# A Numerical Analysis of the Effect of Condensation on Plume Rise

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ABSTRACT

This paper presents the results of a numerical solution of the equations of moist plume rise and compares the trajectories of wet and dry cooling tower and scrubbed industrial plumes under a wide range of atmospheric stability conditions. Similar comparisons have been made previously by Wigley and Slawson, Hanna, and Weil, and the results of these authors are discussed. It is found that their results are all qualitatively correct, but that there are important quantitative differences between their results and the numerical solution. Previous approximate analytic results have shown that the critical lapse rate for the transition to unstable plume behavior for wet plumes is close to the saturated adiabatic lapse rate. The more-complete numerical solution confirms this result when one allows for the variation of the saturated adiabatic lapse rate with height. The approximate analytic formula for the maximum height of rise of dry plumes is also examined and found to overestimate plume rise by 6–20% when compared with the numerical solution.

## 1. Introduction

Under stable atmospheric conditions, buoyant industrial plumes have a maximum height of rise which depends on the lapse rate and the plume's initial flux of buoyancy. The form of this dependence can be predicted using the one-dimensional entrainment theory for plume behavior, and the predictions agree quite well with observations. When one applies these results to "wet" (condensed) plumes, such as cooling tower plumes and scrubbed industrial plumes, the effects of latent heat release (during condensation) and gain (during evaporation) must be considered. Such effects may significantly change plume path and, in particular, maximum height of rise.

A number of papers have been published recently describing various aspects of wet plume behavior (Csanady, 1971; Wigley and Slawson, 1971, 1972; Hanna, 1972; Richards, 1973; Weil, 1974). These papers are largely based on earlier work by Morton (1957). An attempt to compare the behavior of wet and dry plumes was made by Wigley and Slawson (1972; henceforth referred to as WS). They used the entrainment theory approach and suggested that, as an approximation, the effect of condensation could be allowed for by using the dry plume equations, but replacing the Väisälä frequency

$$N \approx \left[\frac{g}{T_a}(\Gamma_{ad}-\Gamma)\right]^{\frac{1}{2}}$$

by a saturated Väisälä frequency

$$N_s \approx \left[\frac{g}{T_a}(\Gamma_{as}-\Gamma)\right]^{\frac{1}{2}}$$

in the energy equation. In the above, $g$ is the gravitational acceleration, $T_a$ the ambient temperature, and $\Gamma$, $\Gamma_{ad}$ and $\Gamma_{as}$ are the environment, dry and saturated adiabatic lapse rates. Briggs and Hanna (1972) criticized this procedure, but in later work by Richards (1973) and Weil (1974) wet-plume results very similar to those predicted by Wigley and Slawson's approximation were derived.

Hanna (1972) suggested an alternative method for finding the maximum rise of wet plumes. Although Hanna modified the governing equations, his method also involved changing the initial conditions (specifically, the initial flux of buoyancy) to incorporate condensation effects. As a first approximation he supposed that all of the initial excess vapor in the plume condenses. He was able to improve on this using an iterative scheme which allowed either more or less than the initial excess vapor to condense.

Both the Hanna (1972) and WS methods showed that a wet plume would rise higher than a corresponding dry plume under the same atmospheric conditions, and that the increase was more pronounced for greater atmospheric lapse rates. The WS method indicated that a wet plume would become unstable if the lapse rate exceeded the saturated adiabatic lapse rate. In addition, the WS method allowed one to compare wet and dry plume trajectories.

In a recent paper, Weil (1974) has examined this question in more detail. His results are very similar to WS (see discussion below and also Fig. 2). By using some approximations in order to obtain an analytic solution Weil finds that under most conditions the critical lapse rate for the transition to unstable plume behavior is slightly greater than the saturated adiabatic lapse rate. The same result was obtained by Richards (1973). Because of the assumptions made by Weil, his results are not much improvement on those of WS, although his analysis and analytic results are a valuable contribution to our understanding of plume dynamics. In order to examine the effect of condensation on plume rise more completely I have solved the complete set of entrainment theory plume equations numerically, eliminating the approximations made by Weil. The results are presented and discussed below.

