Condensation in Jets, Industrial Plumes and Cooling Tower Plumes

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ABSTRACT

The one-dimensional theory for the condensation of buoyant plumes is extended to include supersaturation as an extra variable. An additional equation describing the dynamics of droplet growth is used to make the system tractable. Some simple mathematical results are obtained which allow one to relate the theory to, and so extend, a commonly used graphical representation of the condensation process. The theory is then simplified to a single nonlinear first-order differential equation for the condensed water content. This is solved numerically for a typical jet, scrubbed industrial plume and natural-draft cooling tower plume to obtain down-plume profiles of condensed water content, supersaturation and mean droplet size. High supersaturation is predicted in all three cases, corresponding to mean relative humidities of up to 170% (jet), 150% (scrubbed plume) and 105% (cooling tower). These results may be important in predicting the growth of "foreign" carry-over droplets in plumes from industrial sources or cooling towers. Predictions of plume length in these cases is found to be insensitive to supersaturation, but plume length is noticeably affected by supersaturation in the case of a jet. In the examples considered maximum mean droplet radii never exceed 10 μm which supports the belief that rain-out is caused primarily by carry-over from imperfect mist eliminators.

1. Introduction

The process of condensation within buoyant jets, plumes and clouds has been studied by fluid dynamicists and cloud physicists for more than three decades. Because these phenomena span such a wide range of sizes, and because of the differing needs of scientists in different fields, the development for jets and plumes has differed slightly from that for clouds. In the following I will use the words "jet" to refer to small-scale phenomena with initial radii of the order of centimeters, "plume" to refer to intermediate phenomena with initial radii of the order of meters to tens of meters (including industrial plumes and cooling tower plumes), and "cloud" to refer to larger phenomena (such as natural convective clouds). Considerably more research has been directed toward modelling the condensation process in natural clouds than in phenomena of smaller size, and the development of the theory of condensing plumes and jets still lags behind that for clouds.

In this paper I will present a relatively simple unified theory which is intended primarily for application to intermediate-sized phenomena, but which is also applicable to condensing jets, and is consistent with cumulus cloud theory. The word "plume" will generally be used, but the development presented here applies equally well to jets and clouds. The theory is simpler than that used in models of convective clouds, but more general than the existing theory for condensing plumes, and so helps to bridge the gap that exists between these two. A number of interesting differences will be shown to arise between "plumes" of different sizes.

When a hot, moist gas plume is emitted into a cooler and relatively drier environment both its temperature and moisture content are reduced rapidly by entrainment of ambient air. Because the saturation vapor pressure of water is a nonlinear function of temperature, the rate of cooling, if sufficiently rapid, can cause the plume to condense. Its subsequent motion will then be affected by latent heat liberation and by the presence of condensed droplets of various sizes. A theory to describe these "wet" plumes (as opposed to "dry" or non-condensing plumes) has been developed by Morton (1957), Csanady (1971), Wigley and Slawson (1972), Richards (1973) and others. These authors have used a simple extension of the one-dimensional entrainment theory model used by Morton *et al.* (1956). This theory, through the work of Morton (1957) and Squires and Turner (1962), also provides the basis for most of the recent models of convective clouds.

One major difference between cloud models and wet plume models is that, in the latter, only an equilibrium theory has so far been developed: "equilibrium" in the sense that any vapor excess above the saturation level is assumed to condense instantaneously. Obviously some vapor must remain uncondensed, as supersaturation, in order to initiate and maintain condensation and to act as the driving force for the diffusion of water vapor to the condensation surfaces. This is rather an anomalous situation because the degree of supersaturation should be larger in jets and plumes than in clouds. Numerous measurements of supersaturation in clouds have shown it to be small. In contrast, Hidy and Friedlander (1964) have found that a high degree of supersaturation is reached in condensing jets even before visible condensation occurs. One might therefore expect intermediate-sized plumes to show supersaturations somewhere between these extremes.

The observed supersaturation in jets and natural clouds may be explained qualitatively by considering the time scales for the rate at which moisture becomes available for condensation and the rate at which droplets may grow.

The former is controlled by the rate of growth of the plume which can be characterized by a time scale τ_p defined by

$$\frac{1}{R^2}\frac{dR^2}{dt}=\frac{1}{\tau_p}.$$

Here, R is the plume "radius," identifiable with the standard deviation for dispersion of heat and matter in the plume. In terms of the eddy diffusivity K this becomes, following Slawson and Csanady (1967)

$$\tau_p\approx\frac{R^2}{2K}.$$

In the "initial phase" of plume growth, where mixing with the environment is determined mainly by the plume's own turbulence,

$$K\approx\tfrac{1}{10}wR,$$

where w is the vertical component of the center-line plume speed. Hence

$$\tau_p\sim\frac{5R}{w}.$$

Typical initial values of τ_p are 0.005 s for a jet, 0.75 s for an industrial plume, 30 s for a cooling tower plume, and 250 s or more for a natural convective cloud. τ_p increases rapidly with time.

