On the Condensation of Buoyant, Moist, Bent-Over Plumes

T. M. L. Wigley and P. R. Slawson

Dept. of Mechanical Engineering, University of Waterloo, Ontario, Canada

(Manuscript received 18 August 1970, in revised form 17 November 1970)

ABSTRACT

The theory of growth of dry plumes is extended to include the effects of moisture in both the vapor and liquid forms. A relation determining the point at which a moist plume first condenses is derived. Numerical solutions of this relation indicate that, under most atmospheric conditions, condensation either occurs very close to the stack or not at all. A method for predicting whether or not condensation will occur for arbitrary conditions of atmospheric relative humidity and initial temperature excess and relative humidity of the plume is given. The special case where the plume is initially saturated, corresponding to cooling tower effluents, is considered in more detail.

1. Introduction

The understanding of moist plume behavior is becoming increasingly important as more use is made by industry of stack gas scrubbing techniques for the removal of particulate matter and water-soluble gases. The experience of the Battersea power station in England, where such techniques are thought to have been responsible for increases in pollutant concentrations close to the stacks, has exposed a number of problems in this type of pollution abatement program. Waste heat disposal through cooling tower plume releases is a related problem that will become of paramount importance as the availability of natural heat sinks, such as large lakes, is diminished by industries' awareness of the dangers of thermal pollution.

Most discussions of plume rise are concerned only with "dry" plumes; i.e., plumes which contain no water vapor, or, more strictly, plumes in which there is no transfer of water vapor between plume and environment. Morton (1957) and, more recently, Csanady (1971) have considered the dynamics of "wet" plumes and the present paper is an extension of their work. In this discussion a "moist" plume is one in which there is significant moisture content and where one may refer to the uncondensed and condensed sections of the plume separately. The term wet plume will be reserved for the condensed section, the section containing some liquid water content.

The transition to the wet phase is a most striking feature of the behavior of moist plumes. After condensation, if the liquid water content is sufficiently great, the release or absorption of latent heat, through condensation or evaporation, and the force of gravity acting on the droplets themselves may significantly affect the dynamics of the plume. It is also possible that the liquid water content may become sufficiently large for droplets to precipitate or "rain out" of the plume and so distribute pollutants in excessive concentrations at ground level close to the source. Either of these mechanisms could produce ground-level distributions of pollutants very different from those of dry plumes and it thus becomes desirable to know as much about moist plume behavior as is presently known about dry plume behavior.

The equations which determine how bent-over moist plumes behave, similar to those given by Morton (1957) using entrainment theory, are given in the next section. Two more variables, the specific humidity and the liquid water content of the plume must be introduced and their introduction necessitates the use of two additional equations: one expressing the conservation of water vapor and liquid water, and the other relating the moisture conditions of the plume to other plume variables. In dry plume theory, plume behavior is sensitive to the stability of the atmosphere (Moore, 1966; Slawson, 1967). This environmental parameter is best incorporated through the Väisälä frequency N, which is related to the vertical gradient of potential temperature through

$$N^2 = -\frac{g}{T_a}\frac{\partial\theta}{\partial z}.$$

Here g is the gravitational acceleration, T_a and θ are the environment temperature and potential temperature, respectively, and z is the vertical coordinate. For moist plumes another atmospheric parameter, G, the lapse rate of specific humidity, also appears in the theory.

One of the first questions to be asked is: Where and under what conditions will a moist plume condense? The present paper is concerned primarily with answering this question. Numerical values for the determining parameters, the initial plume conditions of temperature and moisture content, and the corresponding environment conditions are given in graphical form (Fig. 4). From this, knowing the initial relative humidity of the plume, the relative humidity of the atmosphere, and the initial temperature difference between plume and environment, one can predict whether or not a moist plume will condense. The results are independent of the dynamics of the plume. (Fig. 4 has been constructed assuming $T_a=10$C, but does not alter significantly for environment temperatures up to 20C from this value.)

