Measurement of Doppler Spectrum Parameters by Crossing Rate Methods[1]

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(Manuscript received 15 April 1974, in revised form 17 July 1975)

ABSTRACT

The effect of discrete sampling on the measurement of the root mean square frequency f_{rms} of the Doppler spectrum by the bipolar video zero-crossing rate method is discussed. It is shown that discrete sampling results in an underestimate of f_{rms}. A method of correcting for the underestimate is suggested. Conditional positive and negative axis-crossing rates, or the rates at which the signal vector crosses a phase angle in the counterclockwise and clockwise directions, are defined and shown to be equal to $(f_{rms}+\bar{f})/2$ and $(f_{rms}-\bar{f})/2$, respectively, where $\bar{f}$ is the mean of the Doppler spectrum of the signal. This result suggests that the conditional axis-crossing rates may be used for the measurement of spectrum mean and variance. The effect of discrete sampling on the conditional axis-crossing rates is also discussed.

1. Introduction

In many meteorological studies using Doppler radar, the mean and variance of the Doppler spectrum are of primary interest. The mean and variance may be computed by first calculating the Doppler spectrum. However, because of the relatively large amount of calculation required to obtain the Doppler spectrum, techniques which yield the mean and the variance without the intermediate calculation of the Doppler spectrum are of great interest. Among these techniques mention may be made of the pulse-pair method (Rummler, 1968), the phase-change method (Sirmans and Doviak, 1973), and level-crossing rate methods. The pulse-pair method gives both the mean and the variance, while the phase-change method gives only the mean of the spectrum. The intensity-crossing rate method, or the *R*-meter method (Rutkowski and Fleisher, 1955), gives the variance of the spectrum provided the mean signal intensity is known. The bipolar-video zero-crossing rate method gives the root mean square frequency of the spectrum (e.g., Atlas, 1964). In this paper the zero-crossing rate method is extended to the complex plane, and the relationship between the various crossing rates is clarified. The effect of discrete sampling on the crossing rates is also discussed.

2. Signal statistics

Various probability densities associated with the radar signal will be required in the following discussion. In the conventional pulsed-Doppler radar case, the signal received at any particular instant of time is the vector sum of the signals returned by the scatterers contained in a pulse volume determined by the radar characteristics. After envelope detection, the received signal may be represented by its amplitude and phase (A,ϕ), or equivalently as a complex number Z with modulus A and argument ϕ, or by the real and imaginary parts (X,Y) of Z. These quantities are interrelated as follows:

$$Z=X+jY=A(\cos\phi+j\sin\phi)=A\exp(j\phi), \quad (1)$$

where

$$X=\sum a_i\cos\phi_i, \quad (2a)$$

$$Y=\sum a_i\sin\phi_i, \quad (2b)$$

$j=(-1)^{\frac{1}{2}}$, and (a_i,ϕ_i) are the amplitude and phase of the signal returned by the ith scatterer. The sums in (2a) and (2b) are taken over all the scatterers in the pulse volume. For visualization, a graphical representation is helpful (Fig. 1). The signal received at time τ after the instant to which X and Y relate is given by

$$X'=\sum a_i\cos(\phi_i+\omega_i\tau)=\sum a_i\cos\omega_i\tau\cos\phi_i-\sum a_i\sin\omega_i\tau\sin\phi_i, \quad (3a)$$

$$Y'=\sum a_i\sin(\phi_i+\omega_i\tau)=\sum a_i\cos\omega_i\tau\sin\phi_i+\sum a_i\sin\omega_i\tau\cos\phi_i, \quad (3b)$$

where

$$\omega_i=\frac{4\pi v_i}{\lambda_0} \quad (4)$$

is the radian Doppler frequency shift due to the ith scatterer, v_i being its radial velocity and λ_0 the wave-

[1] Research supported in part by the National Science Foundation under Grant GA-38109 and the Office of Naval Research under Contract N00014-67-A-0285-0014, NR 061-188.