A time scale for droplet growth is derived later in the body of this paper as

$$\tau_d\sim\frac{\bar{r}^2}{3kE},$$

where $\bar{r}$ is the mean droplet radius, E the supersaturation (defined as the ratio of the vapor excess as supersaturation to the saturation value), and k a temperature-dependent "constant" (which is of order 10^{-6} cm^2 s^{-1} at 10°C and which decreases as temperature decreases). For droplets of order 1 μm radius

$$\tau_d\sim\frac{1}{300E};$$

for droplets of order 10 μm radius

$$\tau_d\sim\frac{1}{3E}.$$

By comparing τ_p and τ_d one can estimate the expected degree of supersaturation. In a steady-state situation the rate of growth of droplets must account for the rate at which condensation moisture becomes available due to plume growth. If, for example, τ_p were much less than τ_d then this would indicate a moisture supply rate much greater than the rate of droplet growth. This would cause a buildup in supersaturation (and subsequent decrease in τ_d) until a balance was obtained. The order of magnitude of E is thus determined by the relation

$$\tau_p\sim\tau_d.$$

Even ignoring changes in E, both τ_p and τ_d are rapidly changing quantities so only an order of magnitude estimate of E is possible.

For a jet, for small droplet size and small times, E must be very large (of order 1) because τ_p is so small. Initially, droplet growth cannot keep up with the availability of condensation material in this case, and a rapid buildup of supersaturation can occur. For clouds, however, τ_p is sufficiently large that equality of τ_p and τ_d can be achieved by relatively modest supersaturations (of order less than 0.005, a value in agreement with observations). For scrubbed industrial plumes one would expect E to build up to values of order 0.1, while for cooling towers a value between this and typical natural cloud values can be expected. In addition, since k varies directly with temperature, τ_d will tend to be larger at lower temperatures. Higher supersaturations could therefore be expected under these conditions.

In the following sections a more quantitative discussion of these largely qualitative arguments is given. An extension of the one-dimensional theory of buoyant, wet plumes will be developed in which the degree of supersaturation is included as an additional variable. The theory will then be simplified to obtain estimates of E (as a function of time) under various conditions for both jets and plumes. The effect of supersaturation on the prediction of condensed water content will also be evaluated. This latter parameter is of some importance to the problem of rain-out from wet plumes.

2. Extension of vapor plume theory

The equations which describe the behavior of a moisture-containing plume may be derived from the principles of conservation of mass, moisture, momentum and energy. In a one-dimensional model, only variations in plume variables along the centerline of the plume are considered. The plume may be pictured as having sharp boundaries and constant variable values across any cross section. The dominant dispersing agent for energy (heat) and momentum is assumed, for the well-established "initial phase" theory, to be the plume's own "self-generated" turbulence. Results derived from this theory have been fairly successfully correlated with field and experimental results for jets, industrial plumes and natural clouds.

The conservation equations yield a set of four relations linking five variables which describe the plume: its radius R, temperature T_p, specific humidity q_p, condensed water content σ, and vertical velocity component w. Strictly, two other momentum equations should be included to describe the horizontal components of plume motion, but, to eliminate these, it is usually assumed that the plume is either in a calm environment or is bent-over and moves with the prevailing wind. The following discussion will be limited to bent-over plumes, but similar results are expected for other cases.

For a dry plume, or a moist plume prior to condensation, this set of equations can be closed using the condition $\sigma=0$. After condensation, when the plume is "wet," the set may be closed using the equilibrium assumption that the plume's specific humidity is equal to the saturation specific humidity q_{sp}, where q_{sp} is a known function of T_p. More correctly, however, supersaturation (E) should be included in the theory as an additional variable. The two variables q_p and E are related by

$$q_p=q_{sp}(1+E). \tag{1}$$

The equations can then be closed by considering the processes of droplet formation within the plume to derive a sixth relation between the plume variables. This is basically the approach used in recent models of cumulus clouds.

It is more convenient to use the excess vapor content as supersaturation (θ) as a variable instead of E. The supersaturation θ is defined by $\theta=q_{sp}E$ and Eq. (1) becomes

$$q_p=q_{sp}+\theta. \tag{2}$$

The equations representing conservation of moisture and energy in the plume are then

$$\frac{d}{dt}[VR^2(q_{sp}+\theta-q_a+\sigma)]=-wVR^2G, \tag{3}$$

$$\frac{d}{dt}\left[\frac{VR^2}{T_a}\left(T_p-T_a-\frac{L}{C_p}\sigma\right)\right]=-wVR^2\frac{N^2}{g}, \tag{4}$$

where V is the centerline plume speed, L the latent heat, and C_p the specific heat capacity at constant pressure for air; $G=dq_a/dz$ is determined by the environment moisture profile, N^2 is the Väisälä frequency, and g the gravitational acceleration. Subscript a refers to the environment. The temperatures T_p and T_a are virtual temperatures.

As a simplification I will suppose that G and N^2 are both zero, corresponding to a well-mixed environment. Since supersaturation is expected to be significant only for small times (where τ_p is small) this is a valid approximation even in an environment which is not well-mixed. Eqs. (3) and (4) can now be integrated to give

$$VR^2(q_{sp}+\theta-q_a+\sigma)=w_0R_0{}^2\Delta q_0, \tag{5}$$

$$VR^2\left(T_p-T_a-\frac{L}{C_p}\sigma\right)=w_0R_0{}^2\Delta T_0\frac{T_a}{T_{a0}}, \tag{6}$$

where $\Delta T=T_p-T_a$ and $\Delta q=q_p-q_a$, and subscript zero is used for values at the point of efflux (where $V=w_0$). These results assume that $\sigma=0$ at the point of efflux.