In the "initial" phase of plume development [using the terminology of Slawson and Csanady (1967)] it is shown that condensation can only occur either very close to the stack or not at all. Although some quantitative results for the distance downwind from the stack that condensation occurs are also given (Figs. 2 and 3), the qualitative result is of much greater significance since it holds even though the basic equations governing plume rise become less valid close to the stack. As a consequence, only the initial-phase theory is necessary for many moist plume problems. Also, satisfactory results can be expected from a model in which vertical variations in environment specific humidity and potential temperature are neglected. Finally, a knowledge of the conditions conducive to plume condensation enables us to modify initial plume parameters in a direction to eliminate or reduce condensation and minimize any adverse effects of condensation.

2. Plume-rise theory

Using a plume model in which plume properties are assumed constant over the plume cross section ("top hat" profiles of temperature, density, etc.), the mean behavior of buoyant dry plumes can be adequately described by solving the equations of conservation of plume mass, momentum and energy. This approach was first used by Morton *et al.* (1956), applied to bent-over plumes in neutral conditions by Slawson and Csanady (1967), and to stable and unstable atmospheres by Slawson (1967) and other authors. The model has been used by Morton (1957) to discuss moist plumes in neutral and stable environments. Recently, Csanady (1971) has extended Morton's work to bent-over plumes under more general atmospheric conditions.

The plume is described by its radius R, its vertical velocity component w, its vapor and liquid water content, q and σ, respectively, and by the buoyant acceleration b $[=g(\rho_a-\rho)/\rho]$. Following Morton (1957) and Csanady (1971), the corresponding flux variables will be used:

$$M=UR^2w,\quad F=UR^2b,\quad S=UR^2g\sigma,\quad H=UR^2(q-q_a),$$

where U is the mean horizontal wind speed (supposed constant) and subscript a is used for environment variables to distinguish them from those of the plume. The equations of conservation of mass, momentum and energy become

$$U\frac{dR^3}{dx}=3\alpha\frac{M}{U},\tag{1}$$

$$U\frac{dM}{dx}=F-S,\tag{2}$$

$$U\frac{d}{dx}(F-\lambda S)=-MN^2.\tag{3}$$

In (2) the term S arises from the gravitational attraction on condensed water droplets and in (3) the term $d(\lambda S)/dx$ comes from the release (or absorption) of latent heat during condensation (or evaporation) of liquid water droplets. These two terms distinguish Eqs. (1), (2) and (3) from the corresponding dry plume equations. Other parameters in (1)–(3) include: the approximate constant λ $[=L/(C_pT_a)$, where L is the latent heat of vaporization and C_p the specific heat capacity of air at constant pressure]; N^2, the square of the Väisälä frequency reflecting changes in atmospheric stability; and an entrainment parameter α. Eq. (1) applies only in the initial phase of plume growth when the plume's own turbulence is responsible for mixing with the environment (see, e.g., Slawson and Csanady, 1967); in subsequent phases the right-hand side must be modified. Here we are concerned only with the initial-phase behavior.

For moist plumes, conservation of water vapor and liquid water must be added to the theory. This can be expressed in the form given by Csanady (1971), i.e.,

$$U\frac{d}{dx}\left(H+\frac{S}{g}\right)=-MG,\tag{4}$$

where $G=\partial q_a/\partial z$ is the vertical gradient of specific humidity in the atmosphere. To obtain a consistent set of equations a fifth relation between S or H and the other variables is needed. For a moist plume prior to condensation this is simply

$$S=0.$$

After condensation, provided the density of condensation nuclei is sufficiently great to ensure that the plume does not become supersaturated, it can be assumed that $q=q_s$, the saturation specific humidity, and the thermodynamic relation obtained by integrating the Clausius-Clapeyron equation, can be used; *viz*,

$$q_s=q_{sa}\exp[\beta(T-T_a)/T],\tag{5}$$

where $\beta=L/(R_vT_a)$, R_v is the gas constant for water vapor, and q_{sa} the environment saturation specific humidity.

