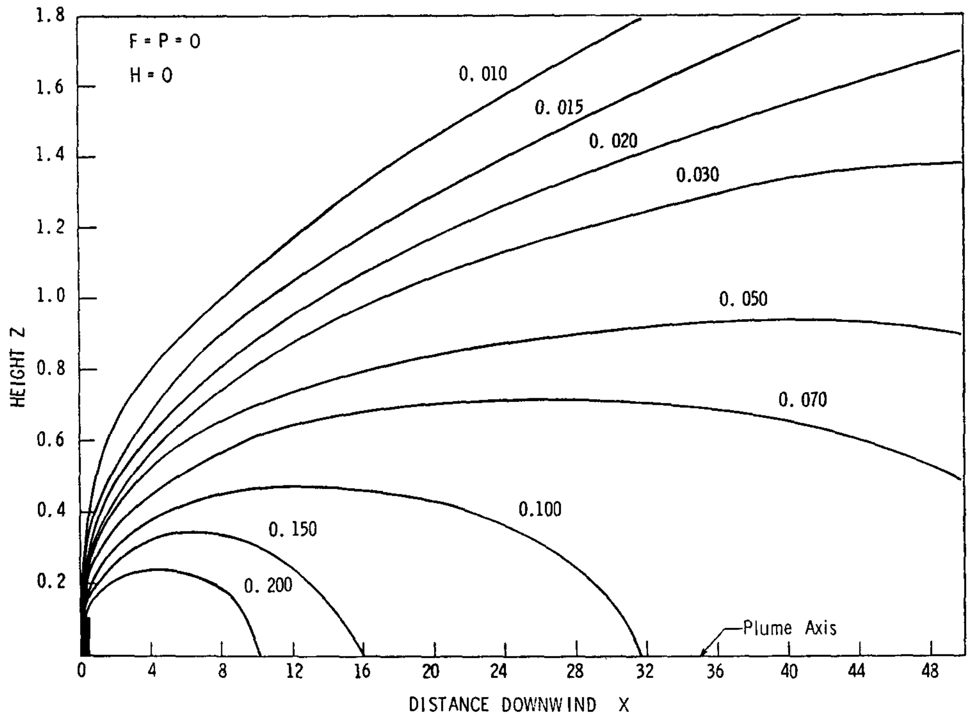

FIG. 6. Solution for the case of the constant wind profile with fall speed, $F=0$, and ground absorption coefficient, $P=0$. The source is in the first cell, effectively on the ground. The lines of constant concentration are plotted for the values 1, 1.5, 2, 3, 5 and 7 for each decade of the concentration ratio S.

