On Some Aspects of Omega Windfinding

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ABSTRACT

A generalized hyperbolic navigation system using Omega signals is defined. For three stations this system is shown to be equivalent to the ordinary hyperbolic navigation system. It is shown that the velocity and position determination are independent of the station selected to be the common station. Furthermore, a formula is given to combine several estimates of the wind vector into a single optimum estimate. The efficiency of using three frequencies instead of a single frequency to derive the wind vector is derived.

1. Introduction

Recently, several algorithms to derive vertical wind profiles have been given (e.g., Acheson, 1974; Passi, 1974; etc.). Most of the available techniques have been ably summarized by Poppe (1973). The results of several system tests show encouraging results. These have been documented by Acheson *et al.* (1973), Acheson (1973), Nybo (1973), Govind (1973), Passi and Olson (1974) and others. However, there are certain aspects of Omega windfinding which have not been covered. Some are alluded to with some confusion and speculation. The purpose of this paper is to give theoretical treatment and clarify two such aspects:

- For various reasons one needs to adopt the hyperbolic technique in determining the receiver position and velocity. It is a common belief that the choice of the common station has a bearing on the results. It will be shown that the position fix as well as the computed velocity is independent of the choice of the common station.
- The use of multiple frequencies in the velocity determination has been proposed. This will provide three times the number of observations available for a single Omega frequency. This will result in greater accuracy, but how much greater is not known. Acheson (1974), in his prediction of the wind-errors during GATE, "assumed that the inclusion of the fourth transmitter and other two frequencies reduces three-transmitter 13.6 kHz error by a factor of two." The reduction in error due to a fourth Omega station alone, depending on the geometry, could be by a factor much greater than two; on the other hand, there are situations where the error reduction is not significant at all (see Passi, 1973b). However, the effect of the use of multiple frequencies is always tangible and, assuming the signal quality for the three frequencies from a single station is the same, the effect is given by a constant factor.

In Section 2 we will give the basic Omega system concept and some mathematical equations which will be used in the later sections. In Section 3 we will deal with the effect of the choice of the common station. Section 4 will deal with the efficiency to be obtained by using multiple frequencies. As a by-product we will give the optimum procedure for combining individual frequency estimates into a single estimate which is better than any individual estimate.

2. Omega system and windfinding equations

The Omega system, when completed, will be a network of eight transmitting stations located in Norway, Liberia, Hawaii, North Dakota, Japan, Argentina, La Reunion in the Indian Ocean, and Australia. Each station transmits an approximately one second long signal successively on 10.2, $11\frac{1}{3}$ and 13.6 kHz once every 10 s, as shown in Fig. 1. In addition to these three frequencies, each Omega station transmits for the remaining time at a unique frequency, between 10.2 and 13.6 kHz, which has been assigned to it. Most of the windfinding work is done at the 13.6 kHz frequency. The other frequencies can be used without any change in the windfinding algorithm. For further explanation of the Omega system the reader is referred to Pierce *et al.* (1966).

We now give the basic equations necessary for windfinding: Let $(\xi_j, \eta_j) =$ (longitude, latitude), $j = 1, \ldots, k$ be the positions of the jth Omega station from which

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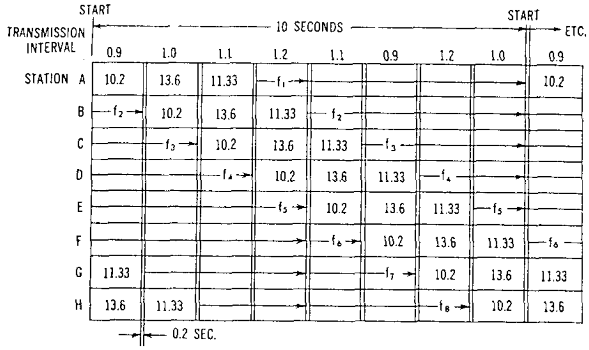

