X-Band Attenuation and Liquid Water Content Estimation by a Dual-Wavelength Radar

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ABSTRACT

Direct measurements of attenuation due to liquid water content (LWC) between two points in the common beam of a dual-wavelength radar system will be possible using a real-time digital processor of radar signals presently under construction. Precise measurements of attenuation are possible since accurate absolute calibration of the radars is not required. A resolution of 0.65 dB km^{-1} is possible in contiguous kilometer sections along the beam axis with an average over 32 independent values of echo power. In terms of LWC this is a resolution of $\sim$2 mg m^{-3} of precipitation using a new relation

$$M=2.23A^{0.787},$$

where M is the liquid water content (gm m^{-3}) and A the attenuation rate (dB km^{-1}) along the common beam.

1. Introduction

Cloud liquid water is a major quantity among those that influence the growth of cloud systems. Accordingly, the liquid water content (LWC) must be precisely known for insertion into the theoretical system-modeling equations for realistic growth and development sequences to result. Immersion sensors such as aircraft, rockets, dropsondes, etc., can provide precise "point" measurements, in principle, but such measurements must be extrapolated massively beyond the tracks of such immersion sensors if values of LWC are to be assigned to all points in the system at all times during its history. On the other hand, radar can provide good "large-volume" or "large-scale" values of LWC applicable throughout the system which can then be studied as an entity. However, peak-point values, such as might cause flameout in jet engines, will not be obtainable from radar data.

Recently, there has been a demand from the designers of satellite and point-to-point microwave communication systems for information concerning the gross features of microwave attenuation and attenuation statistics. A dual-wavelength meteorological radar measures attenuation and is ideal for these purposes.

Radar methods, previously used for measuring the attenuation rate A (dB km^{-1}), involved the measurement of returned power, the calculation of the reflectivity factor Z from the radar equation (which required precise calibration of the radar for equally precise measurements of Z), and the subsequent evaluation of the attenuation rate from empirical relations, involving Z and A, or Z, the rainfall rate R, and A. Liquid water content was obtained by a further empirical relation which was generated by a theoretical derivation from an empirical drop-size distribution.

The method to be discussed involves measurement of the ratio of the average returned powers at two wavelengths at the ends of a path, and subsequent calculation of the ratio of these to yield the attenuation.

The attenuation, itself, is really ***measured*** and is not the result of calculation using empirical relations.

Fluctuations in the average returned powers and, therefore, in the ratio of such powers are the major cause of fluctuations in the measured attenuation, provided that the average returned signals are sufficiently above the noise level so that fluctuations due to system noise can be neglected.

Calculation of LWC from attenuation still requires an empirical relation, and the bases are described for a new empirical relation deduced from calculations on real drop-size data (Mueller and Sims, 1966a), measured at the ground but assumed to exist aloft. Thus, no empirical drop-size distribution contaminates this new relation.

2. Calculations ignoring fluctuations

The chief properties of interest of a dual-wavelength radar are that the two radar beams are matched in

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beam width, are collimated, are reasonably narrow, and that the associated equipment analyzes data from identical ranges.

The *average* echo powers P_2 and P_1 returned at 10 cm (long) and 3 cm (short) wavelengths, respectively, from matched beams illuminating a volume at range r (km), are given by

$$P_2=C_2Z/r^2, \quad (1)$$

$$P_1=(C_1Z/r^2)10^{-0.2\int_{r_0}^{r_0+s}Adr}, \quad (2)$$

where C_1 and C_2 are the radar constants and take account of attenuation up to the range r_0 at the front of the storm, s is the distance from r_0 to the observed volume, and Z the equivalent reflectivity factor due to rain. The longer wavelength is chosen so that the attenuation rate A_2 due to rain is negligible, and hail is assumed absent. Taking the common logarithm of the ratio of (1) and (2), we have

$$y=10\log[(P_2/C_2)/(P_1/C_1)], \quad (3)$$

$$=2\int_{r_0}^{r_0+s}Adr, \quad (4)$$

where the unit of length is the kilometer.

In the absence of hail all of the particles will be Rayleigh scatterers so that $Z_{10}/Z_3=1$.

Now at some other range r_0+s+s' inside the storm, we assume hail is absent in the increment s', we measure P_1' and P_2' (average powers, as before), and obtain an integral involving the incremental attenuation rate A'; thus, we have

$$2\int_{r_0}^{r_0+s}Adr+2\int_{r_0+s}^{r_0+s+s'}A'dr =10\log[(P_2'/C_2)/(P_1'/C_1)]. \quad (5)$$

By subtracting (4) from (5), we arrive at the expression

$$2\int_{r_0+s}^{r_0+s+s'}A'dr=10\log(P_2'P_1/P_1'P_2), \quad (6)$$

and both the effect of the attenuation up to the range r_0+s and the radar constants have vanished.