Prior to condensation we have, in general,

$$U\frac{dR^3}{dx}=3\alpha\frac{M}{U}, \tag{6}$$

$$U\frac{dM}{dx}=F, \tag{7}$$

$$\frac{dF}{dx}=-\frac{MN^2}{U}, \tag{8}$$

$$\frac{dH}{dx}=-\frac{MG}{U}. \tag{9}$$

In a neutral atmosphere Eq. (8) becomes $dF/dx=0$, with solution $F=F_0=$constant. Eqs. (6) and (7) can then be solved to give

$$R=\left(\frac{3\alpha l}{2}\right)^{\frac{1}{3}}x^{\frac{2}{3}}, \tag{10}$$

exhibiting the well-known $\frac{2}{3}$-power dependence on x, the downwind distance from the virtual point source. Here $l=F_0/U^3$ is a length scale of buoyant movements. If R_0 is the stack radius, (10) can be written as

$$R=R_0(x/x_0)^{\frac{2}{3}}, \tag{11}$$

where x_0 is the distance from the virtual source to the stack which is given by

$$x_0{}^2=2R_0{}^3/(3\alpha l). \tag{12}$$

Since $\alpha\approx 0.35$ and l is of order 1 to 20 ft, x_0 must be small and of order R_0.

3. Condensation point

It is convenient to introduce a "distance" variable X defined by

$$R=R_0X^{-\frac{1}{2}}. \tag{13}$$

The plume is defined, therefore, for $0\leqslant X\leqslant 1$. For a moist uncondensed plume in the initial phase in a neutral atmosphere, X and x are explicitly related by (11) which can be written as

$$X=\left(\frac{x}{x_0}\right)^{-\frac{4}{3}}. \tag{14}$$

Many of the results of this paper are given as functions of X. They can be determined explicitly in terms of the distance downwind from the stack in any particular case using (12) and (14). However, much of the following is independent of the precise functional form of the relationship between X and x.

The solution $F=F_0$ can be written as

$$T=T_a+(T_0-T_a)X, \tag{15}$$

where T_0 is the initial temperature of the plume. [Strictly, since F is a function of *moist* air densities, one should interpret these temperatures as virtual temperatures. This is not essential, though, as (16) may be used to recover, as an excellent approximation, the non-virtual temperature form]. Since Eq. (15) is linear in X, using X as an independent variable is therefore equivalent to using the temperature excess $\Delta T=T-T_a$ as an independent variable as Csanady (1971) has done. However, here the explicit dependence on x can be easily obtained through (14).

If it is now assumed that $G=0$, fulfilling the conditions of a well-mixed environment, Eq. (9) becomes $dH/dx=0$, the solution of which is

$$q=q_a+(q_0-q_a)X, \tag{16}$$

where q_0 is the initial specific humidity of the plume. Thus, $q-q_a$ depends linearly on ΔT. This result holds independent of the form of the relation between x and X and so is true during any phase of uncondensed plume development. Since the assumption that a one-to-one correspondence exists between the entrainment of heat and moisture into the plume is inherent in the theory, the linear relation between $q-q_a$ and ΔT is not surprising. Of course, when the plume condenses and condensation, or evaporation, introduces sources or sinks of heat and moisture within the plume, such a relation can no longer hold. Also, except under very special circumstances, it cannot hold if either N^2 or G is non-zero.