length of the transmitted radiation. In the following it will be assumed that the a_i and ω_i are constants, and that the a_i are normalized so that

$$\sum a_i^2 = 1. \tag{5}$$

In meteorological situations, a very large number of terms is involved in the summations in (2a) and (2b). The statistics of X, Y are therefore Gaussian, the joint probability density of (X,Y,X',Y') being given by (see, e.g., Davenport and Root, 1958)

$$p(X,Y,X',Y') = (4\pi^2 D)^{-1} \exp\{-(2D)^{-1}[(X^2+Y^2+X'^2+Y'^2) - 2c(XX'+YY') - 2s(XY'-X'Y)]\}, \tag{6}$$

where

$$D(\tau) = 1 - c^2 - s^2, \tag{7}$$

$$c(\tau) = \sum a_i^2 \cos\omega_i\tau, \tag{8}$$

$$s(\tau) = \sum a_i^2 \sin\omega_i\tau. \tag{9}$$

It may be noted that because of the large number of terms involved, the sums in (8) and (9) may be approximated by integrals, if the Doppler frequency shift is continuously distributed:

$$c(\tau) = \int_{-\infty}^{+\infty} S(\omega) \cos\omega\tau \, d\omega, \tag{10}$$

$$s(\tau) = \int_{-\infty}^{+\infty} S(\omega) \sin\omega\tau \, d\omega, \tag{11}$$

where $S(\omega)d\omega$ is the sum of the squares of the a_i for the scatterers producing Doppler shifts in the range $(\omega, \omega+d\omega)$. Corresponding to (5) we have the normalization

$$\int S(\omega) d\omega = 1. \tag{12}$$

The $S(\omega)$ has the interpretation of the normalized Doppler spectrum. The probability density of Y, Y' obtained directly or by integrating (6) over X, X' is given by

$$p(Y,Y') = \frac{1}{2\pi(1-c^2)^{\frac{1}{2}}} \exp\left[-\frac{Y^2+Y'^2-2cYY'}{2(1-c^2)}\right]. \tag{13}$$

The probability density of (X,X') is similar.

The joint probability density of X, $\dot{X}$, Y, $\dot{Y}$ will also be needed, $\dot{X}$ and $\dot{Y}$ being the time derivatives of X and Y:

$$\dot{X} = dX/dt = -\sum a_i\omega_i \sin\phi_i, \tag{14}$$

$$\dot{Y} = dY/dt = \sum a_i\omega_i \cos\phi_i. \tag{15}$$

The joint probability density of $(X,Y,\dot{X},\dot{Y})$ may be derived directly (Srivastava and Carbone, 1969), or by transforming the probability density of (X,Y,X',Y')

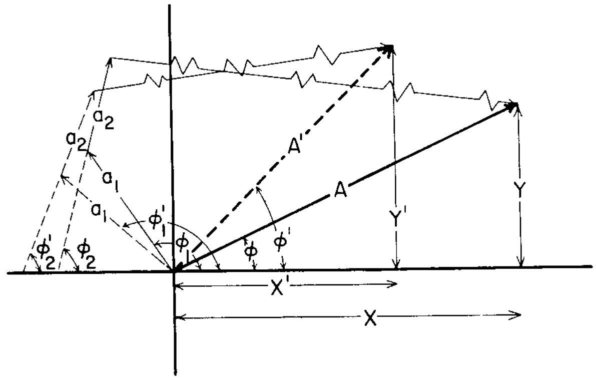

FIG. 1. Representation of signal received from a meteorological target. The amplitude and phase of two of the scatterers are shown. The resultant signal shown by thick lines is the vector sum of the signals from the individual scatterers. The full and dashed vectors rotate according to their Doppler shifts.

given in (6) to one of $[X, Y, (X'-X)/\tau, (Y'-Y)/\tau]$ and then passing to the limit $\tau \to 0$. The result is

$$p(X,Y,\dot{X},\dot{Y}) = \frac{1}{\pi^2\sigma^2} \exp\left\{-\left[(X^2+Y^2) + \left(\frac{\dot{Y}-\bar{\omega}X}{\sigma}\right)^2 + \left(\frac{\dot{X}+\bar{\omega}Y}{\sigma}\right)^2\right]\right\}, \tag{16}$$

where $\bar{\omega}$ and σ^2 are the mean and variance of the Doppler spectrum:

$$\bar{\omega} = \Sigma a_i^2\omega_i = \int \omega S(\omega) d\omega, \tag{17}$$

$$\sigma^2 = \Sigma a_i^2\omega_i^2 - (\Sigma a_i^2\omega_i)^2 = \int \omega^2 S(\omega) d\omega - \left[\int \omega S(\omega) d\omega\right]^2. \tag{18}$$