FIG. 1. Omega signal format.

signals are being received, and let (x,y) be the position of the sonde at any time. The phase of the signal from station j at (x,y) is given by

$$\phi_j = (2\pi R/\lambda) f_j, \tag{1}$$

where

$$\cos f_j = \cos(\xi_j - x)\cos y \cos\eta_j + \sin y \sin\eta_j, \tag{2}$$

R is the radius of the earth, and λ the wavelength. The quantity measured, however, is

$$p_j = \phi_j - \phi_L, \tag{3}$$

where ϕ_L is the phase of the local oscillator. Let

$$\begin{aligned} p_{ij} &= p_i - p_j \\ &= (2\pi R/\lambda)(f_i - f_j) = (2\pi R/\lambda) f_{ij},\quad i \neq j = 1, \ldots, k. \end{aligned} \tag{4}$$

The independent set of $(k-1)$ equations is obtained by fixing $j=k$ in (4). The position solution (x,y) is obtained by minimizing $(\mathbf{p}-\mathbf{u})'\boldsymbol{\Sigma}_1^{-1}(\mathbf{p}-\mathbf{u})$, where $\mathbf{p}'$, the transpose of the vector $\mathbf{p}$, is given by

$$\mathbf{p}' = [p_{1k}, \ldots, p_{k-1,k}], \tag{5}$$

$$\mathbf{u}' = (2\pi R/\lambda)[f_{1k}, \ldots, f_{k-1,k}], \tag{6}$$

and $\boldsymbol{\Sigma}_1$ is the covariance matrix of $\mathbf{p}$. The velocity vector $(U,V)'$ is given by

$$(U,V)' = (\mathbf{F}'\boldsymbol{\Psi}^{-1}\mathbf{F})^{-1}\mathbf{F}'\boldsymbol{\Psi}^{-1}\dot{\mathbf{p}}, \tag{7}$$

where $\mathbf{F}$ is a $(k-1)\times 2$ matrix having in its ith row

$$F_{i1} = (2\pi/\lambda \cos y)\partial f_{ik}/\partial x, \quad F_{i2} = (2\pi/\lambda)\partial f_{ik}/\partial y;$$

the vector $\dot{\mathbf{p}}$ is the time derivative of $\mathbf{p}$; and $\boldsymbol{\Psi}$ is the covariance matrix of $\dot{\mathbf{p}}$. All of these results are taken from Passi (1974).

3. Choice of the common station

Any two equations in (4) which have a common subscript constitute a hyperbolic set of equations; a system which utilizes a hyperbolic set of equations to get a position fix (x,y) is referred to as a hyperbolic navigation system (HNS). The classical Omega navigation system is based on three-station hyperbolic navigation; if the information were available from more than three, then those three were chosen which would give the best results.

Let us call the system using more than three stations with any one of them as the common, as the generalized hyperbolic navigation system (GHNS). The generalized equations to get a position fix (x,y) using GHNS are Eqs. (15) in Passi (1974). First we show that, for $k=3$, the GHNS yields the same equations as in ordinary HNS. Passi's Eqs. (15) can be rewritten as a single equation

$$(\mathbf{p}-\mathbf{u})'\boldsymbol{\Sigma}_1^{-1}\mathbf{M} = \mathbf{O}, \tag{8}$$

where

$$\mathbf{M} = (\partial\mathbf{u}/\partial x, \partial\mathbf{u}/\partial y).$$

For $k=3$, $\mathbf{M}$ is 2×2 matrix and assuming it to be non-singular, Eq. (8) reduces to

$$\mathbf{p} = \mathbf{u} \tag{9}$$

which is the set of equations for the ordinary HNS. Eqs. (9) can be iteratively solved using the Newton-Raphson procedure as given in Passi (1973a).