Now there is some average value $\bar{A}$ of the attenuation rate that applies over this range increment s'(km) such that

$$2\bar{A}s'=2\int_{r_0+s}^{r_0+s+s'}A'dr. \quad (7)$$

From (6) and (7), then, the attenuation (dB) over this interval is

$$\bar{A}s'=0.5[10\log(P_2'P_1/P_1'P_2)], \quad (8)$$

$$=0.5(10\log_{10}e)\ln(P_2'P_1/P_1'P_2),$$

$$\approx 2.1715[(P_2'P_1/P_1'P_2)-1]. \quad (9)$$

This latter approximation $[\ln(1+\delta)\approx\delta]$ is valid to within 10% for all liquid water contents (gm m^{-3}) less than or near $1.5/s'$ [assuming Eq. (21)].[2]

If $\bar{A}$ is small, (9) shows that the fluctuations in $\bar{A}$ are identical to the fluctuations in $(P_2'P_1/P_1'P_2)$. There should, of course, be negligible fluctuations in the time-defined parameter s'.

We emphasize that we are concentrating on the smallest value of $\bar{A}$ which a dual-wavelength radar can detect, so that the assumption that $\bar{A}$ is small is a very relevant one.

3. Statistical fluctuations

It is well known that the mean of k independent observations of the echo power P returned from a set of meteorological scatterers has the probability distribution

$$P(J_k)dJ_k=\frac{k^k}{(\bar{P})^k(k-1)!}J_k^{k-1}\exp(-kJ_k/\bar{P})dJ_k, \quad (10)$$

where

$$J_k=\sum_1^k (P_i)/k.$$

A set of these distributions for different values of k is shown in Fig. 3 of Marshall and Hitschfeld (1953). The ratio $P/\bar{P}$ is a normal variate for large k, but is distributed like $\chi_{2k}^2/(2k)$ for *all* k, as was first pointed out by Kessler (1959). This chi-square distribution is skewed for all k, but is effectively normally distributed for large k with a standard deviation of $\sim(1/k)^{0.5}$. It has a mean of unity for all k.

The experimental quantity that is measured is $(P_2'P_1/P_1'P_2)$. Its statistical properties can be understood if we use the fact, shown in the Appendix, that $(P_2'P_1/P_1'P_2)^{0.5}$ is distributed precisely like an F distribution, $(F|_{4k,4k})$, for the case of nearly equal numbers of independent samples k in each power measurement. Now, in most experiments the time to independence (a function of wind shear and relative radial velocities of precipitation particles) for a standard X-

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[2] Eq. (8) can be written

$$\bar{A}s'=2.1715\ln\left(\frac{P_2'P_1}{\bar{P}_2'\bar{P}_1}\Big/\frac{P_1'P_2}{\bar{P}_1'\bar{P}_2}\right)\left(\frac{\bar{P}_2'\bar{P}_1}{\bar{P}_1'\bar{P}_2}\right)$$

$$=2.1715\ln\left(\frac{\bar{P}_2'\bar{P}_1}{\bar{P}_1'\bar{P}_2}\right)+2.1715\ln\left(\frac{P_2'P_1}{\bar{P}_2'\bar{P}_1}\Big/\frac{P_1'P_2}{\bar{P}_1'\bar{P}_2}\right),$$

where the overbar signifies a population mean, commonly called an infinite average. The first term of this latter expression yields the median value of $\bar{A}s'$ and the second involves the fluctuations of $\bar{A}s'$ about this median. The Appendix gives a rigorous treatment of the statistical fluctuations of the operand in this second term. Therefore, the fluctuations in $\bar{A}s'$ are rigorously known, independent of the approximation (9).

band radar is known to be a factor of 3 shorter than that for an S-band radar. And this difference in independence times, which Reid, (1970) shows can exist only at slow scanning rates and at large ranges for coherent radars, will cause a factor of 3 difference in the number of independent samples. However, Krehbiel and Brook (1968) describe a radar system such that the effective number of samples contributing to an instantaneous (from a single transmitter pulse) measurement of power is very much greater than unity. Further, it is a simple matter to equip a radar with "frequency hopping" capability such that a different frequency is radiated with each transmitter pulse. Thus, this assumption of nearly equal numbers of samples in each measurement of power is realistic, since it is achievable in two types of radar systems. Its great advantage is that it allows a simple tutorial-type analytic derivation of the distribution to be expected from the quotient $(P_2'P_1/P_1'P_2)^{0.5}$ and, therefore, of the distribution of its square.

