An Investigation of the Numerical Properties of the Surface Heat-Balance Equation

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1. Introduction

The interface heat-balance equation has been solved for many years, generally as part of more complex boundary-layer or global-circulation models (c.f., Estoque, 1963; Smagorinsky *et al.*, 1965; Manabe *et al.*, 1965; Pandolfo, 1969; Zdunkowski and Barr, 1972).

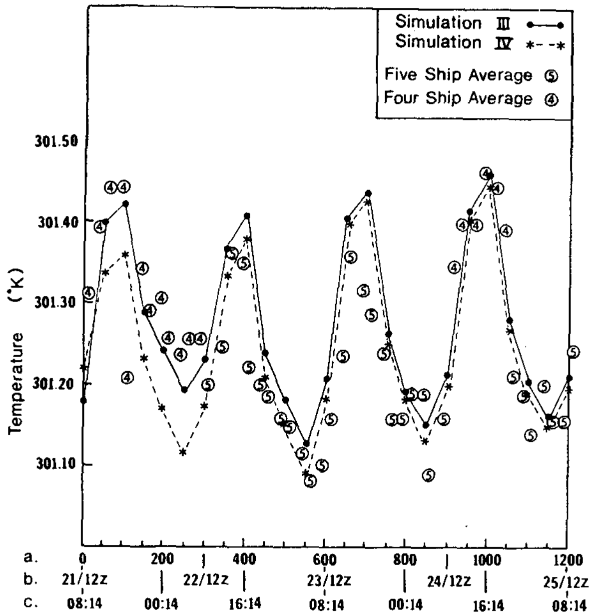

FIG. 1. The sea surface temperature as a function of time over a 4-day period for simulations III and IV. The abscissa has a triple time scale which indicates the time step, a., the day in June GMT, b., and the sun time, c. Spatially averaged bucket temperatures are plotted at 2-hr intervals from five ship stations (*Rainier*, *Oceanographer*, *Mt. Mitchell*, *Rockaway* and *Discoverer*); four-ship averages, however, were used due to lack of data. (See Pandolfo and Jacobs, 1972.)

Outcalt's (1972) arrival at a solution of this equation is certainly not unique, but his attempt to present the details of his solution technique appears to be rare in the literature. A team of scientists[1] at The Center for the Environment and Man, Inc. (CEM) has been solving a similar interface heat-balance equation for more than five years as part of complex one- and three-dimensional numerical boundary-layer models (Pandolfo *et al.*, 1971; Pandolfo and Jacobs, 1972; Atwater, 1972). Fig. 1, reproduced from an earlier paper, shows an example of the validation of such a solution. This figure shows the sea-surface temperature as a function of time over a 4-day period. The two simulations (III and IV) which are displayed differ only in the initial dynamic and thermodynamic states of the atmosphere and ocean. For simulation III, these initial states were constructed from the observed conditions in the BOMEX region for the period simulated. These observations contained erroneous humidity measurements which indicated a drier atmosphere than actually existed. After 4 days of simulation, a new state of the atmosphere and ocean was predicted. This new state was similar to the initial state except that the humidity was higher throughout the atmospheric layer modeled. Simulation IV used this new, model-generated state as initial conditions. It can be seen from Fig. 1 that the interface temperature is noticeably affected by the moisture content of the atmosphere.

[1] This team consists of Dr. J. P. Pandolfo, Dr. M. A. Atwater, Mr. P. S. Brown, Jr., and Mr. C. A. Jacobs.

Until recently, an heuristically derived iterative technique was used in the models to obtain the equilibrium surface temperature. This heuristic technique had two disadvantages: on rare occasions it would fail to converge within the specified number of iterations (25), and the logic of the technique was too complex for tractable explanation and was therefore omitted from journal articles. For these reasons, the character of the interface heat-balance equation was investigated. The investigation has yielded a simple technique based on well-established iterative algorithms.