To show that the choice of the common station is immaterial, let us consider the case of $k=4$. From (5), if station 4 is taken to be common, the vector $\mathbf{p}$ is given by

$$\begin{aligned} \mathbf{p}' &= [p_{14}, p_{24}, p_{34}] \\ &= [p_1 - p_4, p_2 - p_4, p_3 - p_4]. \end{aligned}$$

Now, if station 3 were to be the common station, the corresponding vector $\mathbf{p}^*$ of the phase differences will be

$$\mathbf{p}^* = [p_1 - p_3, p_2 - p_3, p_4 - p_3]'.$$

We note that $\mathbf{p}^*$ can be obtained by an elementary transformation, i.e.,

$$\mathbf{p}^* = \mathbf{Q}\mathbf{p}, \tag{10}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since $\mathbf{Q}$ is an elementary matrix, it is its own inverse, i.e.,

$$\mathbf{Q}^{-1} = \mathbf{Q},$$

which implies that

$$\mathbf{Q}\mathbf{Q} = \mathbf{Q}'\mathbf{Q}' = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. The dispersion matrix $\boldsymbol{\Sigma}^*$ of $\mathbf{p}^*$ from (10) is

$$\boldsymbol{\Sigma}^* = \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}' \quad \text{and} \quad \boldsymbol{\Sigma}^{*-1} = \mathbf{Q}'\boldsymbol{\Sigma}^{-1}\mathbf{Q},$$

where $\boldsymbol{\Sigma}$ is the covariance matrix of $\mathbf{p}$. With station 3 as the common, the position fix (x,y) is obtained by minimizing $(\mathbf{p}^*-\mathbf{u}^*)'\boldsymbol{\Sigma}^{*-1}(\mathbf{p}^*-\mathbf{u}^*)$, which is easily seen to be $(\mathbf{p}-\mathbf{u})'\boldsymbol{\Sigma}^{-1}(\mathbf{p}-\mathbf{u})$. The latter quadratic form is the one to be minimized when station 4 is the common station.

In general, the change of the common station amounts to an elementary transformation which does not make any difference in the computations. However, if the weighting is not done by the covariance matrix, the results will not be independent of the choice of the common station, except for $k=3$. For example, if $\mathbf{\Sigma}$ is replaced by the identity matrix, then

$$(\mathbf{p}^*-\mathbf{u}^*)'(\mathbf{p}^*-\mathbf{u}^*)=(\mathbf{p}-\mathbf{u})'\mathbf{Q}'\mathbf{Q}(\mathbf{p}-\mathbf{u}),$$
$$\neq(\mathbf{p}-\mathbf{u})'(\mathbf{p}-\mathbf{u}),$$

because $\mathbf{Q}'\mathbf{Q}$ is not equal to the identity matrix.

The treatment for the velocity is identical to the above and, hence, is ommitted.

4. Use of multiple frequencies in Omega windfinding

If, instead of one frequency, all three Omega frequencies are utilized to derive the winds, then, at any time we will have three estimates $\mathbf{V}_j$, $j=1, 2, 3$, of the wind vector $\mathbf{V}$, one corresponding to each of the frequencies. Let $\mathbf{\Psi}_j$ be the covariance matrix of the vector $\dot{\mathbf{p}}_j$ [as in (7)] and λ_j be the wavelength corresponding to the jth frequency, $j=1, 2, 3$. Note that the matrix $\mathbf{F}$ in (7) can be written as

$$\mathbf{F}=(1/\lambda)\mathbf{G},$$

where $\mathbf{G}$ is independent of the frequency and depends only on (x,y) and the Omega station locations. From (7), the covariance matrix $\mathbf{\Lambda}_j$ of $\mathbf{V}_j$ is given by

$$\mathbf{\Lambda}_j=(1/\lambda_j^2)(\mathbf{G}'\mathbf{\Psi}_j^{-1}\mathbf{G})^{-1},\quad j=1, 2, 3. \tag{11}$$

Now the three estimates $\mathbf{V}_j$ can be combined optimally according to the following formula (Passi, 1975):