The joint probability density of $(Y,\dot{Y})$ or $(X,\dot{X})$, may be obtained directly or by integrating (16) over X and $\dot{X}$ or Y and $\dot{Y}$. It may also be obtained by transforming $p(Y,Y')$ in (13) to $p[Y, (Y'-Y)/\tau]$ and letting $\tau \to 0$. The result is

$$p(Y,\dot{Y}) = \frac{1}{\pi\omega_{\text{rms}}} \exp[-Y^2 - (\dot{Y}/\omega_{\text{rms}})^2], \tag{19}$$

where ω_{rms} is the root mean square frequency of the Doppler spectrum, i.e.,

$$\omega_{\text{rms}}^2 = \int \omega^2 S(\omega) d\omega = \Sigma a_i^2\omega_i^2 = \bar{\omega}^2 + \sigma^2. \tag{20}$$

The probability density of the rate of change of phase $\dot{\phi}$ is obtained by transforming (16) and performing the necessary integrations (Srivastava and Carbone, 1969):

$$p(\phi)=\frac{1}{2\sigma}\left[1+\frac{(\phi-\bar{\omega})^2}{\sigma^2}\right]^{-\frac{3}{2}}. \qquad (21)$$

3. Zero-crossing rate: Continuous sampling

General expressions for the positive and negative zero-crossing rates, W_+ and W_- of a random variable ξ, were derived by Rice (1945):

$$W_+=\int_0^\infty \dot{\xi}p(0,\dot{\xi})d\dot{\xi}, \qquad (22)$$

$$W_-=\int_{-\infty}^0 |\dot{\xi}|p(0,\dot{\xi})d\dot{\xi}. \qquad (23)$$

In the above $p(\xi,\dot{\xi})$ is the joint probability density of $(\xi,\dot{\xi})$. Using the above expressions with (19), it may be shown easily that the positive and negative zero-crossing rates for Y or X are equal and that each is given by

$$W_+=W_-=\omega_{\mathrm{rms}}/2\pi=f_{\mathrm{rms}}, \qquad (24)$$

a result well known in radar meteorology (Atlas, 1964).

It is also known that the average rate of change of phase of the signal vector is equal to $\bar{\omega}=2\pi\bar{f}$ (Srivastava and Carbone, 1969). This implies that the rate at which the signal vector crosses zero phase (or any given phase) in the counterclockwise direction is equal to $\bar{f}$. The reason for the difference between the zero-crossing rate and the phase-crossing rate may be seen by reference to Fig. 2, which shows the positions of the signal vector at three successive instants of time. Between times 1 and 3, there is a positive and a negative axis-crossing rate but no net phase change. It is clear that in the special case of a signal with zero mean frequency and finite spectral variance, there will be no net phase crossings but a number of positive and negative axis crossings. These considerations suggest that the axis crossings be distinguished according to the sense of rotation of the signal vector in the complex plane. Accordingly, we define the following crossing rates (see Fig. 2 for illustration):

W_1 = rate of crossing from negative Y to positive Y, $X>0$

W_2 = rate of crossing from negative Y to positive Y, $X<0$

W_3 = rate of crossing from positive Y to negative Y, $X>0$

W_4 = rate of crossing from positive Y to negative Y, $X<0$.

It may be seen that

$$W_+=W_1+W_2, \qquad (25)$$

$$W_-=W_3+W_4. \qquad (26)$$

Also by symmetry, or by direct calculation, it may be verified that

$$W_1=W_4, \qquad (27)$$

$$W_2=W_3. \qquad (28)$$

The above equations are, of course, consistent with $W_+=W_-$.