The detailed statistical properties of $(P_2'P_1/P_1'P_2)$, which are needed for an associated experiment, can be inferred from the known properties of its square root, as is shown in the Appendix. Some approximate values of its statistical parameters, which are adequate for this paper, are listed below:

$$\text{variance} \approx 4/(k-1.8) \approx 4/k, \tag{11}$$

$$\text{standard deviation} \approx 2/(k-1.8)^{0.5} \approx 2/(k)^{0.5}, \tag{12}$$

and its median is precisely unity.

From (9) and (11) we can relate the variance $\sigma^2{}_A$ of $\bar{A}$ to the variance $\sigma^2{}_{P^2/P^2}$ of $(P_2'P_1/P_1'P_2)$; thus, we have

$$\sigma^2{}_A = \left(\frac{2.1715}{s'}\right)^2 \sigma^2{}_{P^2/P^2}, \tag{13}$$

$$= \left(\frac{2.1715}{s'}\right)^2 \frac{4}{k}. \tag{14}$$

4. Minimum measurable values of the attenuation coefficient

a. Single sets of measurements

There is a statistic called the effective degrees of freedom (edf) which is a simple measure of the significance that can be given to an estimate of a quantity if both the mean and the variance of the estimate are known. The edf is defined as:

$$\text{edf} = 2(\text{mean})^2/\text{variance}. \tag{15}$$

It is usually required that the edf of an estimate be approximately equal to 10, in which case the standard deviation is somewhat less than $\frac{1}{2}$ of the mean value for that quantity.

Let us define $\bar{A}_{\min}$ such that it has an edf of 10. Then

$$10 = 2(\bar{A}_{\min})^2/\sigma^2{}_A, \tag{16}$$

and

$$\bar{A}_{\min} = \sqrt{5}\sigma_A = 9.72/(s'k)^{0.5}\,[\text{dB km}^{-1}]. \tag{17}$$

Alternatively, we could have given the identical treatment to the product $\bar{A}s'$, formed $(\bar{A}s)_{\min}$, and shown that $(\bar{A}s)_{\min}$ is invariant if k is fixed; in this case, we would have

$$(\bar{A}s)_{\min} = 9.72/k^{0.5}\,[\text{dB}]. \tag{18}$$

In other words, the error (standard deviation, s.d.) in either $\bar{A}$ or the product $\bar{A}s'$ is independent of $\bar{A}$ for reasonable liquid water contents, $M \lesssim 1.5/s'$ [gm m^{-3}].

Now we are in a position to calculate the minimum attenuation rate to which a dual-wavelength radar is sensitive in a single set of measurements, the latter defined as P_1, P_2 and P_1', P_2' at an incremental range s'. Assuming $s'=1$ km and $k=32$ independent observations, we get

$$\bar{A}_{\min} = 1.7 \text{ dB km}^{-1}.$$

If we use the relationship $A=0.013R^{1.15}$ (Wexler and Atlas, 1963), we find that

$$R_{\min} = 70 \text{ mm hr}^{-1}.$$

This is the minimum rainfall to which the dual-wavelength radar is sensitive under these conditions, *in a single set of measurements*.

b. Averaging in range

Averaging in range can substantially improve the measurement accuracy and we show that such averaging is legitimate because there are independent values of $\bar{A}$ available from each range. The result $\bar{A}$, calculated from P_1, P_2 and P_1', P_2' at an incremental distance s' apart, can be considered as the value of $\bar{A}$ applicable to the range $r_0+s+(s'/2)$.

A completely independent value of $\bar{A}$, called $\bar{A}'$, is obtainable from measurements at $r_0+s+(h/2)$ and $r_0+s+s'+(h/2)$, where h is the radar pulse length. This new value of $\bar{A}'$ is applicable to the range $r_0+s+(s'/2)+(h/2)$, or, in other words, to the very next range gate. Therefore, even though $\bar{A}$ and $\bar{A}'$ are averages over grossly overlapping regions in space, they are completely independent estimates of $\bar{A}$ because the values of received power which are used to compute them are completely independent.