Let $\mathbf{Y}_j$, $j=1,\ldots,n$, be n independent unbiased estimates of $\mathbf{\mu}$, then the optimum estimate of $\mathbf{\mu}$ based on $\mathbf{Y}_j$ is given by

$$\hat{\mathbf{\mu}}=(\sum_{j=1}^{n}\mathbf{\Lambda}_j^{-1})^{-1}(\sum_{j=1}^{n}\mathbf{\Lambda}_j^{-1}\mathbf{y}_j),$$

where $\mathbf{\Lambda}_j$ is the covariance matrix of $\mathbf{Y}_j$; and the covariance matrix of $\hat{\mathbf{\mu}}$ is given by

$$\mathrm{cov}(\hat{\mathbf{\mu}})=(\sum_{j=1}^{n}\mathbf{\Lambda}_j^{-1})^{-1}.$$

Based on this result, the optimum estimate $\mathbf{V}^*$ of the velocity vector $\mathbf{V}$ is given by

$$\mathbf{V}^*=[\sum_{j=1}^{3}\lambda_j^2(\mathbf{G}'\mathbf{\Psi}^{-1}\mathbf{G})]^{-1}[\sum_{j=1}^{3}\lambda_j^2(\mathbf{G}'\mathbf{\Psi}^{-1}\mathbf{G})\mathbf{V}_j]$$

and

$$\mathrm{cov}(\mathbf{V}^*)=[\sum_{j=1}^{3}\lambda_j^2(\mathbf{G}'\mathbf{\Psi}_j^{-1}\mathbf{G})]^{-1}$$
$$=[\mathbf{G}'(\sum_{j=1}^{3}\lambda_j^2\mathbf{\Psi}_j^{-1})\mathbf{G}]^{-1}.$$

If we assume that the signals at the different frequencies have the same signal-to-noise ratio, then $\mathbf{\Psi}_j$ are equal, say, to $\mathbf{\Psi}$. In that case,

$$\mathrm{cov}(\mathbf{V}^*)=(\mathbf{G}'\mathbf{\Psi}^{-1}\mathbf{G})^{-1}/\sum_{j=1}^{3}\lambda_j^2 \tag{12}$$

when the covariance matrix of any individual estimate $\mathbf{V}_j$ is

$$\mathrm{cov}(\mathbf{V}_j)=\lambda_j^{-2}(\mathbf{G}'\mathbf{\Psi}^{-1}\mathbf{G})^{-1}. \tag{13}$$

The standard error of the velocity vector is defined to be the square root of the trace of the covariance matrix. Therefore, the ratio of the standard error of the multiple frequency estimate to the standard error of the single frequency estimate is given by

$$r=\lambda_j(\sum_{i=1}^{3}\lambda_i^2)^{-\frac{1}{2}}.$$

If λ_j corresponds to 13.6 kHz frequency, then $r=0.4858$.

5. Concluding remarks

Here we have defined a generalized hyperbolic navigation system, which for $k=3$ reduces to the ordinary hyperbolic system. This system is shown to be independent of the choice of the common station.

A formula to combine several estimates of the wind vector derived using different frequencies is given. The reduction in the standard error of the wind vector thus derived is shown to be by a factor greater than two if all three of the Omega frequencies are used instead of the commonly used frequency of 13.6 kHz.

It has been pointed out by a reviewer that the selection of a common station will not alter the results if editing and/or smoothing is done on p_j [Eq. (3)]; however, if editing and/or smoothing of data is accomplished on p_{ij} [Eq. (4)], the selection of the common station could change the results. This point by the reviewer is well taken. It has been pointed out by Passi and Olson (1974) that, to eliminate the effects due to (i) the frequency offset and/or the frequency drift of the local oscillator, and (ii) the changing communication path length between Omegasonde and the Omega phase comparitor, the phase differencing must be performed with proper time synchronization. This can be accomplished only if editing and/or smoothing is performed on p_j rather than p_{ij}.

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