In the equilibrium case, where $\theta=0$, VR^2, which is proportional to the mass flux within the plume, can be eliminated from (5) and (6) to give

$$\sigma^*=\frac{\dfrac{T_{a0}\Delta q_0}{T_a\Delta T_0}(T_p-T_a)+q_a-q_{sp}}{1+\dfrac{LT_{a0}\Delta q_0}{C_pT_a\Delta T_0}}, \tag{7}$$

where σ^* is the condensed water content assuming no supersaturation. Since q_{sp} is a known function of T_p and environment pressure p, Eq. (7) determines σ^* as a function of plume temperature and environment temperature and pressure. If T_a is assumed constant and the variation of q_{sp} with pressure (and hence with plume rise) is ignored, then (7) is a mathematical formulation of the graphical technique used by Blum (1948) and Overcamp and Hoult (1971) to determine liquid water content in condensing cooling tower plumes. Apart from the conditions on T_a and q_{sp} mentioned above, it can be seen that this graphical method applies strictly only to a well-mixed environment, and it ignores the effects of supersaturation. I will discuss this method further below.

If supersaturation is included (i.e., $\theta\neq0$), then elimination of VR^2 from (5) and (6) yields

$$\sigma=\sigma^*-\theta\left(1+\frac{LT_{a0}\Delta q_0}{C_pT_a\Delta T_0}\right)^{-1}, \tag{8}$$

where σ^* is given by (7). This shows, a rather obvious result, that condensed water content is diminished by the presence of supersaturation. However, the relation is not a simple matter of subtraction, and the factor multiplying θ arises because the latent heat input is reduced if only a part of the vapor available for condensation actually condenses.

Even though it is limited to a well-mixed atmosphere and ignores the variation of q_{sp} with plume rise, some useful insight can be gained by reconsidering the graphical technique used by Blum (1948) and Overcamp and Hoult (1971) and extending it to include, at least qualitatively, the effect of supersaturation. The method, which is based on some early work by Brunt (1935) on the mixing of air masses, is illustrated in Fig. 1.

The plume conditions are initially at O and condensation first occurs at point C. The curve CMNE is the saturation curve. To be precise, the position of this curve must shift as the plume rises because of the dependence of q_{sp} on pressure. The point A corresponds to the prevailing environment conditions which are assumed to be constant and independent of plume rise. Strictly speaking this is an unrealistic assumption since constancy of T_a is inconsistent with the assumption of a well-mixed, neutral environment. If plume rise is relatively small the effect of assuming the saturation curve and point A to remain fixed is minor. Their influence can easily be examined using (7).

If there were no condensation the plume conditions would evolve from O toward the environment conditions A along the straight line OA. In the equilibrium theory (i.e., assuming no supersaturation) the plume conditions would travel along the saturation curve CMNE. At the point M where the tangent to the curve is parallel to OA there will be a maximum in the condensed water content. This will then decrease toward zero at point E which is the end of the wet or visible condensed portion of the plume. The condensate profile can be determined graphically as a function of plume temperature which in turn can be related to other plume parameters such as dilution, plume rise or downwind distance. For more precision, Eq. (7) could be used since this allows q_{sp} and T_a to change as the plume rises.

What changes occur if supersaturation is included? Initially, plume conditions must evolve along a path between the saturation curve CMNE and the no-condensation line OA. Within this region the plume is supersaturated and net condensation must occur. As the plume grows, the degree of supersaturation will rise to a peak and then fall, reaching zero at point N where the evolution path crosses the saturation curve. Beyond N the plume becomes undersaturated and net evaporation of droplets commences. At point N, therefore, where the plume is *at* saturation, the condensed water content σ must be a maximum, and the tangent to the evolution curve at N must therefore be parallel to OA. In addition, σ and σ^* must be equal at this point. These predictions will be confirmed by the analytic solution given later.

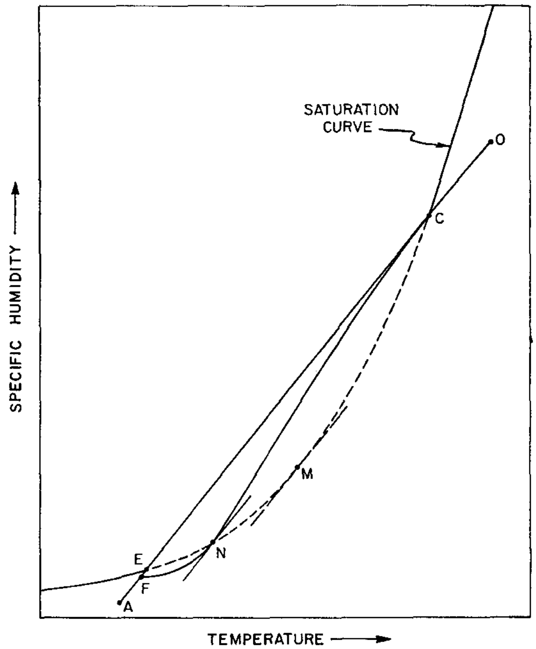

Fig. 1. Schematic representation of the evolution of a condensing plume showing the difference between the equilibrium and the present theory. In the former, after condensation at C, the plume is supposed to evolve along the saturation curve CME to the end of the condensed section at E. Point M, where the slope of the saturation curve is parallel to OA, is the point of maximum condensed water content. If one allows for supersaturation, the condensed plume evolves along CNF to its end at F. The maximum condensed water content occurs at N with net condensation prior to this point and net evaporation beyond it. The points O and A correspond to initial and environment conditions respectively.