There is no accepted terminology for the crossing rates defined here. Rates W_1, W_4 and W_2, W_3 may be called the conditional positive and negative zero-crossing rates. A more descriptive terminology would be to call W_1 and W_4 the counterclockwise zero-crossing rate and W_2 and W_3 the clockwise zero-crossing rate. Alternatively W_1, W_4 and W_2, W_3 may be called the counterclockwise and clockwise phase axis crossing rates.

Let us now compute W_1 using the general expression (22). First, the joint probability density $p(Y,\dot{Y},X)$ is obtained by integrating (16) over $\dot{X}$. The positive crossing rate for any X is then obtained by integrating $\dot{Y}p(0,\dot{Y},X)$ with respect to $\dot{Y}$ from 0 to ∞; the conditional crossing rate W_1 is then obtained by a further integration over X from 0 to ∞. Hence

$$W_1=\int_0^\infty\int_0^\infty \dot{Y}p(0,\dot{Y},X)d\dot{Y}dX =\frac{\omega_{\mathrm{rms}}+\bar{\omega}}{4\pi}=\frac{f_{\mathrm{rms}}+\bar{f}}{2}. \qquad (29)$$

Similarly, it may be seen that

$$W_2=\frac{\omega_{\mathrm{rms}}-\bar{\omega}}{4\pi}=\frac{f_{\mathrm{rms}}-\bar{f}}{2}. \qquad (30)$$

Reference to Fig. 2 suggests an intimate connection between the axis-crossing rate and the rate of change

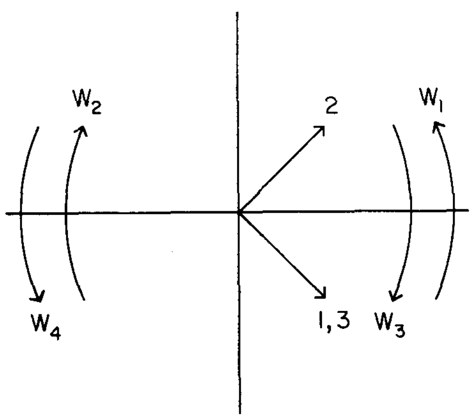

FIG. 2. Resultant vector at three instants of time and the conditional positive and negative axis-crossing rates.

of phase of the received signal. Indeed $W_1=W_4$ is also equal to the rate of crossing any phase angle in the positive (counterclockwise) direction, while $W_2=W_3$ is equal to the rate of crossing any phase angle in the negative (clockwise) direction. This interpretation provides an alternative proof of (29) and (30). Thus using (21) and (22), we have

$$W_1=(2\pi)^{-1}\int_0^{\infty}\dot{\phi}p(\dot{\phi})d\dot{\phi}=\frac{\omega_{\rm rms}+\bar{\omega}}{4\pi}=\frac{f_{\rm rms}+\bar{f}}{2}. \tag{31}$$

Similarly,

$$W_2=(2\pi)^{-1}\int_{-\infty}^{0}|\dot{\phi}|p(\dot{\phi})d\dot{\phi}=\frac{\omega_{\rm rms}-\bar{\omega}}{4\pi}=\frac{f_{\rm rms}-\bar{f}}{2} \tag{32}$$

in agreement with (29) and (30). The graphical representation also suggests that $2\pi(W_1-W_2)$ is equal to the mean rate of phase change $\dot{\phi}$; thus

$$\bar{\dot{\phi}}=2\pi(W_1-W_2)=\bar{\omega}=2\pi\bar{f}. \tag{33}$$

The rate of change of phase, therefore, gives the mean of the Doppler spectrum; this method for the measurement of the mean frequency was implemented by Sirmans and Doviak (1973) at the National Severe Storms Laboratory. Eqs. (29) and (30) suggest that both $f_{\rm rms}$ and $\bar{f}$ may be deduced by measuring the conditional positive and negative axis-crossing rates defined here. It should be pointed out that the above crossing rates apply when the signal is observed continuously. In meteorological applications of pulsed Doppler radar, the signal is usually box-carred so the crossings are measured on the basis of the signal sampled at points of time separated by an interval τ. The effect of the discrete sampling on the crossing rates will now be considered.