2. Heat-balance equation

The interface temperature T is obtained in our one- and three-dimensional models for either a soil or water subsurface through solution of the nonlinear heat-balance equation

$$T=\left[\frac{1}{\epsilon\sigma}(R_H+R_a+R_x+P_T-A-S-LE)\right]^{\frac{1}{4}}, \quad (1)$$

where:

- ϵ emissivity
- σ Stefan-Boltzmann constant
- R_H net solar energy absorbed at the interface minus the amount transmitted to lower layers
- R_a atmospheric radiation incident on the interface
- R_x artificial heat source due to combustion.
- P_T heat energy due to precipitation
- A sensible heat flux from the interface to the atmosphere
- S sensible heat flux from the interface to the water or soil
- L latent heat
- E surface evaporation.

Eq. (1) is essentially the equation Outcalt solves with the exception of the terms R_x and P_T which do not appear in his equation. The details of the calculation of each of these terms have been given by Jacobs (1973). The variables P_T, A, S, L are linear functions of T, while E has the form

$$E=\gamma+\alpha\exp\left(\beta\frac{T-T_1}{T-T_2}\right), \quad (2)$$

where α, β, γ, T_1 and T_2 are constants. Eq. (1) is not a true quartic, in general, since the evaporation term contains an exponential function.

If we define $g(T)$ as

$$g(T)\equiv T-\left[\frac{1}{\epsilon\sigma}(R_H+R_a+R_x+P_T-A-S-LE)\right]^{\frac{1}{4}}, \quad (3)$$

solution of (1) for the interface temperature is achieved through determination of a real root of the equation

$$g(T)=0. \quad (4)$$

Various approaches to the numerical solution of the heat-balance equation have led to difficulties due to the fact that the bracketed quantity in (1) becomes negative for values of T near the solution value, T_0. In terms of $g(T)$ defined above, the behavior of the bracketed term causes $g(T)$ to take on complex values for certain values of T. For typical values of the parameters, $g(T)$ has the form shown in Fig. 2. For $T>T_c$, $g(T)$ is complex, T_c being only 7K to the right of T_0, the required root of (4).

3. The solution procedure

We now present an iterative solution procedure that will converge provided $g(T)$ exhibits the behavior shown in Fig. 2. For this purpose, we define a new function $\tilde{g}(T)$ as

$$\tilde{g}(T)=\begin{cases} g(T), & T\leqslant T_c \\ T, & T>T_c \end{cases} \quad (5)$$

The extension of $g(T)$ to a real-valued function $\tilde{g}(T)$ for $T>T_c$ is shown in Fig. 2 by the dotted line. The interface temperature can be found by solution of the nonlinear equation

$$\tilde{g}(T)=0. \quad (6)$$

The iteration scheme proposed is a combination of the Newton-Raphson iterative procedure with the method of false position; *viz.* for an approximation $T^{(i)}$ to the interface temperature, find the subsequent approximation $T^{(i+1)}$ in accordance with the relations

$$T^{(i+1)}=T^{(i)}-\frac{\tilde{g}[T^{(i)}]}{\tilde{g}'[T^{(i)}]}, \quad T^{(i)}<T_c, \quad (7a)$$

(Newton-Raphson)

$$T^{(i+1)}=T^{(i)}-\left(\frac{T^{(i)}-T^*}{\tilde{g}[T^{(i)}]-\tilde{g}(T^*)}\right)\tilde{g}[T^{(i)}], \quad T^{(i)}\geqslant T_c, \quad (7b)$$

(False position)

where T^* is the most recently computed approximation for which $\tilde{g}[T^{(i)}]\times\tilde{g}(T^*)<0$. Since $\tilde{g}[T^{(i)}]>0$ for $T^{(i)}\geqslant T_c$, the method of false position requires a value T^* for which $\tilde{g}(T^*)<0$. If such a value has not been computed in a previous iteration, we may use an initially prescribed "safe" value for T^*, say T_m, for which $\tilde{g}(T_m)<0$ (e.g., for a water subsurface take T_m to be the freezing point of water if T_0 is expected to be well above freezing). It should be noted that T_c could be obtained by determining the value of T for which the bracketed term is zero, thereby providing an upper bound for the approximations to be used in Newton's method. Such an approach, however, would require the solution of an additional nonlinear equation.