For an uncondensed moist plume the moisture equation (9) is only weakly coupled with the other equations, (6)–(8). An uncondensed plume and a dry plume should behave very similarly. Since this is no longer the case after the plume starts to condense, it is important to determine explicitly if and where condensation first occurs. As a moist plume develops, both its specific humidity and saturation specific humidity change. The former varies linearly with X while the latter depends on the plume temperature, also a linear function of X, through the integrated Clausius-Clapeyron equation (5). Combining this with the solution (15) gives

$$q_s=q_{sa}\exp(\beta\Delta T_0X/T), \tag{17}$$

where ΔT_0 is the value of ΔT at the stack ($\Delta T_0=T_0-T_a$), or

$$q_s\approx q_{sa}(q_{s0}/q_{sa})^X, \tag{18}$$

where q_{s0} is the initial saturation specific humidity of the plume. At the point where the plume first condenses ($X=X_c$), q becomes equal to q_s and (16) and (18) yield

$$q_a+(q_0-q_a)X_c=q_{sa}(q_{s0}/q_{sa})^{X_c}. \tag{19}$$

In the initial phase, then, an explicit solution can be found by solving this equation and using (14) and (12) to find x_c.

An example of the solution is given in Fig. 1. This is equivalent to a similar figure in Csanady (1971) except that the abscissae are given both in units of X and of the

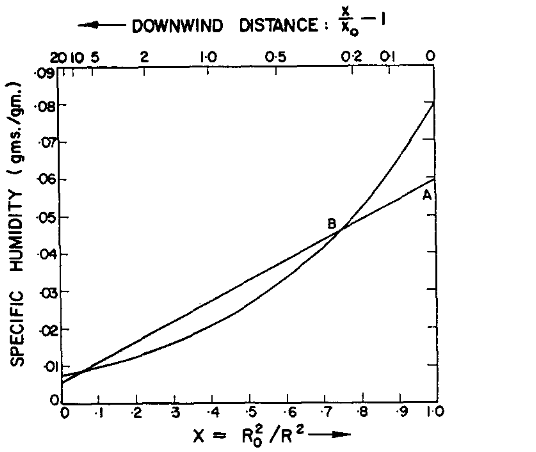

FIG. 1. Determination of the condensation point for $T_a=10$C, $\Delta T_0=40$C, $r_a=0.75$, $r_0=0.75$. From an initial value at point A the specific humidity of the plume follows the straight line until it intersects the saturation curve and condenses at point B.

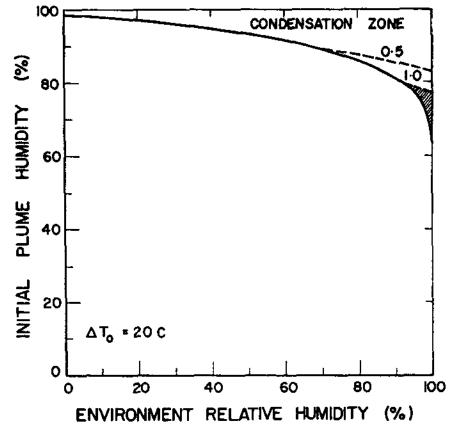

FIG. 2. Variation of condensation point with environment and initial plume conditions for $\Delta T_0=20$C. Distances are measured downwind from the stack in units of x_0 as shown along the dotted lines.

dimensionless length downwind from the stack, $(x/x_0)-1$, rather than ΔT. In the range $0\leqslant X\leqslant 1$ [i.e., $\infty\geqslant(x/x_0)-1\geqslant 0$] it can be seen that there are either zero, one or two solutions to (19), depending on the values of q_0, q_{s0}, q_a and q_{sa}. In the illustrated example, both the condensation point and the re-evaporation point lie close (of order R_0 units downwind) to the stack. This is not obvious if ΔT is used as independent variable. It is a most important qualitative result.

It is certainly impractical to solve Eq. (19) graphically for all possible boundary conditions. One must therefore resort to other methods in order to determine whether the conclusions drawn from Fig. 1 are generally valid, and also to determine exactly under what conditions condensation of a moist plume can occur. Eq. (19) was solved by iteration on a computer[1], and the results are shown in Figs. 2–4. In terms of r_0[the initial (fractional) relative humidity RH of the plume ($r_0=q_0/q_{s0}\approx$RH/100)] and r_a [the (fractional) relative humidity of the atmosphere], (19) becomes

$$X_c=\frac{1}{Z_0}\ln[r_a(1-X_c)+X_c r_0\exp(Z_0)], \qquad (20)$$

where

$$Z_0=\beta(T_0-T_a)/T_0=\beta\Delta T_0/(T_a+\Delta T_0). \qquad (21)$$

Condensation is determined by only three variables, r_0, r_a and ΔT_0, since variations in T_a do not significantly alter Z_0.