## 2. Plume equations and solution

The equations describing moist plume dynamics and growth in an environment of arbitrary structure, but containing no condensed water, may be written in the form:

$$\frac{d}{dt}(VR^2)=\frac{2v_e}{R}VR^2 \tag{1}$$

$$\frac{d}{dt}[VR^2(\Delta q+\sigma)]=-GVR^2w \tag{2}$$

$$\frac{d}{dt}\left[VR^2g\left(\frac{\Delta T^*}{T_a{}^*}-\frac{L\sigma}{C_pT_a{}^*}\right)\right]=-N^2VR^2w \tag{3}$$

$$\frac{d}{dt}(VR^2w)=gVR^2\left(\frac{\Delta T^*}{T_a{}^*}-\sigma\right) \tag{4}$$

$$\frac{d}{dt}(VR^2v_x)=U\frac{d}{dt}(VR^2) \tag{5}$$

$$\frac{dz}{dt}=w \tag{6}$$

$$\frac{dx}{dt}=v_x. \tag{7}$$

The terms are identified as follows:

| | |
|---|---|
| $C_p$ | specific heat capacity at constant pressure |
| $G$ | atmospheric specific humidity gradient |
| $L$ | latent heat |
| $q$ | specific humidity |
| $R$ | plume radius |
| $T$ | temperature (°K) |
| $U$ | wind speed |
| $v_e$ | entrainment velocity |
| $V, v_x, w$ | centerline, horizontal and vertical plume speeds |
| $x, z$ | horizontal and vertical plume coordinates |
| $\sigma$ | liquid water content. |

*Subscripts*

$a$ atmospheric
0 initial
$p$ plume
$s$ saturated

*Superscripts*

* virtual temperature

The prefix $\Delta$ indicates the excess of a plume variable over its atmospheric value. The independent variable ($t$) is the time of travel along the centerline plume path so that $ds=Vdt$ is the distance increment along the plume path. The equations can be closed by using the equilibrium assumption that $q_p=q_{sp}$ if $\sigma>0$; non-equilibrium effects (i.e., including supersaturation as an additional variable) have been discussed by Wigley (1975). Eqs. (1)–(7), except for minor differences, are formally equivalent to those used by Weil (1974). Solid-body drag has not been considered in the horizontal momentum equation [Eq. (5)] and the possibility of downwash has been neglected.

These equations were solved numerically using a fifth-order Runge-Kutta method incorporating a variable step length. A standard library program was used. I assumed that plume dispersion was in the initial phase (see, for example, Slawson and Csanady, 1971) where

$$v_e=\alpha|w|$$

and used a value 0.3 for the entrainment parameter $\alpha$. This procedure is consistent with Weil (1974) and earlier authors, but is questionable beyond the maximum height of rise since the plume should enter an atmospheric-turbulence-dominated phase at or before this point. In solving these equations I computed saturated specific humidity using the extremely accurate formula given by Richards (1971; see also Wigley, 1974).

I considered two specific examples corresponding to a typical natural-draft cooling tower ($R_0$= 30 m, $w_0$= 3 m s$^{-1}$, $T_{p0}$= 25°C) and scrubbed industrial plume ($R_0$= 3 m, $w_0$= 20 m s$^{-1}$, $T_{p0}$= 55°C) in an atmosphere with constant wind speed of 7 m s$^{-1}$, constant relative humidity and constant lapse rate. Lapse rates were varied from −0.01 to 0.009°C m$^{-1}$. To characterize the wet and dry plume types I used extreme conditions of $\mathrm{RH}_{p0}=\mathrm{RH}_{a0}=100\%$ and $\mathrm{RH}_{p0}=\mathrm{RH}_{a0}=0\%$ (where RH is relative humidity). The atmospheric temperature was taken as 0°C at plume efflux point (i.e., $x=z=t=0$). The results are shown in Figs. 1 to 4.