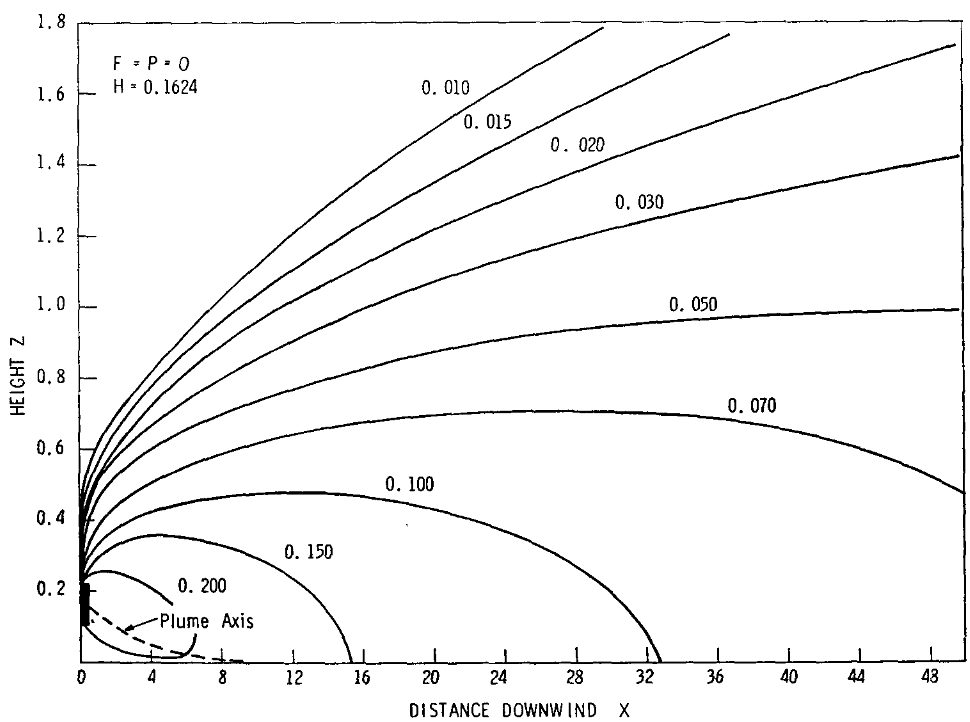

FIG. 7. Plot of the solution for the constant wind profile with fall speed $F=0$, ground absorption coefficient $P=0$. Source height $H=0.1624$.

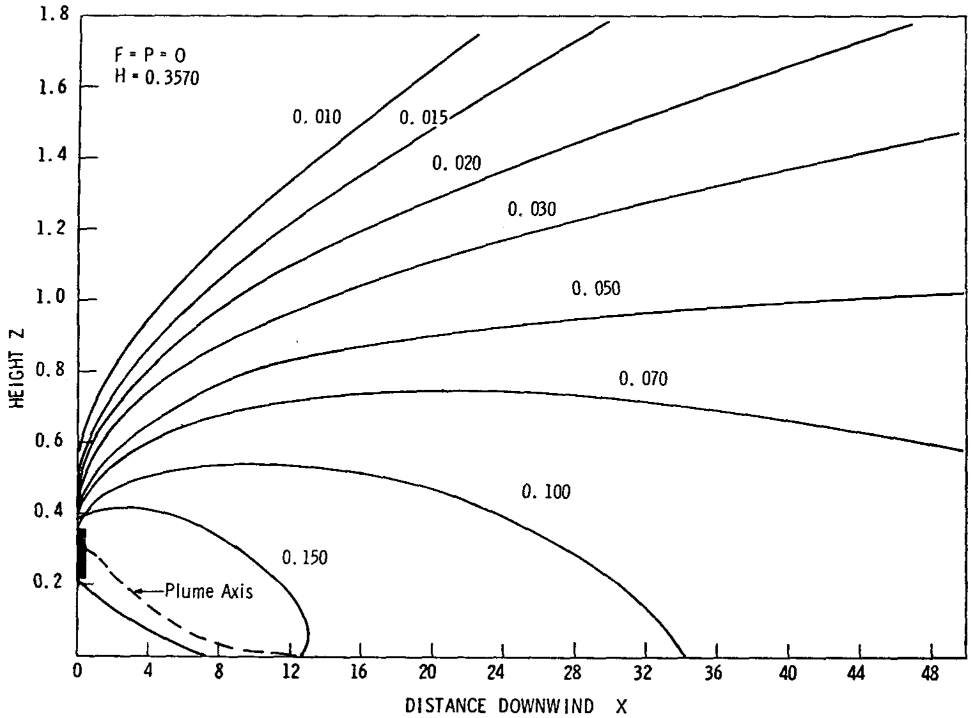

FIG. 8. Plot of the solution for the constant wind profile with fall speed $F=0$, ground absorption coefficient $P=0$. Source height $H=0.3570$.