Thus, it is possible to average a set of u contiguous values of $\bar{A}$ and emerge with an estimate whose variance is decreased by a factor of $u-1$ and whose standard deviation is less by a factor of $\sqrt{u-1}$. This new minimum value $(\bar{A}_{\text{contig}})_{\min}$ can be written as

$$(\bar{A}_{\text{contig}})_{\min} = 9.72/s'[k(u-1)]^{0.5} \approx 0.65 \text{ dB km}^{-1},$$

if $s'=1$ km, $u=8$ gates km^{-1} and $k=32$ independent observations.

From the relation $M=2.23A^{0.787}$ (given later in this paper) for precipitation, we obtain

$$(M_{\text{contig}})_{\min} \approx 1.6 \text{ gm m}^{-3}$$

for LWC due to precipitation.

c. Averaging in space

We now consider that the radar beam illuminates a "cube" made up of u volume elements or bins per side, and assume that meteorologists are interested in the attenuation or LWC for this entire cube. For the radar case this will be, in fact, a volume of u beamwidths in azimuth, u in elevation, and u range bins deep, but will not be a geometrical cube. Since the estimate of $\bar{A}$ in each of the u^3 elements in this volume are independent, one obtains a value of $(\bar{A}_{u^3})_{\min}$ such that

$$(\bar{A}_{u^3})_{\min} = 9.72/(u-1)[(u-1)k]^{0.5}, \approx 9.72/u^{1.5}k^{0.5} \text{ [dB km}^{-1}]. \quad (19)$$

d. Averaging over pie-slice volume

Naturally, the radar cannot be restricted to viewing a cube if it is to be used for operations or for obtaining data in other volumes. Thus, (19) can only apply to post-analysis.

In real-time the radar produces data on one slice, one beamwidth wide (or high depending whether the mode of scan is RHI or PPI) and is thus capable of producing data on u^2 independent elements.

For real-time scanning, we obtain

$$(\bar{A}_{u^2})_{\min} = 9.72/s'(u-1)k^{0.5} \quad (20)$$

for the average attenuation coefficient in a pie-slice volume. Once again assuming $s'=1$ km, $u=8$ range bins per km, and 32 independent observations per power measurement, we obtain a minimum measurable one-way attenuation coefficient of 0.22 dB km^{-1}. This would arise from a rainfall rate of 10.6 mm hr^{-1} (using $A=0.013\ R^{1.15}$).

The minimum measurable LWC due to precipitation can be obtained using the relation (21) (given later in this paper), i.e.,

$$M_{\min} = 0.67 \text{ gm m}^{-3}.$$

Note that the precision of measurement of A can be improved by a factor of n^2 if both s' and u are increased by a factor of n. But, naturally, the increase in precision is obtained at the cost of a loss in spatial resolution of A. Further, the increase in time of observation, or k, by a factor of n is far less productive of further precision, since the latter is increased by only $\sqrt{n}$.

5. Estimates of the variance of LWC predicted from attenuation

In order to estimate the variance introduced by drop-size spectra, data from three locations have been utilized. These were collected in Illinois, New Jersey and North Carolina using a photographic technique. Table 1 indicates the relationships which were determined between the attenuation coefficient at 3.2 cm and the liquid water content. Each sample represents a drop distribution as measured in 1 m^3 of space (Mueller and Sims, 1966a). From this distribution for a water temperature of 10C the attenuation coefficient at 3.2 cm was calculated using what is known as the Mie relationship. The liquid water content was obtained by summing the volume of the raindrops. Since linear relationships of the logarithms of the variables yield greatly superior fits to the data than the linear regressions of the variables, they are therefore used, and the residual variance column refers to units of the common logarithms. The data from Illinois exhibit the greatest residual variances and are used throughout the paper. The standard error for the logarithm of the liquid water content is 0.138. This yields a standard error of about ±35% when reduced to non-logarithmic units.

Thus, we take

$$M = 2.23\ A^{0.787}. \quad (21)$$

As a means of comparison between this method of radar determination of liquid water and the conventional estimate from the reflectivity, the regression of liquid water content and 10-cm reflectivity was performed for Illinois data. This resulted in a residual variance of the logarithm of 0.0315. This value is larger than any of the variances of Table 1. Further, this value of variance is only the value due to drop-size variability and does not include any additional variances due to the radar calibrations or measurement techniques.

TABLE 1. Relationship of liquid water content to calculated attenuation.