## 3. Discussion and conclusions

In Fig. 1, as a typical example, I have compared the trajectories of a dry and a wet cooling tower plume in an isothermal atmosphere. The result is almost identical to that given by Weil (1974, Fig. 3) and also to that given earlier by Wigley and Slawson (1972, Fig. 1c). The main points to be noted are the greater rise and longer oscillation wavelength for the wet plume. These differences become more pronounced as the lapse rate increases. It should be remembered that the

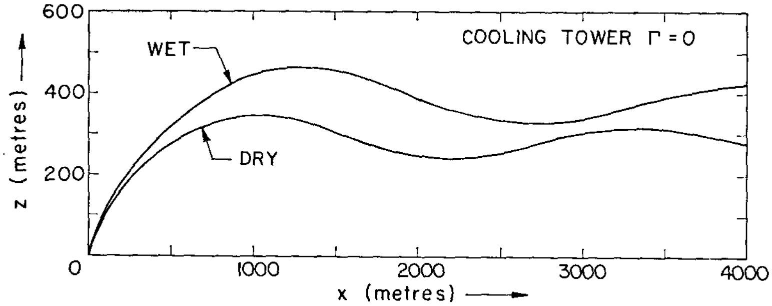


FIG. 1. Comparison of a condensed plume in an isothermal saturated atmosphere with a corresponding dry plume in an isothermal dry atmosphere. The scales show actual, rather than nondimensional, plume rise and downwind distance.

initial-phase assumption employed here is suspect beyond the maximum height of rise.

In Figs. 2 and 3 I have shown how the effect of condensation on maximum height of rise ($z_m$) varies with stability by plotting the ratio of wet-plume value ($z_{ms}$) to dry-plume value ($z_{md}$) as a function of atmospheric stability. For comparison, the approximate ratio given by WS is shown (in Fig. 2) together with specific values calculated by Weil using his approximate solution and using Hanna's method (see Weil, 1974, p. 440). The Weil and Hanna results apply to slightly different plume and environment conditions from those used here. The WS approximation was calculated using

$$\frac{z_{ms}}{z_{md}}=\left[\frac{\Delta T_0^*}{\Delta T_0}\frac{(\Gamma_{ad}-\Gamma)}{(\Gamma_{as}-\Gamma)}\right]^{\frac{1}{3}}, \tag{8}$$

which incorporates the effect of virtual temperature (WS, p. 336; see also Wigley and Slawson, 1971) into the formula quoted by WS (p. 339), and using a constant value of $\Gamma_{as}=0.0065°\text{C m}^{-1}$.

Fig. 2 shows the situation for a natural-draft cooling tower. The complete numerical solution is similar in form to that predicted by (8) and shows significantly greater plume rise for condensed plumes at lapse rates greater than isothermal. The ratio $z_{ms}/z_{md}$ increases as lapse rate increases. These qualitative results are also predicted by Hanna (1972) and Weil (1974). However, all three methods (WS, Hanna and Weil) show significant quantitative deviations from the numerical solution.

The numerical solution shows that the ratio $z_{ms}/z_{md}$ is finite even at lapse rates $>0.0065°\text{C m}^{-1}$, although at

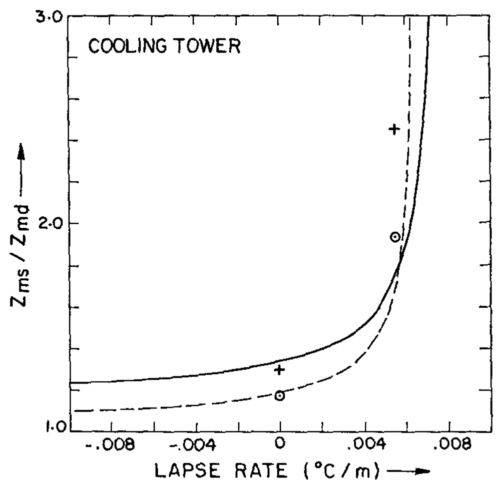


FIG. 2. Ratio of wet plume maximum rise ($z_{ms}$) to dry plume maximum rise ($z_{md}$) for various stabilities for a typical natural-draft cooling tower. The solid curve is the computed result. The dashed curve is that predicted by Wigley and Slawson. The circles are values calculated by Weil using his own theory, and the crosses are values obtained using the method suggested by Hanna (also calculated by Weil). The Wigley and Slawson curve rises asymptotically to infinity as the lapse rate approaches the (assumed constant) saturated adiabatic lapse rate (0.0065°C m$^{-1}$ in this case) whereas the computed result is still finite at $\Gamma=0.009°\text{C m}^{-1}$. Cooling tower initial conditions and other atmospheric conditions are given in the text.