Figs. 2 and 3 show how $(x_c/x_0)-1$ varies with r_0 and r_a for two values of ΔT_0, 20C and 40C. The r_0, r_a space is divided into two regions according to whether condensation occurs or not, the dividing line corresponding to the conditions under which the two curves in Fig. 1 intersect tangentially. As ΔT_0 increases so the area which admits condensation increases. More importantly, $(x_c/x_0)-1$ is nearly always small in the condensation region. Since x_0 is small (of order R_0) this means that under nearly all plume and environment conditions, condensation may be supposed to occur either very close to the stack or not at all. This agrees with our tentative conclusion from Fig. 1 and is an observed feature of moist plume behavior. (It must be noted, however, that this conclusion is only qualitatively correct. Since the basic theory used here neglects the initial size of the plume in supposing a point source to exist, and the initial momentum of the gases, quantitative results close to the stack can only be treated with some reservation. In any event, the qualitative result that condensation always occurs close to the stack is one which negates the need for more precise information on the location of the condensation point.)

Because of this result the determination of the conditions under which condensation can occur, for any value of ΔT_0, is even more fundamental to moist plume theory. This could be achieved by trial and error using the solution to Eq. (20). Alternatively we can use the fact

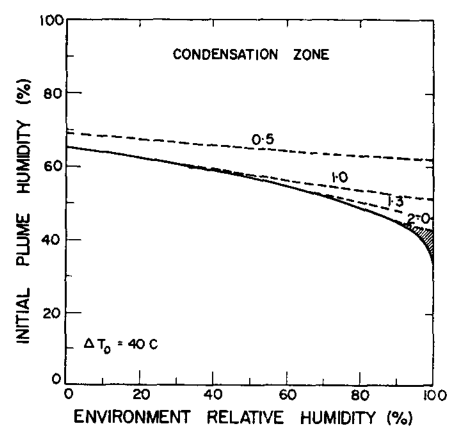

FIG. 3. Same as Fig. 2 except for $\Delta T_0=40$C.

[1] See Appendix A.

that the limiting value of r_0 (i.e., r_c) for specified Z_0 and r_a is given by

$$r_c=(Y_0Z_0+r_a)\exp(-Z_0), \tag{22}$$

where Y_0 is the solution[2] of

$$Y_0=r_a/(1-\ln Y_0). \tag{23}$$

Condensation will never occur if $r_0<r_c$. Eqs. (22) and (23) are thermodynamic results and independent of the actual dynamics of the plume. Their validity is independent of the position of the condensation point and as such is not subject to the limitations noted above. Eq. (23) can easily be solved by iteration on a computer and the results are illustrated in Fig. 4.

Fig. 4 has considerable engineering significance since it can be used as a criterion for the prevention of condensation of moist plumes. The plume parameters ΔT_0 and r_0 are, in principle, controllable and can be varied with the aid of Fig. 4 to ensure that condensation does not occur. As an example, consider the data illustrated in Fig. 1. The atmospheric relative humidity is 75% and the initial plume parameters are excess temperature 40C, relative humidity 75%. This is well into the condensation zone which lies above the 40C isopleth in Fig. 4. It can be seen that a reduction of almost 30% in plume humidity, to something less than 46%, would be necessary to prevent condensation of the plume. Referring to Fig. 3 shows that the condensation point is very close to the stack, only 0.3 x_0 units downwind, in agreement with Fig. 1.