Before N, σ must be less than σ^*, while beyond N, σ must exceed σ^*. Since σ is a maximum here, this means that, if supersaturation is allowed for, the maximum condensed water content must be less than that predicted if supersaturation is ignored. In addition, since $\sigma > \sigma^*$ beyond N, the neglect of supersaturation must lead to an underestimate of wet plume length. The magnitude of these effects can only be determined by an analytic solution. Before such a solution is possible another equation involving the dynamics of droplet growth must be derived.

3. Droplet growth

When a single droplet grows by condensation there must be simultaneous diffusion of moisture to the droplet surface and diffusion of liberated latent heat away from the surface. A balance of these two processes can be used to derive an expression for the rate of growth of a droplet of radius r in the form (see, for example, Fletcher, 1962)

$$r\frac{dr}{dt}=k\xi\left(E-\frac{K_1}{r}+\frac{K_2}{r^3}\right), \tag{9}$$

TABLE 1. Proportionality constant in Eq. (10).

Plume temperature (°C)	−30	−20	−10	0	10	20	30	40	50
k^* (cm² s⁻¹)	0.0007	0.0014	0.0030	0.0060	0.0095	0.0133	0.0172	0.0214	0.0259

where k is a constant (which depends on temperature) and ξ is a ventilation factor. The second and third terms in the parentheses arise from curvature and solute effects, and the derivation of the equation assumes that growth rate is not too rapid, that mean-free-path effects are negligible, and that droplets are not electrically charged.

For droplets of radius $>0.1\,\mu$m the curvature and solute terms in (9) may be neglected. Similarly, for droplets smaller than about 20 μm, the ventilation factor may be taken as unity: for droplets much larger than this coalescence becomes the dominating growth mechanism, rather than condensation. If $n(r)$ is the distribution function of droplet radii, then

$$\frac{d\sigma}{dt}\approx\frac{4\pi\rho_w}{3\rho_p}\int_0^\infty \frac{d}{dt}(r^3)n(r)dr,$$

where ρ_w and ρ_p are the densities of water and the plume. Combining this with the simplified form of Eq. (9) yields

$$\frac{d\sigma}{dt}\approx\frac{4\pi\rho_w kE}{\rho_p}\int_0^\infty rn(r)dr,$$

or, in terms of the mean droplet radius $\bar{r}$ and the droplet number density N,

$$\frac{d\sigma}{dt}\approx k^*NE\bar{r}, \tag{10}$$

where $k^*=4\pi\rho_w k/\rho_p$. The constant k^* may be calculated using tables of k as a function of temperature given by Byers (1965). For $\bar{r}$ in centimeters and N in droplets cm^{-3}, k^* is given in Table 1. The analysis leading to (10) is common in cloud physics (see, for example, Fletcher, 1962). The droplet-growth time scale τ_d given in the Introduction can now be derived. Since

$$\sigma\sim N(\tfrac{4}{3}\pi\bar{r}^3\rho_w)/\rho_p,$$

Eq. (10) becomes

$$\frac{1}{\sigma}\frac{d\sigma}{dt}\sim\frac{3kE}{\bar{r}^2},$$

so that one immediately obtains

$$\tau_d\sim\frac{\bar{r}^2}{3kE}$$

as given earlier.

For the present purpose a more precise relation between σ and $\bar{r}$ is required. If $\bar{r}_0$ is the mean condensation nucleus radius, then

$$\bar{r}=\left(\frac{3\rho_p\sigma}{4\pi N\rho_w}+\bar{r}_0^3\right)^{\frac{1}{3}}. \tag{11}$$

Substituting (11) into (10), and using $\theta=q_{sp}E$, yields

$$\frac{d\sigma}{dt}=\frac{k^*N\theta}{q_{sp}}\left(\frac{3\rho_p\sigma}{4\pi N\rho_w}+\bar{r}_0^3\right)^{\frac{1}{3}}. \tag{12}$$

This is the key result of this section since it is the additional equation which was demanded by the introduction of supersaturation into the problem as an additional variable. Now k^*, N, q_{sp} and ρ_p may all be expressed in terms of the plume variables previously defined.

Eq. (12) also verifies one of the qualitative conclusions derived earlier. Since, by virtue of Eq. (8), θ is proportional to $\sigma^*-\sigma$, Eq. (12) requires that $d\sigma/dt$ is also proportional to $\sigma^*-\sigma$. Hence the maximum value of σ must occur at the point where the condensed water content predictions allowing for and neglecting supersaturation coincide, i.e., at point N in Fig. 1.