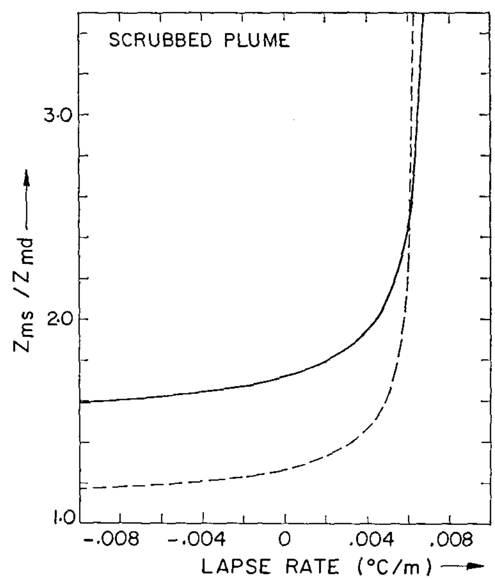


FIG. 3. Ratio of wet plume maximum rise to dry plume maximum rise for various stabilities for a typical scrubbed industrial plume. The solid curve, which is the computed result, is finite even at $\Gamma=0.009°\text{C m}^{-1}$. The dashed curve, which is that predicted by Wigley and Slawson, rises asymptotically to infinity as the lapse rate approaches the (assumed constant) saturated adiabatic lapse rate (0.0065°C m$^{-1}$). Plume initial conditions and other atmospheric conditions are given in the text.

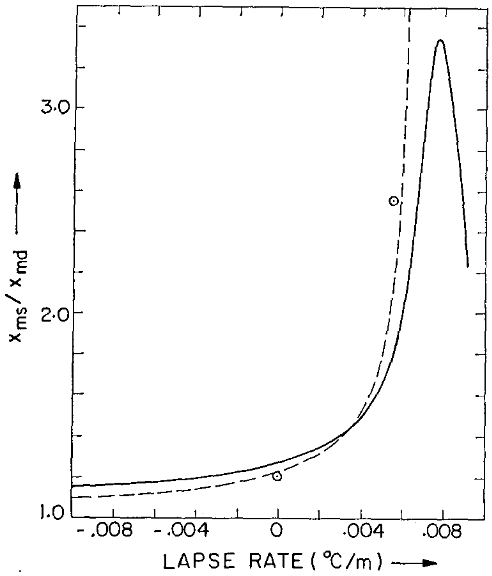


FIG. 4. Ratio of the downwind distances to the maximum height of rise for a wet plume ($x_{ms}$) to that for a corresponding dry plume ($x_{md}$). The dashed curve is that predicted by Wigley and Slawson. It rises asymptotically to infinity as the lapse rate approaches 0.0065°C m$^{-1}$, and is independent of all plume and environment parameters (except the lapse rate). The solid curve is the result of a complete numerical solution of the plume equations for a typical natural-draft cooling tower. For a typical scrubbed industrial plume a very similar, but slightly greater, curve results. The circles are values calculated by Weil for a cooling tower.

$\Gamma=0.008$ and 0.009°C m$^{-1}$ the ratio is so large (5.0 and 6.7, respectively) that plume rise is high enough for the plume behavior to be effectively unstable. This result is contrary to the WS prediction in Fig. 2. However, it is consistent with Eq. (8) when one recalls that $\Gamma_{as}$ is not a constant. For plumes which rise sufficiently high the ambient $\Gamma_{as}$ approaches $\Gamma_{ad}$ and plume behavior remains technically stable even at lapse rates close to $\Gamma_{ad}$. This possibility was predicted by WS who state that "because the saturated adiabatic lapse rate increases with decreasing temperature...a wet plume would tend to become less unstable as it rises" (WS, p. 336).