An instance of particular importance involves the application of the theory to cooling towers. Here the effluent is generally saturated on exit so that $r_0=1.0$. Fig. 4 can be used to determine a critical value of ΔT_0, below which the effluent will not condense. The value depends on the relative humidity of the environment, but can never exceed 16C. A more precise estimate of the critical value of ΔT_0 can be obtained from (23) and (22) with $r_c=1.0$ to give

$$Z_0=1-r_a\exp(-Z_0). \tag{24}$$

The solution is shown graphically in Fig. 5. Csanady (1971) has solved an equivalent relation in a particular case and obtains a result consistent with that obtained from Fig. 5. An approximate solution is given by

$$\Delta T_0=15.6\left(\frac{6-5r_a}{6-3r_a}\right),$$

which gives excellent results particularly for $r_a<0.9$. Fig. 5 shows that condensation will occur under any conditions if $\Delta T_0>16$C. In accord with previous results it should be remembered that condensation, if it does occur, will always do so close to the cooling tower mouth.

[2] See Appendix B.

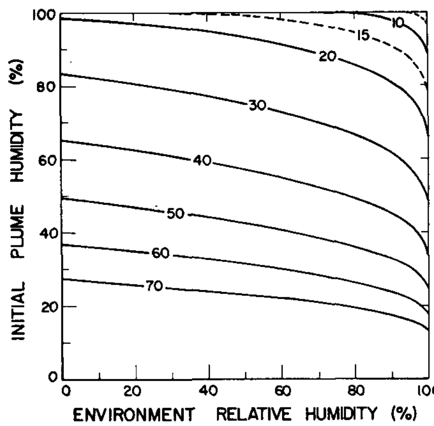

FIG. 4. Critical conditions for various values of ΔT_0(°C). For given ΔT_0, condensation will occur if the environment and plume humidity give a point above that particular ΔT_0 isopleth.

4. Conclusions

The main results of this paper are the evaluation of explicit conditions under which condensation of a moist plume will occur and the proof that such condensation always occurs close to the stack. Apart from their engineering significance, these results serve as *a posteriori* justification for the use of two simplifications in moist plume theory. The first of these is that the initial phase assumption may be used to excellent effect since condensation of a moist plume will generally occur before the subsequent phases of plume development are encountered. Regardless, it should be noted that the qualitative results of this paper and those results presented in Figs. 4 and 5 are, in fact, independent of phase.

Second, although the results of this paper apply strictly only to a well-mixed environment, they can be used under more general environment conditions. Condensation occurs so close to the stack that the effects of vertical variations in q_a and θ for practical applications are very likely to be negligible in the uncondensed part of the plume. Even after condensation, since the re-evaporation point is frequently very close to the

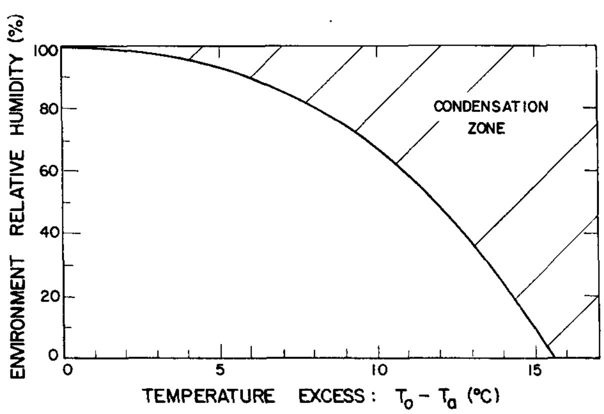

FIG. 5. Critical conditions for cooling tower effluents where $r_0=1$.

stack (see Fig. 1), the approximation of a well-mixed environment should yield satisfactory results.