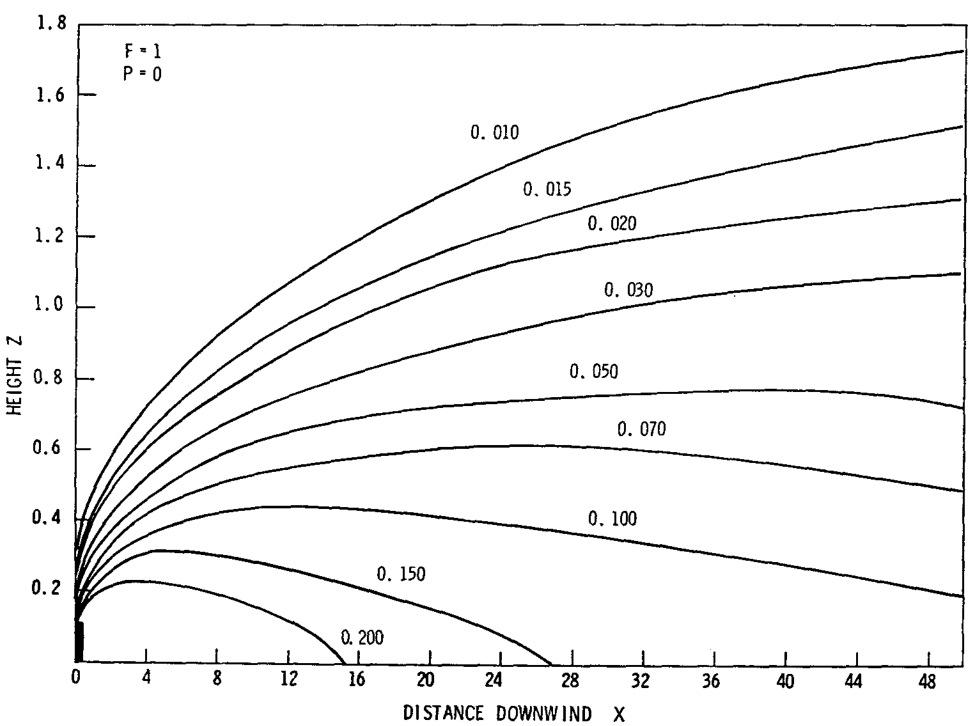

FIG. 9. Plot of the solution for the constant wind profile with fall speed $F=1$, ground absorption coefficient $P=0$. Source is on the ground.

If in a typical problem, $u_1=500$ cm sec^{-1}, $K_1=5\times10^3$ cm^2 sec^{-1}, $z_0=10^4$ cm, $\chi=1.899\times10^{-6}$ QS gm cm^{-3}, $x=100$ X meters, and $z=10^4$ Z cm. Then, the concentration in the middle of the source cell, $z=5.163$ m, falls to a tenth of its value at $x=0$ when $x=3.1$ km.

(b) *Constant wind profile with gravitational settling.* The solution for a particulate material with a fall speed ratio $F=1$ and with a reflecting ground $P=0$, is shown in Fig. 9. The relevant parameters for this case as well as for (c) and (d) are $u(z)=u_1$, $G_1(Z)=1$, $K(z)=K_1$, $G_2(Z)=1$, $C=1/100$. To evaluate F and P, use the relations $f=BF/z_0$ and $p=BP/z_0$. The boundary condition, Eq (9) becomes $S(0)=[1+\Delta Z(1)]\,S(1)$.

The specification of a completely reflecting ground might be realized in the case where the particulates are immediately refloated upon hitting the ground. The pattern is markedly different from the one observed in Fig. 6 since the lowest layers are enriched by settling from the upper layers but do not lose material to the ground.

(c) *Constant wind profile with ground absorption.* Fig. 10 shows the solution for the diffusion of a gas, $F=0$, when the ground is acting as a sink, $P=1$. The boundary condition Eq (9) becomes $S(0)=[1-\Delta Z(1)]\,S(1)$. Here the plume axis rises since gas is being lost from both the top and the bottom layers.

(d) *Constant wind profile with gravitational settling and with ground absorption.* The diffusion of particulate matter when the ground acts as a sink is shown in Fig. 11. In this case $F=P=1$, $S(0)=S(1)$, and note that the plume does not rise. The pattern is similar to that in Fig. 6 but the concentrations are much lower. This situation might be observed when the particulates settle out at a steady rate, land on the ground and are not refloated. The model cannot distinguish between deposition of material on a boundary and passage through it.