4. Zero-crossing rate: Discrete sampling

In the pulsed Doppler case, the zero-crossing rate for the box-carred signal may differ from that for the continuously sampled signal because crossings may occur during the inter-pulse period. The effect of discrete sampling on the intensity crossing rate was discussed by Lob (1968). The effect of discrete sampling on the unconditional and conditional zero-crossing rates is considered below.

The discrete zero-crossing rate in the positive direction, S_+, is clearly obtained by dividing the probability that $Y=Y(t)<0$ and $Y'=Y(t+\tau)>0$, by the sampling interval τ; thus,

$$S_+=\frac{1}{\tau}\int_0^{\infty}\int_{-\infty}^{0}p(Y,Y')dYdY'. \tag{34}$$

Substituting from (13) and carrying out the integration, we have

$$S_+=\frac{1}{2\pi\tau}\arccos c. \tag{35}$$

It may be easily shown that the discrete zero-crossing rate in the negative direction is the same. Let us now assume that the Doppler frequencies ω_i are small compared to the sampling rate, i.e.,

$$\omega_i\tau\ll 1. \tag{36}$$

Reference to Eq. (8) then shows that c is close to 1. Expanding the $\cos(\omega_i\tau)$ in (8) and the $\arccos c$ in (35) in power series, it may be shown that

$$S_+=M_2^{\frac{1}{2}}-\frac{\tau^2\pi^2}{3}M_2^{\frac{3}{2}}(1+M_4/M_2^2)+\cdots, \tag{37}$$

where M_n is the nth moment of the Doppler spectrum, so that $M_2^{\frac{1}{2}}$ is the root-mean-square frequency $f_{\rm rms}$[Hz]. It is seen from (37) that as $\tau\rightarrow 0$, S_+ tends to $f_{\rm rms}$ or W_+, the crossing rate for the continuous case. If condition (36) is satisfied then the discrete crossing rate will be a good approximation to $f_{\rm rms}$. However, this condition may be too stringent since the larger ω_i will usually be associated with smaller a_i and the smaller a_i's contribute little to c.

In order to demonstrate the quantitative differences between S_+ and W_+, the ratio S_+/W_+ is plotted against $\sigma_f\tau$ in Fig. 3 for selected spectra, σ_f[Hz] being the

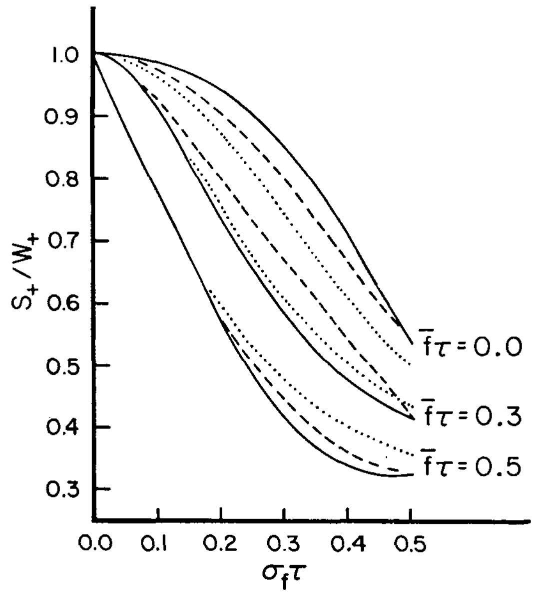

FIG. 3. Ratio S_+/W_+ of the root-mean-square frequencies estimated by the discrete and continuous zero-crossing rates plotted against the normalized standard deviation $\sigma_f\tau$ of the Doppler spectrum. Curves are shown for uniform (full), triangular (dashed), and Gaussian (dotted) spectra for three values of the normalized mean $\bar{f}\tau$ of the Doppler spectrum.

TABLE 1. Some properties of selected Doppler spectra.

