Steady-State Solution of the Semi-Empirical Diffusion Equation for Area Sources

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ABSTRACT

Turbulent transport of material emitted from a surface may be described by the steady-state, two-dimensional, semi-empirical diffusion equation. It is shown that, with wind velocity and eddy diffusivity expressed as power functions of the vertical coordinate, this equation can be solved exactly by introducing a similarity variable. The solution gives the vertical distribution of concentration for area sources in terms of the incomplete gamma function. Implications of the solution are discussed.

1. Introduction

In steady conditions transport of gases in the atmosphere may be described by

$$u(z)\frac{\partial c}{\partial x}=\frac{\partial}{\partial z}K(z)\frac{\partial c}{\partial z} \tag{1}$$

if the effect of diffusion in the horizontal direction is neglected. Here u is the turbulent-mean wind, assumed to be in the x direction, K the coefficient of eddy diffusivity in the vertical direction, and $c(x,z)$ the concentration of the transported substance. The z dependence of the wind and the diffusivity may be represented by empirically determined power laws:

$$u(z)=u_0z^\alpha, \tag{2}$$

$$K(z)=K_0z^\beta. \tag{3}$$

Eq. (1) has been solved previously with these coefficients, for boundary conditions appropriate to dispersion of effluents from point and line sources. References to these works may be found in Monin and Yaglom (1971). An application of this equation to diffusion over areas has been made by O. G. Sutton (1934) for estimating evaporation from a water surface when surface-concentration is specified. A detailed discussion of the equation with various boundary conditions has been given by W. G. L. Sutton (1943).

Eq. (1) may be used also for describing the dispersion of effluents from a uniform area source by using the boundary condition

$$K(z)\frac{\partial c}{\partial z}=-Q, \quad z=0, \tag{4}$$

where Q is the flux from the surface. This is of much interest because of the need for calculating the flow of air pollution in urban atmospheres. The steady-state equation can be useful in predicting air pollution dispersion over short periods when Q, u and K may be approximately invariant; or it can be applied to the calculation of long-term averages of contaminant concentrations if meaningful averages of Q, K and u can be obtained. An approximate solution of Eqs. (1)–(4) has been given, and extensively applied, by Gifford and Hanna (1971). The problem has been solved numerically by Ragland (1973) who used different velocity profiles. Another approximate solution of the same problem has been obtained by an integral method by Lebedeff and Hameed (1975).

In this paper we show that the problem represented by Eqs. (1)–(4) can be solved exactly in a simple manner if the equations are written in terms of a similarity variable. We also use the exact solution to explore two questions of interest in modelling of urban air pollution, *viz.* the distribution of surface concentration downwind of an area source, and the distribution of concentration with height.

2. Solution of the diffusion equation

We consider a uniform area source extending from $x=0$ to $x=\infty$, with a constant flux Q given by (4). The boundary condition at the upwind edge of the source is

$$c(x,z)=0, \quad x=0, \tag{5a}$$

and for large values of z

$$c(x,z)=0, \quad z\rightarrow\infty. \tag{5b}$$

The solution of the problem represented by Eqs. (1)–(5) is facilitated if we make the following ob-

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servations. First, we note that the problem is linear in Q and the solution should, therefore, also be linear in Q. Consideration of the lower boundary condition (4) then implies that the solution c is of the form

$$c(x,z)=\frac{Q}{K_0}z^{1-\beta}\phi(x,z), \tag{6}$$

where ϕ is a dimensionless function, independent of Q, which satisfies the equations:

$$u_0 z^{1+\alpha-\beta}\frac{\partial\phi(x,z)}{\partial x}=K_0\frac{\partial}{\partial z}z^{\beta}\frac{\partial}{\partial z}[z^{1-\beta}\phi(x,z)], \tag{7}$$

$$z^{\beta}\frac{\partial}{\partial z}[z^{1-\beta}\phi(x,z)]=-1,\quad z=0, \tag{8}$$

$$\phi(x,z)=0,\quad x=0, \tag{9a}$$

$$\phi(x,z)=0,\quad z\to\infty. \tag{9b}$$

Let us now consider the scaling transformations

$$x=\delta x',\quad z=\lambda z',$$

where δ and λ are constants. Eq. (7) then becomes

$$\frac{\lambda^{2+\alpha-\beta}}{\delta}u_0 z'^{1+\alpha-\beta}\frac{\partial}{\partial x'}\phi(\delta x',\lambda z') =K_0\frac{\partial}{\partial z'}z'^{\beta}\frac{\partial}{\partial z'}[z'^{1-\beta}\phi(\delta x',\lambda z')]. \tag{10}$$