4. Supersaturation and condensate profiles

Eqs. (5), (6) and (12) relate the plume variables VR^2, θ, σ and T_p to each other and to the independent variable t. The variables q_{sp} and N are also involved. These can be eliminated because q_{sp} is a known function of T_p and pressure (or plume rise, which is written as a function of t in the Appendix), and N is determined by VR^2. A fourth equation is needed to make this system tractable: in fact, equations expressing conservation of mass and of the vertical component of momentum (which introduce w as an additional variable) should be necessary. Fortunately, these extra equations can be avoided by using a semi-empirical result given by Hanna (1972) which expresses VR^2 as a function of plume rise. Using the vertical momentum equation one can then write VR^2 as a function of t and so eliminate this variable from the system. This leaves three equations in three unknowns (θ, σ, T_p) which, after some algebraic manipulation, allow Eq. (12) to be written as a nonlinear first-order differential equation for σ:

$$\frac{d\sigma}{dt}=f(\sigma,t). \tag{13}$$

The details are given in the Appendix.

TABLE 2. Plume parameters used in the solution.

Plume type	T_{a0} (°C)	T_{p0} (°C)	w_0 (m s^{-1})	R_0 (m)	α
Jet	10	50	10	0.01	0.1
Scrubbed plume	10	50	20	3	0.3
Scrubbed plume	−10	50	20	3	0.3
Cooling tower	10	27.5	2.5	30	0.3

Eq. (13) can easily be solved on a computer. However, it contains a large number of "arbitrary" parameters which are determined by the environment conditions and by the initial conditions of the plume. These are: the initial plume radius (R_0), efflux speed (w_0), temperature excess (ΔT_0), specific humidity excess (Δq_0), condensation nucleus density (N_0) and mean radius ($\bar{r}_0$); the environment temperature at efflux (T_{a0}) and specific humidity (q_a), the wind speed (U) and the environment condensation nucleus density (N_a); and the value of the initial phase entrainment parameter (α). Even though the problem has been simplified by choosing a well-mixed (i.e., neutral and constant specific humidity) environment, it is obviously not possible to discuss in detail more than a few specific, and representative, examples.

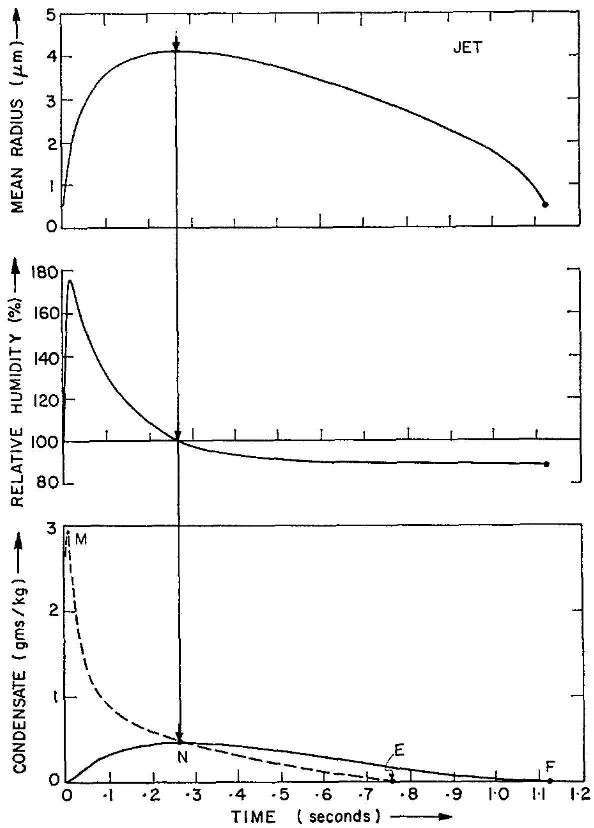

FIG. 2. Mean droplet radius, relative humidity and condensate content for a condensing jet. The letters M, N, E and F are keyed to Fig. 1. The dashed line in the lower diagram is the condensate content which would be expected if supersaturation were ignored (σ^* in the text). The maximum value of the true condensed water content (σ in the text) occurs at N where $\sigma=\sigma^*$. The maximum mean droplet radius is seen to occur at this point also, and the jet is supersaturated prior to it and undersaturated after it.

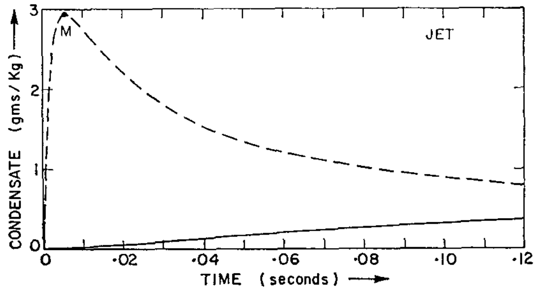

FIG. 3. Condensate content for a condensing jet using an expanded time scale to show the difference between the equilibrium theory (dashed curve) and the present theory (bold curve).