An extension of this case is shown in Fig. 12 where the fall speed is maintained, $F=1$, but the ground sink effect is doubled, $P=2$. The boundary relation is $S(0)=[1-\Delta Z(1)]S(1)$. The plume axis again is above the ground and the whole plume is much lower than before. This combination could be the result of the absorbing action of some agent at or near the ground.

(e) *Power law wind profile.* More realistic profiles of the wind speed and eddy diffusivity are used to produce the solution shown in Fig. 13. A gaseous material is assumed with a reflecting ground so that $F=P=0$. The seventh root wind profile is given by

$$u(z)=u_1\left[\frac{z}{z_1}\right]^{1/7}=u_1\left[\frac{z_0}{z_1}\right]^{1/7}Z^{1/7}$$

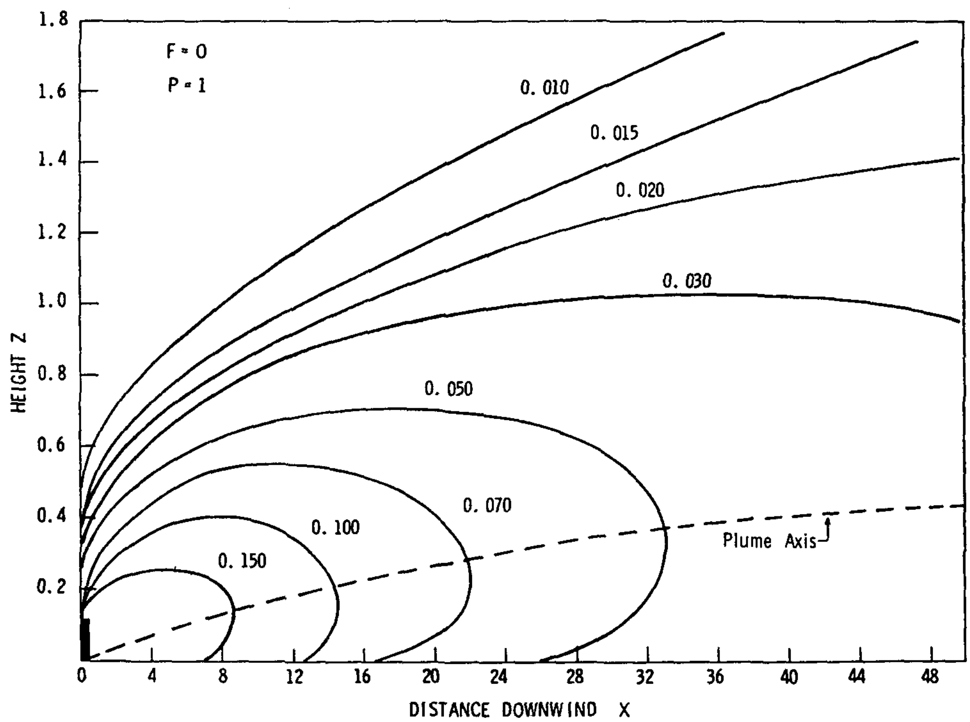

Fig. 10. Plot of the solution for the constant wind profile with fall speed $F=0$, ground absorption coefficient $P=1$. Source is on the ground.