Since the maximum liquid water content, $\sigma_{\max}$, determines whether or not rain-out will occur, it is a most important characteristic of the condensed phase of a moist plume. To calculate $\sigma_{\max}$ the complete equations (1)–(3), together with (5), must be solved, although the approximation $N^2=G=0$ may be used. Csanady (1971) has estimated $\sigma_{\max}$ in a particular case and found it to be small. The more general solution is not difficult and preliminary calculations reveal that $\sigma_{\max}$ may be somewhat higher, of order q_{sa}, and that its point of occurrence, and the "source" of subsequent rain-out, is usually close to the stack. One would expect some critical value of $\sigma_{\max}$ to exist above which rain-out would occur. This, in turn, will depend on details of droplet size spectra, number density and dynamics which are beyond the scope of the present work and will be the subject of a subsequent paper.

Acknowledgments. This work was supported in part by the U. S. Atomic Energy Commission under Contract NYO–3685, and the National Research Council of Canada under Grant A-7107.

APPENDIX A

Iterative Procedure for Solution of Eq. (19)

Eq. (19) was solved by iteration using an IBM 360/75 computer. The method of solution is not completely straightforward since the equation may either have no solution, or may have a solution which lies outside the physically meaningful range $0\leqslant X_c\leqslant 1$. Also, if the equation does have a solution in the range $0\leqslant X_c\leqslant 1$, then it will have, in general, two solutions in this range corresponding to the condensation and re-evaporation points. It is necessary, therefore, to eliminate non-convergent sequences in the iteration and to ensure that convergent sequences tend to the correct of the two possible solutions.

Eq. (19) has been written in a form which is suitable for iteration [i.e., Eq. (20)], using

$$X_c(n+1)=\frac{1}{Z_0}\{\ln[r_a(1-X_c(n))+r_0X_c(n)\exp(Z_0)]\}. \quad \text{(A1)}$$

It can now be proved by induction that, provided $X_c\leqslant 1$ and $X_c\leqslant X_c(1)\leqslant 1$, then $X_c\leqslant X_c(n)\leqslant 1$ for all n. Hence, by choosing $X_c(1)=1$ we can ensure that the iteration will always converge to the required solution if a solution exists such that $X_c\leqslant 1$. If $X_c(n)>1$ or <0 during the iteration, we know that there can be no solution in the range $0\leqslant X_c\leqslant 1$ and so terminate the iteration.

Eqs. (A1) and (19) are based on the approximation (18) which affords considerable algebraic simplicity. Moreover, Eq. (18) is based on a further application of the Boussinesq approximation in replacing T by T_0 in the exact expression (17) and so is consistent within the framework of the theory. However, its use at this stage is equivalent to using $b=g(\rho_a-\rho)/\rho_a$ as the buoyancy acceleration, a procedure which is known to yield less accurate results in dry-plume theory. As a consequence the more precise expression

$$X_c=\frac{\beta+Z_0(X_c-1)}{\beta Z_0}\ln[r_a(1-X_c)+X_cr_0\exp(Z_0)] \quad \text{(A3)}$$

was used in the numerical calculations. The iterative procedure in solving this equation is unchanged [X_c is replaced by $X_c(n+1)$ on the left-hand side and by $X_c(n)$ on the right-hand side of the equation], and the arguments presented above in relation to the approximate equation are easily generalized to cover the more complex case.

To reduce the volume of computation, two simple and easily proved inequalities were also used, *viz.*,

$$r_0\exp(Z_0)>1,$$
$$r_0\exp(Z_0)-r_a>Z_0.$$

If these conditions do not hold it can be shown that condensation can never occur.

APPENDIX B

Derivation of Expressions Based on Eq. (17)

The results (22) and (23) which determine the limiting conditions for condensation of a moist plume are based on the approximate expression (18). In the computations made in constructing Fig. 4, more detailed expressions derived from the exact equation (17) were used. Their derivation is given here for completeness.