Location	Number of cubic meters in sample	Regression coefficients $M=KA^b$		Residual variance of log M	Minimum M	Maximum M	Correlation coefficient
		K	b		gm^{-3}	gm^{-3}	
Illinois	395	2.23	0.787	0.0191	10^{-1}	6.88	0.92
North Carolina	928	2.64	0.728	0.0182	10^{-1}	9.56	0.876
New Jersey	280	2.22	0.70	0.0180	10^{-1}	7.08	0.870

6. Total variance in liquid water content

The total variance in the measurement of liquid water content is a result of the contribution due to the measurement of attenuation and that due to natural variations of the drop-size spectra. As indicated in the previous section, the drop-size variability is naturally in units of the logarithm. When such units are used, the distribution of the points around the regression line are more nearly normal. To combine the variances due to measurement error and drop-size spectra, both must be in the same units and the estimates of the liquid water content must be distributed normally around the average estimates; such will not be strictly true in this instance.

Since the logarithms of the deviations appear nearly symmetric, the variations themselves are skewed toward the higher values of liquid water. An average of the upper and lower values of the exponentiated standard deviations is used in Table 2.

Using (14) and Table 1 for drop-size variability, and assuming the variances can be added to obtain an estimate of the total variance of M, Table 2 is constructed. The last column lists the edf expected with the parameters which are listed in the caption.

Table 2 provides an estimate of the precision which can be obtained. As expected, the variance in the measurement of attenuation is more important at small values of liquid water content, and the drop-size spectrum variance is more important for large values. Thus, the poor values for the edf (asymptotically approaching 20) at large liquid water contents cannot be improved by increasing the precision of measurement of A (e.g., by increasing the radar observation time).

The variance of M arises primarily from the variation among drop-size spectra that have equal attenuations. In an analysis of a similar problem, using the same camera-derived drop-size distributions, Mueller and Sims (1966b) showed that less than 6% of the residual standard deviation for Z–R relations could be attributable to sample size limitations in the 1 m^3 samples. Thus, we feel that the standard deviations listed in Table 2 (due to drop-size spectra) will not contain a contribution much greater than 6% due to sample size limitations. Thus, neither an increase in sample size, even up to radar pulse volume sizes, nor an improvement in the precision of A, will increase the edf of M significantly beyond 20.

TABLE 2. Total expected standard deviations (gm m^{-3}) in estimates of liquid water content ($s'=1$ km, $k=32$, $u=8$).

Attenuation (dB km^{-1})	LWC	Standard deviation due to radar measurements	Standard deviation in LWC due to drop-size spectra	Total standard deviation	edf
10^{-1}	0.364	0.29	0.12	0.31	3
3.10^{-1}	0.865	0.19	0.28	0.34	13
10^{0}	2.23	0.15	0.72	0.74	18
3.10^{0}	5.29	0.12	1.71	1.71	19
10^{1}	13.7	0.09	4.43	4.43	19

7. One possible configration

One possible configuration[3] of a dual-wavelength LWC sensitive radar processor is shown in Fig. 1.

A quantity proportional to

$$y'=10\log[(P_2'/C_2)/(P_1'/C_1)]$$

is formed by generating the true average values of P_2'/C_2 and P_1'/C_1 (rather than averaging their logarithmic values), taking the logarithm, and then subtracting. After subtracting y' from y by using the appropriate range delay s' of the nearer power measurements (P_1,P_2) at r, we have