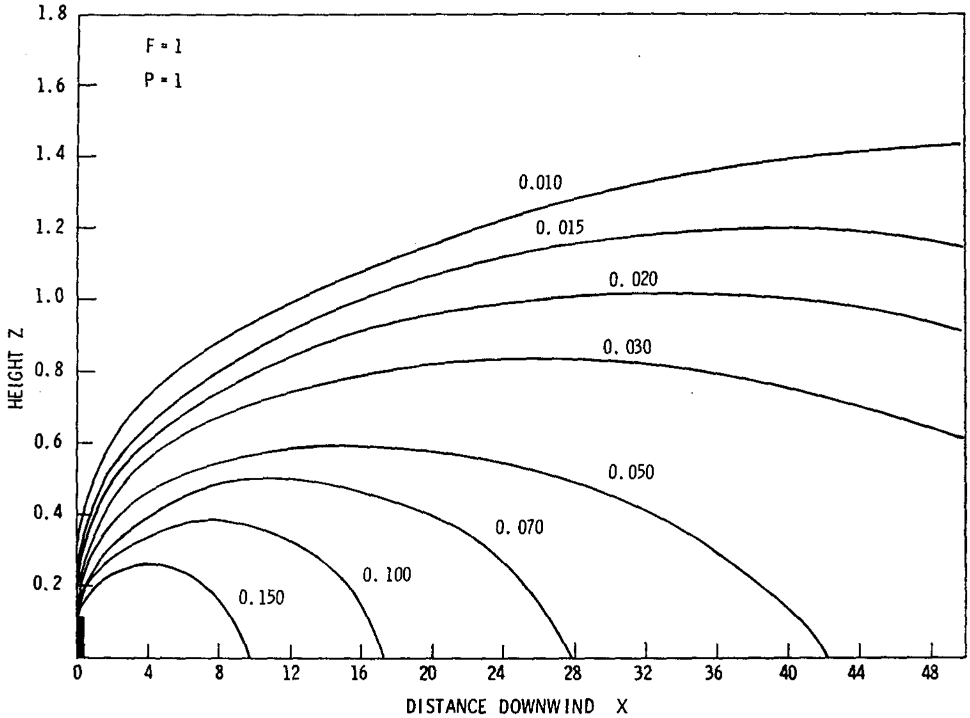

FIG. 11. Plot of the solution for the constant wind profile with fall speed $F=1$, ground absorption coefficient $P=1$. Source is on the ground.

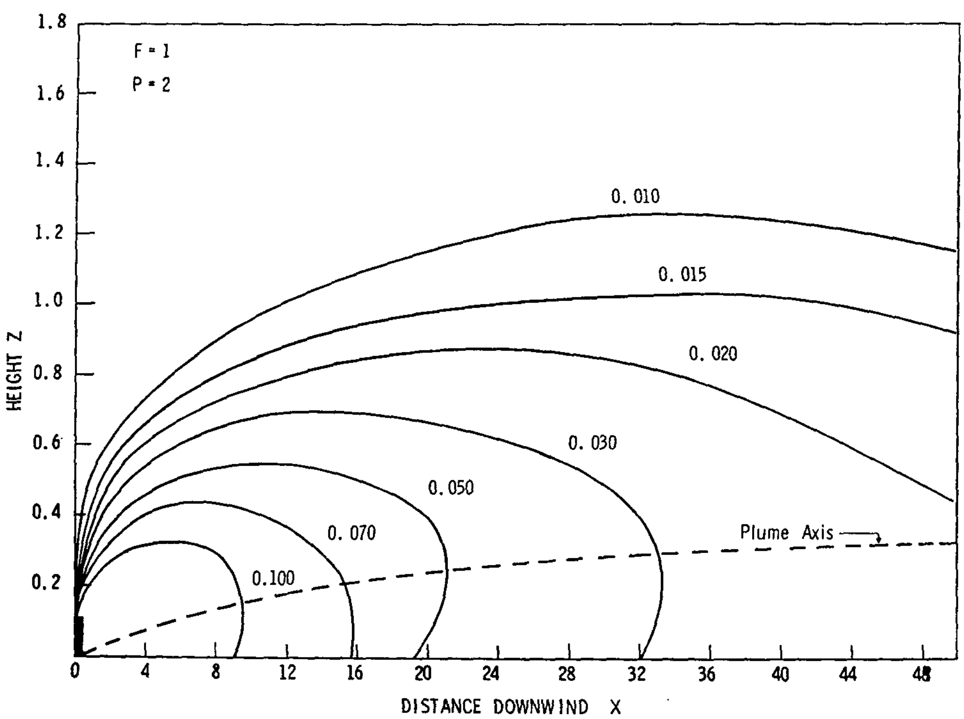

FIG. 12. Plot of the solution for the constant wind profile with fall speed $F=1$, ground absorption coefficient $P=2$. Source is on the ground.