Now, if we choose $\delta=\lambda^{2+\alpha-\beta}$, then $\phi(\delta x,\lambda z)$ satisfies the same equations (7)–(9) as $\phi(x,z)$. Since the solution to these equations is unique, it follows that

$$\phi(\lambda^{2+\alpha-\beta}x,\lambda z)\equiv\phi(x,z). \tag{11}$$

Since λ is an arbitrary constant Eq. (11) can hold if, and only if, ϕ contains x and z in the combination $(z^{2+\alpha-\beta}/x)$. We therefore define the dimensionless variable

$$\omega=\frac{u_0}{K_0}\frac{z^{2+\alpha-\beta}}{x} \tag{12}$$

and write the equations for ϕ in terms of w. Eq. (7) then reduces to the ordinary differential equation

$$\phi''(\omega)+\left[\frac{(1+\nu)}{\omega}+\frac{1}{(2+\alpha-\beta)^2}\right]\phi'(\omega)=0, \tag{13}$$

where

$$\nu=\frac{1-\beta}{2+\alpha-\beta}.$$

This equation may be solved immediately to get

$$\phi'(\omega)=\phi_0\frac{1}{\omega^{1+\nu}}\exp\left[-\frac{\omega}{(2+\alpha-\beta)^2}\right], \tag{14}$$

where ϕ_0 is a constant of integration. We integrate this equation and satisfy the asymptotic boundary condition [Eq. (5b)] to obtain

$$\phi(\omega)=\frac{\phi_0}{(2+\alpha-\beta)^{2\nu}}\Gamma\left[-\nu,\frac{\omega}{(2+\alpha-\beta)^2}\right], \tag{15}$$

where Γ is the incomplete gamma function (Jahnke *et al.*, 1960).

The constant ϕ_0 is chosen to satisfy the lower boundary condition [Eq. (8)]. In terms of w, this condition may be written as

$$(2+\alpha-\beta)\omega^{1-\nu}\frac{d}{d\omega}[\omega^{\nu}\phi(\omega)]=-1,\quad \omega=0. \tag{16}$$

We now substitute Eq. (15) in Eq. (16) and use the expansion for the incomplete gamma function:

$$\Gamma(a,y)=\Gamma(a)-y^a\sum_{n=0}^{\infty}\frac{(-1)^n y^n}{n!(a+n)},\qquad a\neq 0,-1,-2,\cdots. \tag{17}$$

This gives, in the limit $\omega\to 0$,

$$\phi_0=\frac{(2+\alpha-\beta)^{2\nu-1}}{\Gamma(1-\nu)}. \tag{18}$$

Hence,

$$\phi(\omega)=\frac{1}{(2+\alpha-\beta)\Gamma(1-\nu)}\Gamma\left[-\nu,\frac{\omega}{(2+\alpha-\beta)^2}\right]. \tag{19}$$

Substituting this in Eq. (6) gives us the required solution:

$$c(x,z)=\frac{Q}{K_0}\frac{z^{1-\beta}}{(2+\alpha-\beta)\Gamma(1-\nu)}\Gamma\left[-\nu,\frac{\omega}{(2+\alpha-\beta)^2}\right]. \tag{20}$$

In the study of dispersion of air pollution one is particularly interested in the concentration of the pollutant near the surface, $z=0$. This may be obtained by substituting the expansion (17) in (20) and taking the limit $z\to 0$. We find that for $\beta<1$ the concentration on the ground is

$$c(x,0)=\frac{Q}{K_0}\frac{(2+\alpha-\beta)^{2\nu-1}}{\nu\Gamma(1-\nu)}\left(\frac{K_0 x}{u_0}\right)^{\nu},\quad \beta<1. \tag{21}$$

For $\beta\geqslant 1$ the concentration at $z=0$ is infinite.

In the derivation given above we have assumed the area source to extend from $x=0$ to $x=\infty$ with a constant Q. It can be generalized directly to the case of a variable source with strength $Q(x)$ for

$x_0 < x < x_1$. The solution (20) is then replaced by

$$c(x,z)=\frac{1}{K_0}\frac{z^{1-\beta}}{(2+\alpha-\beta)\Gamma(1-\nu)}\times\int_0^{(x_1-x_0)}\Gamma\left[-\nu,\frac{\omega}{(2+\alpha-\beta)^2}\right]dQ(x_1-x), \quad (22)$$

and for concentration on the surface we obtain

$$c(x,0)=\frac{1}{K_0}\frac{(2+\alpha-\beta)^{2\nu-1}}{\nu\Gamma(1-\nu)}\left(\frac{K_0}{u_0}\right)^\nu\times\int_0^{(x_1-x_0)}x^\nu dQ(x_1-x). \quad (23)$$

The integrals in (22) and (23) are defined as Riemann-Stieltjes integrals.