$$\log_2(P_2'P_1/P_1'P_2)=\bar{A}s'/(5\log_{10}2). \tag{22}$$

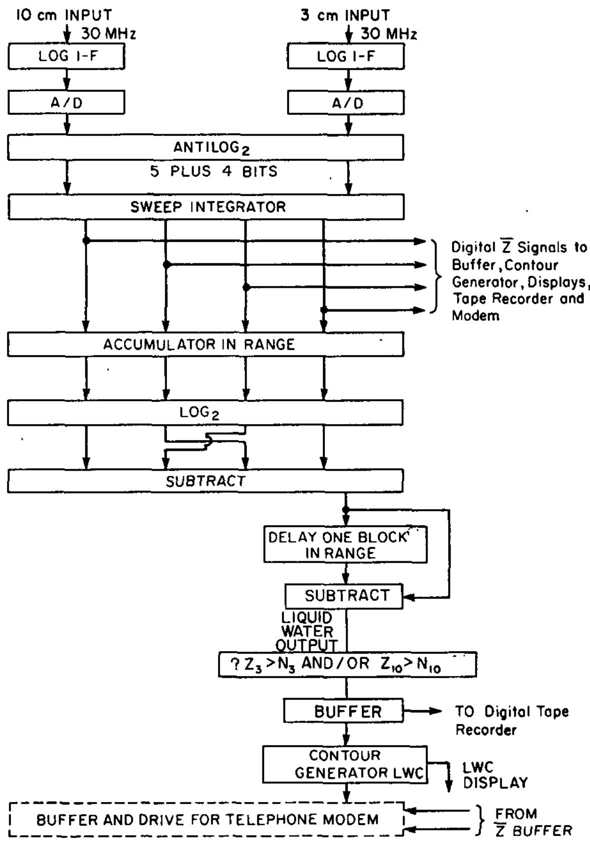

FIG. 1. Schematic diagram of a real-time digital processor of dual wavelength radar signals. The process involves time averaging of values of Z at two wavelengths and subsequent averaging in range. Attenuation is proportional to Δy (see text) and is calculated by the following sequence of operations: logarithm, subtract, delay and subtract. Noise is suppressed and contours of attenuation (a function of liquid water content) are produced. IBM compatible digital magnetic tapes and outputs to real-time computers are produced.

[3] This simplified form of the processor, originally given in Eccles and Mueller (1970), is due to W. Ehrich and R. Trussel of Control Data Corp.

Since the range s' is known, we have A, the attenuation rate. This is applicable to the range $r+s'/2$.

As is mentioned in Section 4b, independent estimates of $\bar{A}$ are available from each range, and thus n' (approximately) values of $\bar{A}$ are available from each beam position (of n' independent range elements), even if s' is equivalent to 10 or 50, say, range elements. Adjacent sets (blocks) of these $\bar{A}$ estimates are averaged, if desired, in the "Accumulator in Range" so that smoother data can be fed to the contour generator in Fig. 1.

Averaging over pie-slice volumes (see Section 4d) is obtained by increasing the amount of averaging in the first integrator beyond one (or part of one) beamwidth up to the number of beam widths desired.

Since the output from the buffers are continuous, the attenuation coefficient can be displayed in exactly the same manner as any video signal on PPI or RHI scopes if D/A converters are used. Alternatively, a contour generator can yield contours suitable for display on PPI or RHI scopes. It is even possible to provide digital processing to yield LWC values at the input of the contour generator for LWC display using the relation $M=2.23A^{0.787}$, since the attenuation coefficient data is in digital form.

A useful display for operations and field analysis is LWC contoured against hail.

The complete design of Fig. 1 also permits bypassing of any of the integration steps if this is desired. Further, since the attenuation coefficient displayed is that due to phenomena at the point in question, peculiar results due to reflectivity differences from permanent echoes and other interfering artifacts do not propagate through the residue of the display.

8. Summary

The basic technique proposed to measure liquid water content using a dual-wavelength radar is to measure the 3-cm attenuation and relate this attenuation to the liquid water content over the path length of the attenuation measurement. The 3-cm attenuation along a path can be deduced from measurements of the power returned at both end points of the path from both the 3- and 10-cm radars. The amount of 3-cm attenuation can be determined since 1) both radars are viewing identical volumes, and 2) since, within the range of scatterers involved in rain and cloud, the difference between the backscattering cross sections at 10 and 3 cm is practically entirely due to the wavelength dependency dictated by the Rayleigh scattering law.

By using carefully designed equipment, it is shown that an on-line minimum sensitivity of about 0.22 dB km^{-1}, with a standard deviation of about 0.1 dB km^{-1} in the measurement of the attenuation rate, can be anticipated if *all* of the data in a pie-slice volume of about 1 km^2 area is properly averaged. Less averaging will probably be undesirable unless the radar operator is certain that spikes that will be observed are due to real local concentrations of LWC and are not due to fluctuations in the data. The statement that spikes will be observed can be made with some confidence since it has been shown that the square root of the attenuation rate as measured has an $F_{4k,4k}$ distribution, which is a long-tailed distribution as compared to the normal (Gaussian) distribution. Therefore, the attenuation rate itself has a *very* long-tailed distribution.

A further problem is the conversion of this measured attenuation rate to LWC by an empirical formula derived from data using real drop distributions from the data of Mueller and Sims (1966a).

It is believed that in the absence of hail and rain and if sufficient signal at the end points is available, an excellent assessment of liquid water content of a cloud can be made. Temperature effects would appear to be important but could be accounted for by using environmental temperatures.

Likewise, in a rain-only environment, a reasonably good estimate of liquid water content should be possible. Here temperature would appear to be of little consequence but the drop-size spectra would be important. Estimates within about ±35% should be possible below cloud base.