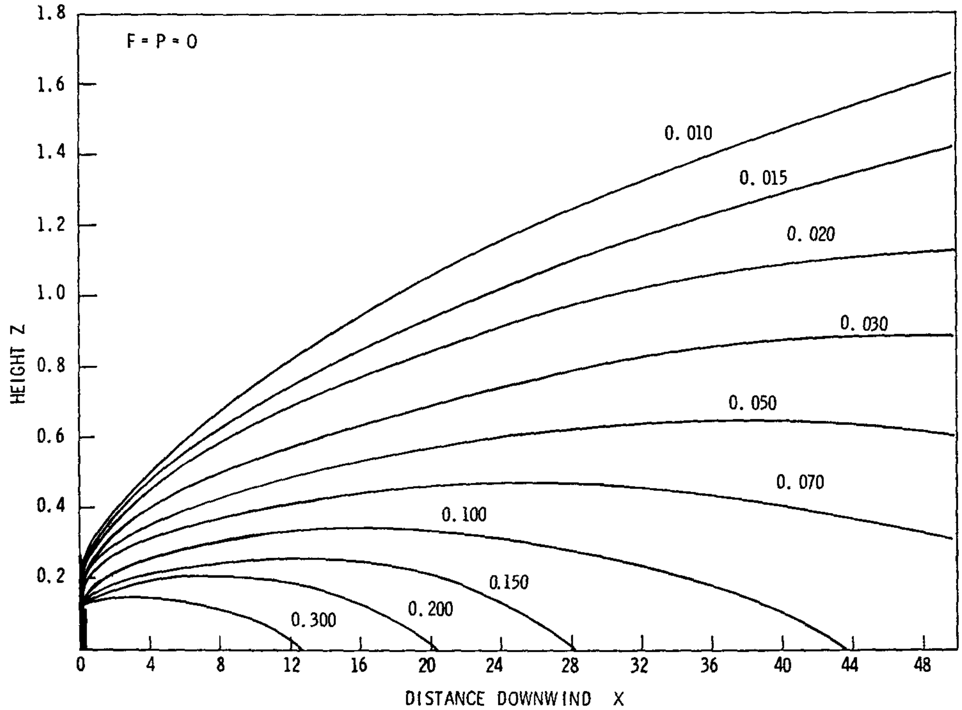

FIG. 13. Plot of the solution for the seventh-root power law wind profile with $F=P=0$.