We have noted that for $\beta \geqslant 1$, the concentration at $z=0$ becomes infinite. In the discussion in the following section, therefore, we consider the range $0 \leqslant \beta < 1$, which means that the parameter ν takes values between 0 and 0.5, for positive α.

3. Discussion of the solution

We now discuss the nature of the solution to the diffusion equation obtained in the previous section. This is of particular interest in the calculation of dispersion of contaminants in urban areas which are characterized by grid-like patterns of area sources.

a. Distribution of surface concentration in the wind direction

In air pollution studies values of concentration on the ground at various locations in a region are of primary interest. Ground concentration within the area source is given by Eq. (21). To obtain the concentration at points outside the source, we add to the equation the solution for a source of strength $(-Q)$ for $x>L$, and obtain

$$c(x)=\frac{Q}{K_0}\frac{(2+\alpha-\beta)^{2\nu-1}}{\nu\Gamma(1-\nu)}\left(\frac{K_0}{u_0}\right)^\nu[x^\nu-(x-L)^\nu], \quad x>L. \quad (24)$$

We have plotted $c(x)/C(L)$ as a function of x in Fig. 1, for $\nu=0.45$ (solid curve) and $\nu=0.05$ (dashed curve). We note that the distribution of ground concentration is sensitive to ν.

For small ν (i.e., $\beta \approx 1$) the concentration rises quickly from the upwind edge ($x=0$) and its gradient in the source region ($0<x<L$) is small. Immediately outside the source there is a very sharp drop in concentration, such that $c(2L)=0.035\ c(L)$. Thus, the assumptions of the box model that the distribution be uniform horizontally in the source region, and zero outside, hold good when ν approaches zero.

For large ν (i.e., $\beta \approx 0$) the rise of concentration from the upwind edge is relatively more gradual in the source region. For $x>L$, although initially the decrease in concentration is considerable $[c(2L)=0.37\ c(L)]$, the gradient of the curve becomes very small for larger values of x. This slow decrease is important because it means that in a grid of area sources, such as a city, distant successive sources add nearly equal amounts to concentration at a point. In this case, therefore, the total contribution of upwind sources may add up to be comparable to the contribution of the local source. In fact, it may be shown, for an array of identical sources, that the contributions to average ground concentration over an area by the upwind sources and the local source are related by the expression

$$\frac{\text{contribution of } N \text{ upwind area sources}}{\text{contribution of local area source}}=(1+\nu)N^\nu-1.$$

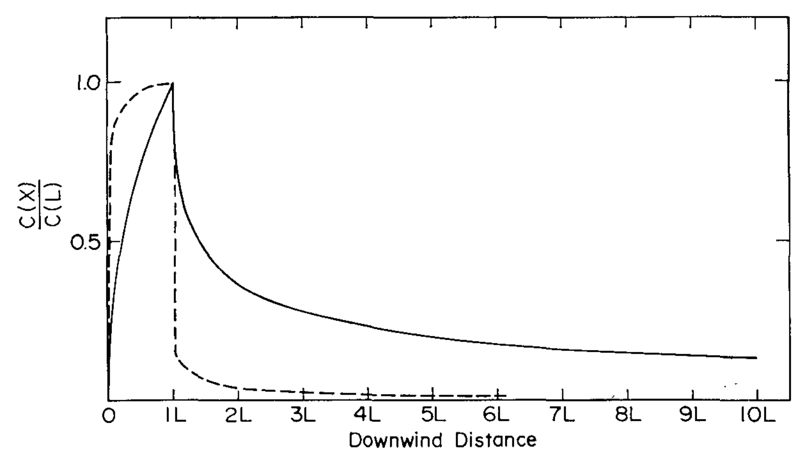

FIG. 1. Distribution of surface concentration along the wind direction. The source extends from $x=0$ to $x=L$. The solid curve is for $\nu=0.45$ ($\beta \approx 0$, i.e., diffusivity nearly constant with height), and the dashed curve is for $\nu=0.05$ ($\beta \approx 1$, i.e., diffusivity increasing approximately linearly with height).