Within the cloud and in the presence of rain drops and cloud droplets, the determination of liquid water is more difficult. Nonetheless, an estimate is possible and is certainly made more appropriately with the dual-wavelength system than by only a single frequency. It is always possible to obtain an upper bound on the liquid water content by assuming all of the water is in cloud droplet sizes and calculating the liquid water content from the measured attenuation rate.

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APPENDIX

The Probability Density Distribution of $P_2'P_1/P_1'P_2$

The probability density of the numerator of $(P_2'P_1/P_1'P_2)$, from (9) in the text, is given by

$$p(P_2'P_1)d(P_2'P_1), \qquad \text{(A1)}$$

where p represents the amplitude of the probability density at $P_2'P$. Let us make the definitions

$$P_2'=P_2'/\bar{P}_2'$$
$$P_1=P_1/\bar{P}_1$$

where $\bar{P}_1$ and $\bar{P}_2'$ are the expected values of P_1 and P_2'.

In the derivation that follows, then, P_2' and P_1 are precisely chi-square distributed, and include the terms $\bar{P}_2'$ and $\bar{P}_1$ implicitly.

If we make the substitution

$$w = P_2' P_1, \tag{A2}$$

each of P_2' and P_1 are now independent of one another. Therefore,

$$\delta[p(w)dw] = p(P_2')dP_2' p(P_1)dP_1, \tag{A3}$$

where δ is the contribution to the probability density at w due to P_2' and P_1. Contributions to this density occur for all values of P_1 and associated values of P_2' such that $P_2' = w/P_1$.

Therefore,

$$\int_0^\infty p(w)dw = \int_0^\infty \int_0^\infty p(P_2')dP_2' p(P_1)dP_1. \tag{A4}$$

Substituting t for P_1 and w/t for P_2', we obtain

$$\int_0^\infty p(w)dw = \int_0^\infty \int_0^\infty p(t)dt p\left(\frac{w}{t}\right)\frac{1}{t}dw. \tag{A5}$$

$$= \int_0^\infty \left[\int_0^\infty p(t)p\left(\frac{w}{t}\right)\frac{dt}{t}\right]dw. \tag{A6}$$

We let m and n be the number of independent samples in P_2 and P_1, respectively, and remembering that $w/t = P_2'$ [which is $\chi_{2m}^2/2m$ (Section 3)], we expand the integrand to obtain

$$p(w) = \int_0^\infty \left[\frac{(t/2)^{n-1}\exp(-t/2)}{2\Gamma(n)}\right] \times \left[\frac{(w/2t)^{m-1}\exp(-w/2t)}{2t\Gamma(m)}dt\right]$$

$$= \frac{w^{m-1}}{2^{m+n}\Gamma(n)\Gamma(m)} \times \int_0^\infty t^{-0.5}\exp[0.5(-t - w/t)]dt, \tag{A7}$$

provided that we assume $2m = 2n - 1$. This is an assumption of nearly equal numbers of samples in each average power and is justified in the text.

Now, if we change the variable, by substituting $x^2 = t/2$, the integral becomes a standard form (see, for example, Weast and Selby, any edition) and we obtain

$$p(w)dw = \frac{\sqrt{\pi}}{2^{2m}\Gamma(n)\Gamma(m)} w^{m-1}\exp(-\sqrt{w})dw. \tag{A8}$$

This shows that the quantity $2\sqrt{w}$ has a probability density given precisely by $\chi_{4m}^2/4m$ since we can make the substitution $u = 2\sqrt{w}$ and obtain[4]

$$p(w)dw = \frac{\sqrt{\pi}}{2^{2m}\Gamma(n)\Gamma(m)}(u/2)^{2m-1}\exp(-u/2)du,$$

$$= \frac{(u/2)^{2m-1}\exp(-u/2)du}{2\Gamma(2m)},$$

$$= f(u)du. \tag{A9}$$

We know that $f(u)du$ of (A9) is the expression for the probability density of a $\chi_{4m}^2/4m$ distribution, and therefore that

$$\int_0^\infty p(w)dw = \int_0^\infty f(u)du = 1.0, \tag{A10}$$

which is a necessary condition for a probability distribution and indicates that there are no arithmetic errors in the derivation of $p(w)dw$.

Further, consider the identities

$$\int_0^\infty f(u)d(u) = \int_0^\infty f(2\sqrt{w})d(2\sqrt{w})$$

$$= \int_0^\infty f(\sqrt{w})d(\sqrt{w}), \tag{A11}$$

and the fact that $\sqrt{w} = \sqrt{P_2' P_1}$. Clearly, the quantity $(P_2'P_1/P_1'P_2)^{0.5}$ has a numerator with a $\chi_{4k}^2/4k$ distribution if we assume $2m \approx 2k \approx 2n - 1$.