We define σ_0 by

$$\sigma_0=q_a+(q_0-q_a)X-q_{sa}\exp(\beta\Delta T_0X/T),$$

so that σ_0 is the difference between the straight line and the saturation curve in Fig. 1. Note that σ_0 is *not* the liquid water content σ; although the two are related σ is always considerably less than σ_0. The condensation point is found by solving $\sigma_0=0$; beyond this point σ_0 increases to a maximum at a point where $\partial\sigma_0/\partial X=0$. If σ_{0m} is the maximum value then it is easy to show that

$$\sigma_{0m}=q_a+q_{sa}Y(\ln Y-1), \quad \text{(B1)}$$

where

$$\left.\begin{aligned} Y&=\frac{q_0-q_a}{q_{sa}Z}=\frac{r_0\exp(Z_0)-r_a}{Z}\\ Z&=\frac{\beta\Delta T_0}{T}\end{aligned}\right\}. \quad \text{(B2)}$$

The limiting condition for condensation is $\sigma_{0m}=0$, which, using (B1), yields

$$r_a = Y(1-\ln Y). \tag{B3}$$

If the approximation $T=T_0$, used to obtain (18) from (17), is used, then

$$Z=\frac{\beta\Delta T_0}{T_0}=Z_0,$$

and hence

$$Y=\frac{r_0\exp(Z_0)-r_a}{Z_0}=Y_0. \tag{B4}$$

Eqs. (B3) and (B4) are equivalent to (23) and (22).

It is not necessary to use this approximation. Using (15), it can be shown that

$$Z_0=Z\left[1+(X-1)\frac{Z_0}{\beta}\right],$$

and, since from (B2) and (B4)

$$Z_0Y_0=ZY,$$

we have

$$Y=Y_0\left[1+(X-1)\frac{Z_0}{\beta}\right].$$

At the point where σ_0 is a maximum (i.e., $\partial\sigma_0/\partial X=0$)

$$X=(1/Z)\ln Y,$$

so that

$$Y=Y_0\left[1+\left(\frac{1}{Z}\ln Y-1\right)\frac{Z_0}{\beta}\right] \tag{B5}$$

at this point. If $\sigma_{0m}=0$, then (B3) may be substituted into the right-hand side of (B5) to yield the following relation between Y_0 and Y:

$$Y(\beta-1)=Y_0(\beta-Z_0)-r_a. \tag{B6}$$

For given conditions of r_a and ΔT_0, Eqs. (B3), (B4) and (B6) may be solved to give the critical value for the initial plume relative humidity r_c. Eq. (B3) must first be solved for Y; Y_0 can then be obtained from (B6) and the critical value for r_0 found by solving (B4). The additional complexity that this exact treatment introduces is in the solution of Eq. (B6). With the approximation $T=T_0$, (B6) is simply replaced by identity (B4), i.e., $Y=Y_0$.

There is a noticeable difference between the approximate and exact results. For example, if $\Delta T_0=20$C and $r_a=1.0$, we find $r_c=0.665$; solving Eqs. (22) and (23) yields an approximate solution of $r_c=0.640$. For $\Delta T_0=40$C and $r_a=0.5$, the corresponding values for r_c are 0.570 (exact) and 0.524 (approximate.)

It should be noted that Eq. (24) is unchanged if the exact formulation is used. This is to be expected since if condensation of a saturated cooling tower effluent is to occur then it must do so at $X=1$ where the "approximation" $T=T_0$ is exact.

REFERENCES

Csanady, G. T., 1971: Bent-over vapor plumes. *J. Appl. Meteor.*, **10**, 36–42.

Moore, D. J., 1966: Physical aspects of plume models. *Intern. J. Air Water Pollution*, **10**, 411–417.

Morton, B. R., 1957: Buoyant plumes in a moist atmosphere. *J. Fluid Mech.*, **2**, 127–144.

——, G. I. Taylor and J. S. Turner, 1956: Turbulent gravitational convection from maintained and instantaneous sources. *Proc. Roy. Soc. London*, **A23**, 1–23.

Slawson, P. R., 1967: Observations of the mean path of buoyant bent-over plumes from large industrial stacks. Ph.D dissertation, University of Waterloo.

——, and G. T. Csanady, 1967: On the mean path of buoyant, bent-over chimney plumes. *J. Fluid Mech.*, **28**, 311–322.