b. Vertical distribution of concentration

Vertical distribution of pollutant concentration over an area source is of interest in air pollution studies. In the absence of knowledge of the vertical profile of concentration, ad hoc assumptions about the profile and its extent, i.e., the "mixing depth," have been introduced in most dispersion calculations in the literature. In the solution of the diffusion equation obtained above, the vertical profile is given in terms of the incomplete gamma function of the similarity variable $\omega=(u_0/K_0)(z^{2+\alpha-\beta}/x)$. The functional dependence of the solution on z may be conveniently represented by

$$\frac{c(x,z)}{c(x,0)}=\nu\xi^{\nu}\Gamma(-\nu,\,\xi),\quad \xi=\frac{\omega}{(2+\alpha-\beta)^2},\qquad (25)$$

where $c(x,0)$ is the concentration at $z=0$. This function is displayed in Fig. 2 for $\nu=0.45$, 0.32, 0.08 and 0.05. We can see that the decrease in concentration near $\xi=0$, i.e., $z=0$, becomes very sharp as ν approaches zero. For instance, $c(x,z)$ becomes half of $c(x,0)$ at $\xi=0.058$ for $\nu=0.32$, and at $\xi=0.51\times10^{-6}$ for $\nu=0.05$. To illustrate the difference in the profile for different values of ν we consider the case of $\beta=1-\alpha$ and assume that the magnitudes of u_0 and K_0 are equal. Then at a downwind distance of 10 km, $c(x,z)$ becomes half of the ground concentration $c(x,0)$ at

$$\left.\begin{array}{llll} z=20 & \text{m}, & \text{for} & \nu=0.32\ (\alpha=0.90\ ,\beta=0.10)\\ z=\ 0.01 & \text{m}, & \text{for} & \nu=0.05\ (\alpha=0.056,\beta=0.944)\end{array}\right\}.$$

It is clear that for $\nu\approx0$, that is, $\beta\approx1$, the solution predicts that almost all of the concentration is located at $z=0$. In such a case $K(z)$ near the surface approaches zero so rapidly that vertical diffusion becomes negligible and the emitted material accumulates near the surface.

c. Application of the equation

In application of the diffusion equation one needs the parameters defining the wind velocity and the diffusivity given in (2) and (3). A power law can usually be fitted to measured values of wind velocity as a function of height. Eddy diffusivity may be obtained by the use of boundary layer theory and also approximated by a power law. Two interesting semi-empirical methods for obtaining $u(z)$ and $K(z)$ as power laws are given by Gee (1966) and Vaughan (1961).

Another possibility is that, if measurements of wind velocity and pollutant concentrations are available, $K(z)$ may be obtained from the solution of the diffusion equation. Parameters defining $K(z)$ may be obtained in this manner for different classes of atmospheric stability. These may then be used later to calculate pollutant concentrations.

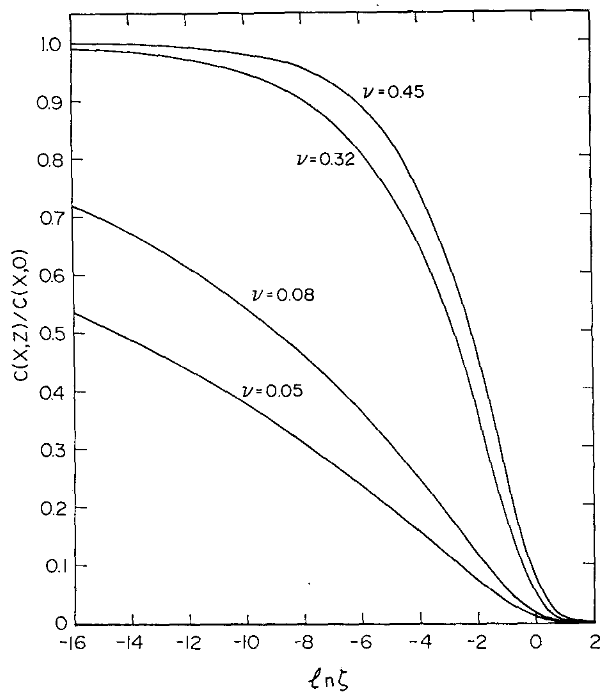

FIG. 2. Vertical distribution of concentration for a given x. The horizontal coordinate is

$$\xi=\frac{u_0}{K_0}\frac{1}{(2+\alpha-\beta)^2}\frac{Z^{2+\alpha-\beta}}{x}.$$

Note that as ν approaches zero (i.e., $\beta\rightarrow1$) the concentration drops to nearly half of the ground concentration for extremely small values of ξ.

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