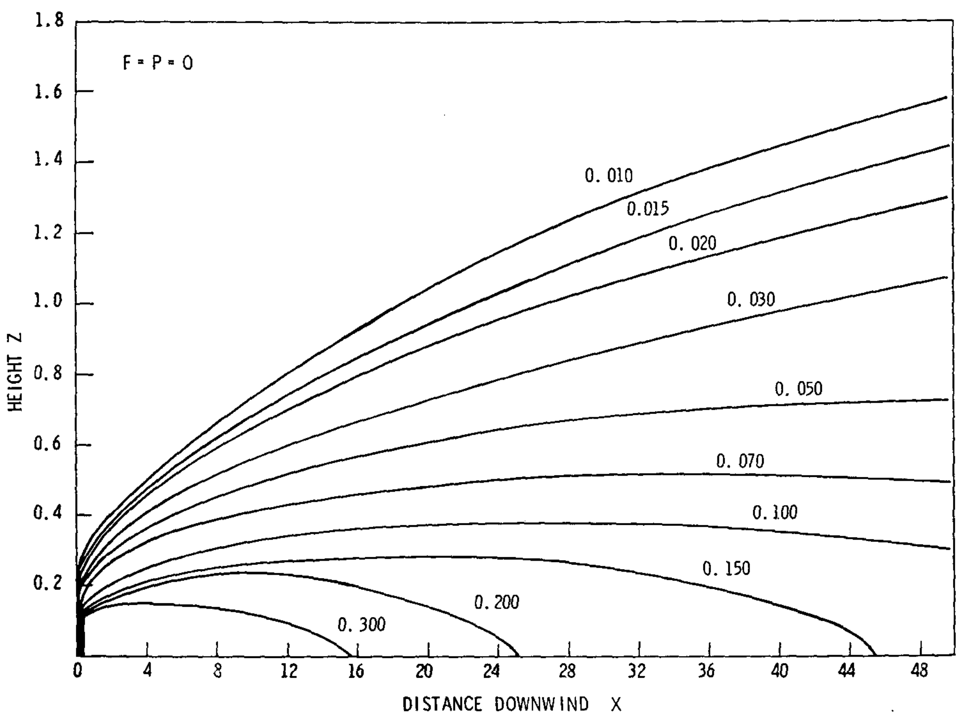

FIG. 14. Plot of the solution for the logarithmic wind profile with $F=P=0$.

and the eddy diffusivity by

$$K(z)=K_1\left[\frac{z}{z_1}\right]^{6/7}=K_1\left[\frac{z_0}{z_1}\right]^{6/7}Z^{6/7}.$$

Here $A=u_1(z_0/z_1)^{1/7}$, $G_1(n)=Z(n)^{1/7}$, $B=K_1(z_0/z_1)^{6/7}$, $G_2(n)=Z(n)^{6/7}$, $C=1/97.06$ so that

$$\chi=\frac{14.50z_1^{1/7}Q}{u_1z_0^{8/7}}S$$

$$x=\frac{u_1z_0^{9/7}z_1^{5/7}}{97.06K_1}X$$

while $z=z_0Z$. Note that despite the different appearance of these relations, they are dimensionally consistent with the previous ones.

This profile enhances the concentration in the lower layers producing a very different pattern.

(f) *Logarithmic wind profile.* Using the logarithmic wind profile given by

$$u(z)=u_1\ln\left[\frac{z+z_2}{z_2}\right]=u_1\ln\left[\frac{z_0}{z_2}Z+1\right]$$

where the roughness length z_2 is assumed equal to $z_0\,10^{-3}$ so that $A=u_1$, and $G_1(n)=\ln[10^3Z+1]$. The eddy diffusivity profile is

$$K(z)=K_1z/z_1=K_1z_0/z_1Z$$

so $B=K_1z_0/z_1$ and $G_2(n)=Z(n)$. The other constants are $F=P=0$ and $C=1/12.68$, so

$$\chi=\frac{2.372Q}{u_1z_0}S$$

$$x=\frac{u_1z_0z_1}{12.68K_1}X$$

and $z=z_0Z$. The solution is shown in Fig. 14. The pattern is very similar to that in Fig. 13. The concentration ratios observed in this and in the preceding case are much higher than in Fig. 6. Fig. 14 has a 0.300 isopleth, Fig. 13 a 0.200 isopleth, while the highest isopleth drawn in Fig. 6 is 0.100. This does not mean that the actual concentrations are higher since the concentration ratio is a function of the wind speed in the source layer. This wind speed was the same in cases (a) through (d) but the power and logarithmic profiles should produce lower wind speeds close to the ground.

7. Summary and conclusions

Solutions to the diffusion equation for the case of the infinite line source have been found for three types of wind speed profile and for the case of diffusion of particulates with an appreciable fall speed. These are sample problems that were chosen to demonstrate the method. In the problems selected, the functions of wind speed and eddy diffusivity with height were specified in analytical form and then translated to discrete steps in height. Since they are used in this form, one could as easily start with functions expressed graphically or in any arbitrary manner. Indeed, using the passive network analogy as a guide, it is possible to design the computing circuit directly from the physical problem. Even the layer spacing used need not follow some mathematical rule; it could be chosen to fit the problem.