Four examples have been chosen as typical of a jet, a scrubbed industrial plume (at two different environment temperatures), and a natural draft cooling tower. Because these span a wide range of the time scale for condensation moisture availability (τ_p), the variable t was non-dimensionalized by using the initial value of τ_p. The numerical solution of Eq. (13) was accomplished using a Runge-Kutta program and incorporating a variable step length. In all cases the condensation nucleus densities, N_0 and N_a, were taken to be 2000 cm^{-3} and the mean condensation nucleus radius $\bar{r}_0$ was taken as 0.5 μm. The environment conditions were specified as $T_{a0}=10$°C, relative humidity at the efflux point=70%, and wind speed=5 m s^{-1}. The scrubbed plume case was also run for an environ-

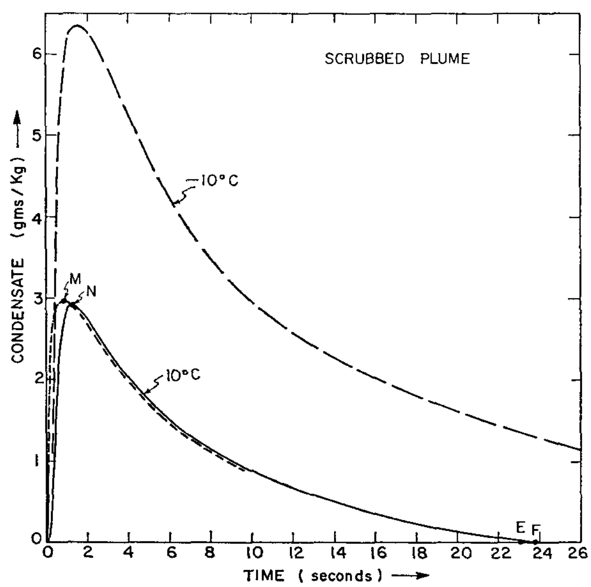

FIG. 4. Condensate content for a typical condensing scrubbed industrial plume at two different environment temperatures. The letters M, N, E and F are keyed to Fig. 1. The effect of supersaturation is much less pronounced than in the case of a jet.

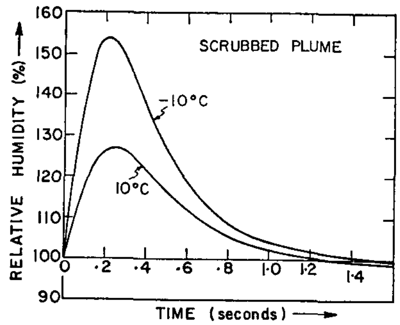

FIG. 5. Supersaturation levels in a condensing scrubbed industrial plume for environment temperatures of 10 and −10°C. The levels are very high and greater at lower temperatures, but supersaturation is only maintained for a time of order 1 s after efflux. Beyond this point the buffering effect of evaporating droplets maintains a high relative humidity (around 99%) throughout the condensed section of the plume.

ment temperature of −10°C. In the examples considered here it has been assumed for simplicity that the plumes were saturated on efflux. Other initial plume parameters are given in Table 2.

The results are shown in Figs. 2–7. These figures have two noticeable features in common: they all show that an appreciable level of supersaturation occurs and that the maximum condensed water content is reached very rapidly.

Supersaturation is illustrated using a "mean" relative humidity defined by the ratio q_p to q_{sp} expressed as percent. For the jet (Fig. 2) a maximum of over 170% is reached in less than 0.02 s. For the plume (Fig. 5), although a somewhat lower peak supersaturation is attained [the actual value being quite sensitive to the environment temperature (or more correctly to ΔT_0)], the values are still high. For the cooling tower the peak relative humidity is 101.4%

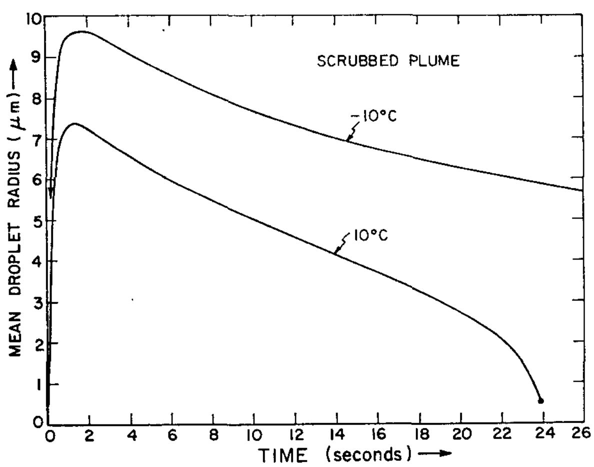

FIG. 6. Mean droplet radius in a condensing scrubbed industrial plume for environment temperatures of 10 and −10°C. The maximum droplet size is achieved very rapidly due to the high level of supersaturation in the plume close to the efflux point.

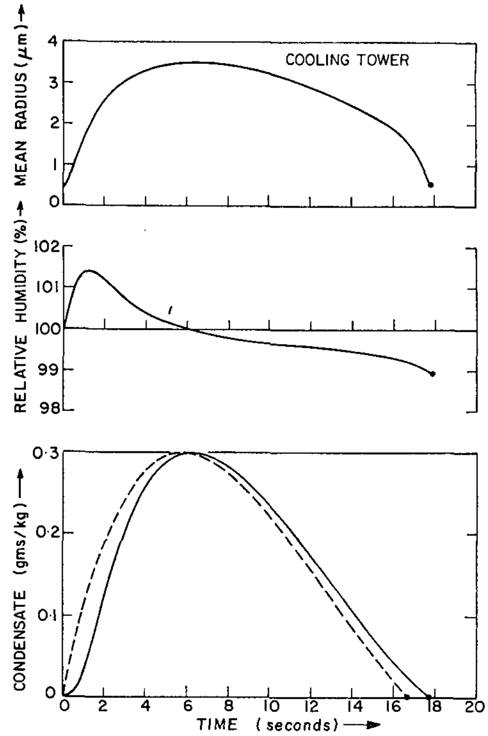

FIG. 7. Mean droplet radius, relative humidity and condensed water content in a typical natural-draft cooling tower plume. The dashed condensate curve corresponds to the equilibrium theory prediction. This example corresponds to a fairly short visible plume; considerably larger droplets and higher supersaturations can be attained in plumes under more extreme atmospheric conditions.