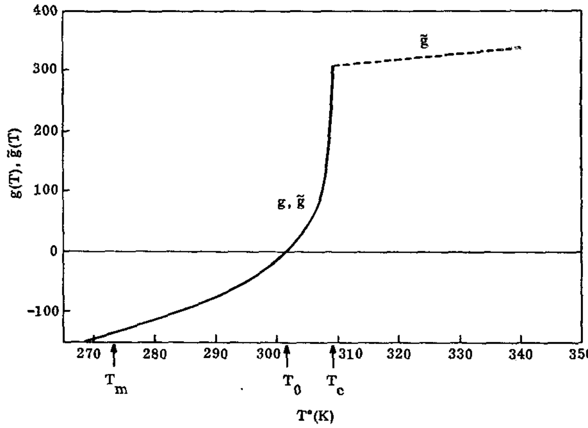

FIG. 2. Extension of the function $g(T)$ to a real-valued function $\tilde{g}(T)$ for $T>T_c$.

For the function shown in Fig. 2, Figs. 3a and 3b show how the iterative scheme proceeds using an initial guess $T^{(1)}$ for which (a) $T^{(1)}<T_0$, and (b) $T^{(1)}>T_c$. For $T_0<T^{(1)}<T_c$, it is clear from the curvature of $g(T)$ that the Newton-Raphson method will converge.

The function $g(T)$ as calculated from Eq. (3) has been plotted (as in Fig. 2) for a wide range of physically realistic meteorological conditions. The character of this function always remained the same although the curvature varied. A significant feature of $g(T)$, as can be seen in Fig. 2, is the sharp change in slope between the

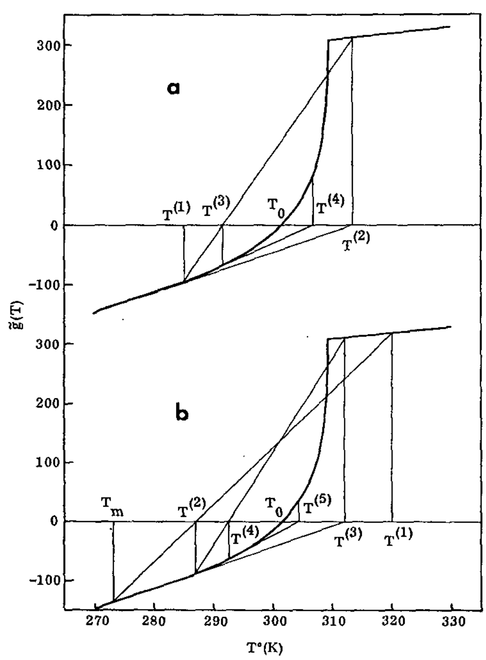

FIG. 3. Geometric interpretation of how the iterative scheme proceeds using an initial guess $T^{(1)}$ for which $T^{(1)}<T_0$, a., and $T^{(1)}>T_c$, b.

temperatures T_0 and T_c. Under certain conditions, the difference between T_0 and T_c can be only a few degrees. This characteristic of $g(T)$ can create problems with many iterative solution techniques particularly if an approximation $T^{(i)}$ falls in the complex region of $g(T)$. The solution technique presented above is comparable in efficiency to Outcalt's, taking on the order of 10 or less iterations to converge for each time step. Furthermore, modification of the parameters is not necessary to obtain a solution (see Outcalt, 1972). The method combines the rapid convergence of the Newton-Raphson procedure with the guaranteed convergence of the method of false position.

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