The wind speed and eddy diffusivity were assumed constant with distance downwind. Changing them with respect to x would involve changing resistors in time which is not difficult to implement with servo multipliers which are usually found in analog computer installations. In the same way, the boundary conditions could be made a function of x. Thus one could simulate diffusion over undulating ground.

The accuracy obtained here was better than 5 per cent over most of the field. This could be improved to 1 per cent by increasing the number of cells inasmuch as analog computers are available with component quality of 0.01 per cent of full scale. Extension beyond 1 per cent solution accuracy is not practical with this technique, nor is there any theoretical or practical need for greater accuracy than this. The advantages of the analog technique in problems of this type are the speed and convenience with which a solution may be obtained and the ease with which atmospheric parameters may be varied. In addition to its value as a research tool, the analog method should be a valuable teaching aid in the analysis of atmospheric diffusion. Taking advantage of the direct relation between mathematical operations and computer components, an instructor could demonstrate before a class the effects of changing parameters on the solution obtained.

This method may be extended to problems of finite line sources, area sources both horizontal and vertical, and to elevated point sources.

Acknowledgments. The author wishes to acknowledge the many helpful suggestions and encouragement given by Professor E. Wendell Hewson. The wholehearted cooperation and assistance of Mrs. Anne C. Rivette is also appreciated.

APPENDIX

The following tables list the equation coefficients that were applied to the computer in the form of coefficient potentiometer settings. The format for the coefficients is obtained by reducing the general finite difference differential equation to

$$\left.\frac{dS}{dX}\right|_n = a_n S(n+1) - b_n S(n) + c_n S(n-1).$$

The following numerical values used in conjunction with Figs. 3 and 4 should enable duplication of the work reported in this paper.

TABLE A-1. Constant wind profile. The column headed IC indicates the initial condition voltage that should be applied to the appropriate integrator to set the source at various heights.

n	a	b	c	IC
1	0.8571	0.8571	0	1.000
2	0.6780	1.444	0.7661	0.8939
3	0.5226	1.089	0.5665	0.7864
4	0.3887	0.8422	0.4545	—
5	0.2735	0.6011	0.3276	—
6	0.1782	0.4015	0.2233	—
7	0.1034	0.2416	0.1382	—
8	0.0483	0.1218	0.0735	—
9	0	0.0413	0.0282	—

TABLE A-2. Constant wind profile with ground absorption. Same as Table A-1 except $b1 = 0.9521$ for $P = 1$.

TABLE A-3. Constant wind profile with gravitational settling.

n	a	b	c
1	0.8996	0.8091	0
2	0.7205	1.444	0.7236
3	0.5595	1.089	0.5292
4	0.4202	0.8422	0.4220
5	0.3009	0.6011	0.3002
6	0.2006	0.4015	0.2007
7	0.1208	0.2416	0.1208
8	0.0607	0.1218	0.0611
9	0	0.0413	0.0210

TABLE A-4. Constant wind profile with gravitational setting and ground absorption.

Same as Table A-3 except

$b1 = 0.8996$ for $P = 1$
$b2 = 0.9901$ for $P = 2$.

TABLE A-5. Seventh root power low wind profile.

n	a	b	c
1	0.1956	0.1956	0
2	0.2499	0.3982	0.1483
3	0.2663	0.4690	0.2027
4	0.2534	0.4721	0.2187
5	0.2213	0.4255	0.2042
6	0.1759	0.3494	0.1735
7	0.1240	0.2551	0.1311
8	0.0714	0.1549	0.0836
9	0	0.0650	0.0398

TABLE A-6. Logarithmic wind profile.

n	a	b	c
1	0.1796	0.1796	0
2	0.2337	0.3584	0.1470
3	0.2598	0.4450	0.1853
4	0.2575	0.4684	0.2109
5	0.2337	0.4402	0.2064
6	0.1926	0.3753	0.1827
7	0.1412	0.2847	0.1435
8	0.0848	0.1812	0.0965
9	0	0.0790	0.0474

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