(Fig. 7), but this value is also quite sensitive to environment temperature attaining a value of over 105% for $T_{a0} = -10$°C. All of these values are much greater than those predicted (by the present and by more elaborate theories) and observed in natural clouds. In the evaporating portions of the scrubbed plume and cooling tower case relative humidity remains high and close to 100% until the plume's condensed portion disappears. The evaporating droplets act as a buffer to compensate for the mixing of drier environment air.

Condensate content is found to be quite dependent on whether or not supersaturation is included in the theory, especially for small times. This is very marked in the case of a jet (see Figs. 2 and 3). If supersatura-

TABLE 3. The effect of environment variables on scrubbed plume properties.

T_{a0} (°C)	RH (%)	σ_{max} (g kg^{-1})	$\bar{r}_{max}$ (μm)	RH_{max} (%)	Length (s)	Length (m)
10	70	2.91	7.36	127.1	23.9	120
10	90	3.09	7.50	127.8	69.5	348
−10	70	6.34	9.62	153.8	114.0	570
−10	90	6.41	9.65	154.0	314.5	1573

tion is ignored (i.e., using the equilibrium theory), the maximum condensate content is approximately 3 g kg^{-1} and occurs only 0.006 s after efflux. The effect of including supersaturation is to reduce this maximum by a factor of 6 and to displace it to occur about 0.3 s after efflux. In addition, the approximate equilibrium theory underestimates the length (duration) of the condensed portion of the jet by almost 50%. These same features are also observed in the scrubbed plume (Fig. 4) and cooling tower (Fig. 7) cases, but to a much lesser degree. If one moves sufficiently far away from the efflux point, the plume is able to "forget" its relatively brief period of supersaturation. Other factors (such as environment stability or moisture profile) are therefore expected to be much more important in predicting plume length or condensed water content than is the consideration of supersaturation for these cases.

Figs. 2, 6 and 7 show the variation in mean droplet radius for the jet, scrubbed plume and cooling tower plume. In each case the *mean* droplet size remains below the limit at which growth by coalescence becomes important (~20 μm). These mean values are, of course, only representative of a *spectrum* of droplet sizes so that some droplets will be considerably larger than the means. Even so, it is doubtful that coalescence can be an important factor except under extreme environment conditions so that its neglect in the preceding development is justified *a posteriori.*

Fig. 6 shows the effect of changing ΔT_0 (through T_{a0}) on droplet size for a scrubbed plume: a 20°C increase in ΔT_0 causes a 30% increase in peak mean radius. The effect of changing environment relative humidity (from 70% to 90%) was also examined and found to be very small for a scrubbed plume although it has a marked effect on plume length (see Table 3). For a cooling tower plume the effect on droplet size and condensate content is somewhat greater; an increase to 90% relative humidity causing an increase of 20% in the peak mean radius and 65% in the maximum condensate content.

In the examples considered here it is extremely doubtful that any droplets would reach a size large enough for them to rain out of the plumes. For scrubbed industrial plumes it appears marginally possible that rain-out may occur under extreme atmospheric conditions. There still remains the possibility that droplets which were in the plume at time zero due to spill-over or carry-over could contribute to rain-out. (These conclusions are fairly generally accepted.) With the present theory it is a relatively simple procedure to compute the growth and trajectory of a "test" spill-over droplet, since the supersaturation profile down the plume is now known. For "salt drift" droplets the buffering effect of other evaporating droplets may be quite significant. Droplets of high ionic strength can continue to grow in relative humidities of as low as 99%, a value which is exceeded throughout most of the condensed portion of a cooling tower plume. No attempt has been made to study this important problem here, although the present development provides a relatively straightforward and self-consistent basis for further work.

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APPENDIX

Derivation of Eq. (13)

A number of approximations have been made in developing this theory. Not the least of these are the use of a one-dimensional model and the neglect of environmental stability and moisture profile effects (although the latter is certainly acceptable for the case of a condensing jet). In addition I have restricted discussion to the bent-over case for which there is no theory available which describes plume behavior both close to and far from the point of efflux.

In the spirit of these points, Eqs. (5), (6) and (12) [involving five time-dependent variables (VR^2, θ, σ T_p, N)] are solved by employing an empirical approximation for VR^2 based on the work of Hanna (1972). In his work on cooling towers, Hanna suggests the relation

$$VR^2=w_0R_0^2\left[1+\frac{\alpha}{R_0}\left(\frac{U}{w_0}\right)^{\frac{1}{2}}z\right]^2 \tag{A1}$$

to describe VR^2 as a function of plume rise z, under neutral conditions for a bent-over buoyant plume. Here α is an entrainment parameter appropriate to the initial phase of plume rise. This result is particularly useful because it is consistent for both large and small values of z; in other words, it has the correct limiting form for these extreme cases.