The case of a typical scrubbed industrial plume is shown in Fig. 3. Here the WS solution is again of a similar form to the more detailed solution, but the quantitative difference between the two is much greater than for the cooling tower example. The values of $z_{ms}/z_{md}$ are also significantly higher than for the cooling tower (compare the values at $\Gamma=0.0$°C m$^{-1}$ of 1.3 and

TABLE 1. Comparison of approximate and numerical solutions for maximum height of rise for dry plumes.

| Lapse rate $\Gamma$(°C m$^{-1}$) | −0.008 | −0.004 | 0 | 0.004 | 0.008 |
|---|---|---|---|---|---|
| $z_{md}/Z_{md}$ (cooling tower) | 0.83 | 0.84 | 0.86 | 0.88 | 0.91 |
| $z_{md}/Z_{md}$ (scrubbed plume) | 0.89 | 0.90 | 0.91 | 0.92 | 0.94 |

TABLE 2. Comparison of approximate and numerical solutions for the distance to the maximum height of rise for dry plumes.

| Lapse rate $\Gamma$(°C m$^{-1}$) | −0.008 | −0.004 | 0 | 0.004 | 0.008 |
|---|---|---|---|---|---|
| $x_{md}/X_{md}$ (cooling tower) | 0.84 | 0.86 | 0.88 | 0.90 | 0.93 |
| $x_{md}/X_{md}$ (scrubbed plume) | 0.89 | 0.90 | 0.92 | 0.94 | 0.96 |

1.7 for example), and this difference can be attributed to the greater values of $\Delta T_0$ and $\Delta q_0$ which occur in the scrubbed plume. As in the cooling tower case the ratio $z_{ms}/z_{md}$ does not become infinite even at $\Gamma=0.009$°C m$^{-1}$ (the values at 0.008 and 0.009°C m$^{-1}$ being 8.9 and 12.3, respectively) although it is so large that the plume behavior is effectively unstable.

In Fig. 4 I have examined the effect of condensation on the distance ($x_m$) to $z_m$ through the ratio $x_{ms}/x_{md}$ at various stabilities for the cooling tower case. The WS prediction

$$\frac{x_{ms}}{x_{md}}=\frac{N}{N_s} \tag{10}$$

is shown for comparison with values quoted by Weil (1974). (The values of $x_{ms}/x_{md}$ obtained for the scrubbed plume were very similar to those obtained for the cooling tower). Eq. (10) gives slightly better agreement with the complete numerical solution than Weil's results. The peak in the computed $x_{ms}/x_{md}$ curve does not conflict with the effectively unstable behavior of the wet plume at high lapse rates when it is remembered that $x_{md}$ increases very rapidly as $\Gamma$ approaches $\Gamma_{ad}$.

In order to estimate parameters such as $z_{ms}$ and $x_{ms}$ using the above results, one must be able to estimate the corresponding dry plume parameters. Simple approximate formulas are available for this purpose. These may be derived from the dry plume equations using the "bent-over" plume assumption (i.e., $v_x\approx U$) and ignoring initial momentum. Incidentally, the same assumptions have been made by Weil (1974) in his treatment of wet plumes. The formulas are

$$Z_{md}=\left(\frac{6F_0}{UN^2\alpha^2}\right)^{\frac{1}{3}}, \tag{11}$$

where $F_0$ is the initial flux of buoyancy, and

$$X_{md}=\frac{\pi U}{N}. \tag{12}$$

I have compared these formulas with the complete numerical solution values for both the cooling tower and scrubbed plume cases in Tables 1 and 2.

Table 1 shows that the approximate formula *overestimates* plume rise. Since the derivation of (11) ignores initial momentum one might have expected the opposite. Such an unexpected result can in part be attributed to the neglect of the non-zero source radius in the approximate formula (this was pointed out by a referee) and, possibly, to the "bent-over" plume assumption, which is a severe and unrealistic approximation close to the source.

The "bent-over" assumption certainly explains the discrepancy between $x_{md}$ and $X_{md}$ shown in Table 2. Since in the constant-$U$ model used here the true downwind plume speed $v_x$ is always less than or equal to the environmental wind speed $U$, one would expect $X_{md}$ to overestimate $x_{md}$ as Table 2 indicates. In addition, when continued far downwind, the computed values of the plume oscillation wavelength were found to increase asymptotically towards $X_{md}$.

In summary, this paper shows that the predictions of the effects of condensation on plume rise given by Wigley and Slawson (1972), Hanna (1972) and Weil (1974) are all qualitatively correct. The more detailed discussion of the problem given by Weil does not appear to give significantly better results *within the framework and validity of the initial-phase entrainment model for plume rise*. It is important to note that, as yet, no detailed observational checks of this theory have been made for wet plumes, although preliminary results for cooling towers quoted by Slawson *et al.* (1975) are encouraging.

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