Now the quotient of two χ^2 distributions $[(\chi_a^2/a)/(\chi_b^2/b)]$ is known to be an $F|_{a,b}$ distribution. Hence, we find that the quantity $(P_2'P_1/P_1'P_2)^{0.5}$ has a distribution which is precisely $F|_{4k,4k}$. The mean is $1 + (0.5/k)$, the variance $1/(k - 1.8) \approx 1/k$, and the standard deviation $\sim 1/k^{\frac{1}{2}}$. Its detailed statistical properties are well known and can be found in any sophisticated statistical text or comprehensive mathematical tables (Abramowitz and Stegun, 1955, Chap. 26).

Now, there is a one-to-one correspondence between $(P_2'P_1/P_1'P_2)^{0.5}$ and $(P_2'P_1/P_1'P_2)$. Therefore, the properties of this latter function can be found from the known properties of the former. For example, if we suppose that $1 + Q$ is the upper value of the $F|_{4k,4k}$ distribu-

[4] The derivation makes use of the following properties of the gamma function:

$$\Gamma(2m) = \Gamma(2n-1) = (2n-2)\Gamma(2n-2),$$
$$= 2(n-1)2(n-2)\cdots 2(2)2(1)2(n-1-\tfrac{1}{2}) \times 2(n-2-\tfrac{1}{2})\cdots 2(\tfrac{3}{2})2(\tfrac{1}{2})2\Gamma(\tfrac{1}{2})\frac{1}{2\Gamma(\tfrac{1}{2})}$$
$$= \frac{[2^{n-1}\Gamma(n)][2^n\Gamma(m)]}{2\sqrt{\pi}} \quad [\text{Note: } 2\sqrt{\pi} = 2\Gamma(\tfrac{1}{2})].$$

Therefore,

$$\frac{1}{2\Gamma(2m)} = \frac{\sqrt{\pi}}{2^{2m}\Gamma(n)\Gamma(m)}.$$

tion such that 1% of the values of $(P_2'P_1/P_1'P_2)^{0.5}$ exceed $1+Q$, then $(1+Q)^2$ will be the value of the $P_2'P_1/P_1'P_2$ distribution which is exceeded by 1% of its estimates. This detailed knowledge of the tails of the distribution is essential in order to predict false alarm frequency in the hail detector described by Eccles and Atlas (1972), but for the purposes of LWC measurement it is adequate to consider the approximate properties of $(P_2'P_1/P_1'P_2)$ near its mean. However, one statistic is known with precision. The median value of $(P_2'P_1/P_1'P_2)$ is unity, since the median value of a symmetric F distribution is 1.0.

Let Q be the standard deviation of the $F|_{4k,4k}$ distribution. Then, from above, the equivalent value for the $(P_2'P_1/P_1'P_2)$ distribution is $1-(1+Q)^2 \approx 2Q$ for small Q. Similarly, we find the other parameters of $P_2'P_1/P_1'P_2$; thus,

$$\left.\begin{array}{r}\text{standard deviation} \approx 2/(k-1.8)^{0.5} \approx 2/k^{0.5} \\ \text{variance} = \sigma^2 \approx 4/(k-1.8) \approx 4/k \\ \text{median} = 1.0.\end{array}\right\}.$$

It is a convenient check on the precision of the above analysis to recall that P_2' (still implicitly including the $\bar{P}_2'$), for example, has a chi-square distribution, and, therefore, is approximately normally distributed with mean of unity, a standard deviation of $1/(k)^{0.5}$ and a variance of $1/k$. Therefore, the numerator $P_2'P_1$ is also approximately normally distributed with a mean of ~ 1.0 and a variance of $2/k$. Now the quotient of two normal distributions is very badly behaved, in general, because of the finite probability of the denominator being zero. However, our function has a zero probability density at zero amplitude; in this case the variance, $2/k$, of the denominator adds to that, $2/k$, of the numerator (since they both have the same means) in the same manner as in any estimate of experimental error from a formula with measured physical quantities in the denominator.

In this very approximate treatment, then, $(P_2'P_1/P_1'P_2)$ is approximately normally distributed with a mean of ~ 1.0, a variance of $\sim 4/k$ and a standard deviation of $\sim 2/k^{0.5}$, all of which agree with the analytic deviation given earlier.

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