To obtain VR^2 as a function of time it is necessary to use an integrated form of the vertical momentum equation. The latter is

$$\frac{d}{dt}(VR^2w)=gw_0R_0^2\frac{\Delta T_0}{T_{a0}}=F_0, \tag{A2}$$

which integrates to

$$VR^2w=R_0^2w_0^2+F_0t \tag{A3}$$

provided F is constant (which it is in a neutral environment). VR^2 can now be eliminated from (A1) and (A3) to obtain w (i.e., dz/dt) as a function of z and t. This first-order equation can, in turn, be integrated to yield

$$\left[1+\frac{\alpha}{R_0}\left(\frac{U}{w_0}\right)^{\frac{1}{2}}z\right]^3=1+\frac{3\alpha}{R_0}\left(\frac{U}{w_0}\right)^{\frac{1}{2}}\left(w_0t+\frac{F_0t^2}{2w_0R_0^2}\right). \quad \text{(A4)}$$

Back-substitution into (A1) gives VR^2 as a function of time as required:

$$VR^2=w_0R_0^2\left[1+\frac{3\alpha}{R_0}\left(\frac{U}{w_0}\right)^{\frac{1}{2}}\left(w_0t+\frac{F_0t^2}{2w_0R_0^2}\right)\right]^{\frac{2}{3}}. \quad \text{(A5)}$$

Furthermore, the droplet number density can be written directly as a function of VR^2. If spontaneous nucleation is neglected, and if N_a is the environment condensation nucleus density, then the quantity $VR^2(N-N_a)$ should be conserved. Hence

$$N=N_a+\frac{(N_0-N_a)w_0R_0^2}{VR^2}, \quad \text{(A6)}$$

where N_0 is the initial condensation nucleus density in the plume.

Eqs. (A5) and (A6) now allow VR^2 and N to be eliminated from (5), (6) and (12) to leave three equations in three unknowns. The resulting equations are simple in principle, but algebraically fairly complicated. Eq. (6) may be considered as relating T_p, σ and t, i.e.,

$$T_p=T_p(\sigma,t). \quad \text{(6,A7)}$$

Also, since q_{sp} is a function of T_p and pressure, and since pressure is determined by plume rise and hence, through (A4), by time, Eq. (5) may be written in the form

$$\theta=\theta(T_p,\sigma,t). \quad \text{(5,A8)}$$

[For computational purposes an accurate formula for saturation vapor pressure, such as that given by Richards (1971), must be used.] Equations (6,A7) and (5,A8) can be combined to give θ as a function of σ and t alone:

$$\theta=\theta(\sigma,t). \quad \text{(A9)}$$

Returning to (12), this can be written symbolically as

$$\frac{d\sigma}{dt}=f(k^*,N,\theta,q_{sp},\rho_p,\sigma).$$

Here k^* is a function of T_p (see Table 1) which is a function of σ and t through (6,A7), N can be written as a function of t using (A6) and (A5), $\theta=\theta(\sigma,t)$ is given by (A9), and both q_{sp} and ρ_p are known functions of T_p and pressure (or plume rise) and so can be expressed in terms of σ and t using (6,A7) and (A4). Thus, after considerable algebraic manipulation, Eq. (12) may be reduced to the form

$$\frac{d\sigma}{dt}=f(\sigma,t).$$

This may be solved extremely rapidly using a Runge-Kutta technique on a digital computer.

REFERENCES

Blum, A., 1948: Drizzle precipitation from water cooling towers. *The Engineer*, **102**, 128–130.
Brunt, D., 1935: Condensation by mixing. *Quart. J. Roy. Meteor. Soc.*, **61**, 213–217.
Byers, H. R., 1965: *Elements of Cloud Physics*. The University of Chicago Press, 191 pp.
Csanady, G. T., 1971: Bent-over vapor plumes. *J. Appl. Meteor.*, **10**, 36–42.
Fletcher, N. H., 1962: *The Physics of Rainclouds*. Cambridge University Press, 390 pp.
Hanna, S. R., 1972: Rise and condensation of large cooling tower plumes. *J. Appl. Meteor.*, **11**, 793–799.
Hidy, G. M., and S. K. Friedlander, 1964: Vapor condensation in the mixing zone of a jet. *AIChE J.*, **10**, 115–124.
Morton, B. R., 1957: Buoyant plumes in a moist atmosphere. *J. Fluid Mech.*, **2**, 127–144.
——, G. I. Taylor and J. S. Turner, 1956: Turbulent gravitational convection from maintained and instantaneous sources. *Proc. Roy Soc. London*, **A234**, 1–23.
Overcamp, T. J., and D. P. Hoult, 1971: Precipitation in the wake of cooling towers. *Atmos. Environ.*, **5**, 751–765.
Richards, J. M., 1971: Simple expression for the saturation vapor pressure of water in the range −50° to 140°. *J. Phys.*, **D4**, L15–L18.
——, 1973: The stability of wet and dry bent-over plumes. *J. Appl. Meteor.*, **12**, 133–139.
Slawson, P. R., and G. T. Csanady, 1967: On the mean path of buoyant, bent-over chimney plumes. *J. Fluid Mech.*, **28**, 311–322.
Squires, P., and J. S. Turner, 1962: An entraining jet model for cumulonimbus updraughts. *Tellus*, **14**, 422–439.
Wigley, T. M. L., and P. R. Slawson, 1972: A comparison of wet and dry bent-over plumes. *J. Appl. Meteor.*, **11**, 335–340.