Spectrum	Uniform	Gaussian	Triangular
Expression $S(f)$	$S(f)=\begin{cases}\frac{1}{f_2-f_1}, & f_1<f<f_2\\ 0, & f\leqslant f_1,\ f\geqslant f_2\end{cases}$	$S(f)=\frac{1}{\sqrt{2\pi}\sigma_f}\exp\left[-\frac{(f-\bar f)^2}{2\sigma_f^2}\right]$	$S(f)=\begin{cases}\frac{2(f-f_1)}{(f_2-f_1)^2}, & f_1<f<f_2\\ 0, & f\leqslant f_1,\ f\geqslant f_2\end{cases}$
Mean $\bar f$	$\frac{f_1+f_2}{2}$	$\bar f$	$\frac{f_1+2f_2}{3}$
Standard deviation σ_f	$\frac{f_2-f_1}{\sqrt{12}}$	σ_f	$\frac{f_2-f_1}{\sqrt{18}}$
$c=\int S(f)\cos(F)df$ $F=2\pi f\tau$	$\cos\bar F\frac{\sin(F_2-F_1)/2}{(F_2-F_1)/2}$	$\cos\bar F\exp[-2\pi^2\sigma_f^2\tau^2]$	$\frac{2\sin F_2}{F_2-F_2}+\frac{2}{(F_1-F_2)^2}[\cos F_2-\cos F_1]$
$s=\int S(f)\sin F df$ $F=2\pi f\tau$	$\sin\bar F\frac{\sin(F_2-F_1)/2}{(F_2-F_1)/2}$	$\sin\bar F\exp[-2\pi^2\sigma_f^2\tau^2]$	$-\frac{2\cos F_2}{F_2-F_1}+\frac{2}{(F_2-F_1)^2}[\sin F_2-\sin F_1]$

standard deviation of the Doppler spectrum. The analytical expressions for the spectra and some of their properties are listed in Table 1 for reference. It is seen that the $f_{\rm rms}$ deduced from the discrete zero-crossing rate is an underestimate. This is because some zero-crossings may occur during the inter-pulse period, and therefore are not counted in the discrete crossing rate. Referring to the curves for $\bar f=0$, it is seen that in order that $f_{\rm rms}$ deduced from the discrete zero-crossing rate be within 90% of the true value, we must have $\sigma_f\tau \lesssim 0.18$ for the Gaussian spectrum, $\sigma_f\tau<0.21$ for the triangular spectrum, and $\sigma_f\tau<0.25$ for the uniform spectrum. If an accuracy of 80% is acceptable, the corresponding numbers are $\sigma_f\tau<0.26$ for the Gaussian spectrum, $\sigma_f\tau<0.30$ for the triangular spectrum, and $\sigma_f\tau<0.34$ for the uniform spectrum. It is seen that in the examples considered, the value of $\sigma_f\tau$ is the least for the Gaussian spectrum. This is perhaps associated with the fact that the Gaussian spectrum contains arbitrarily large Doppler frequencies, whereas the triangular and uniform spectra have a definite upper bound to the maximum Doppler frequency. Reference to the curves for $\bar f\neq 0$ shows that the error in $f_{\rm rms}$ deduced from the discrete zero-crossing rate increases as $\bar f$ increases. If an estimate of the variance of the spectrum is available, for example, from R-meter measurement, an approximate correction to the measured $f_{\rm rms}$ may be applied on the basis of Fig. 3, assuming a model Doppler spectrum. This method should work well as $\bar f$ increases because the curves for the different spectra are almost identical especially for $\sigma_f\tau<0.1$. It may be pointed out that with a typical pulse-repetition rate of 1000 $\rm s^{-1}$, $\tau=1$ ms, $\sigma_f\tau=0.1$ corresponds to $\sigma_f=10^2$ Hz. A radar wavelength of 10 cm then implies a standard deviation of the Doppler spectrum equal to 5 m $\rm s^{-1}$, a value which is not likely to be exceeded in meteorological situations.

5. Conditional zero-crossing rate: Discrete sampling

We shall now consider the discrete analogs of the conditional axis-crossing rates. Let T_{ij} denote the crossing rate from the ith quadrant to the jth quadrant. By direct calculation, or symmetry, it may be shown that only three of the $T_{ij}(i\neq j)$ are distinct. We shall designate these as S_1, S_2, S_0 as follows:

$$S_1=T_{12}=T_{23}=T_{34}=T_{41}, \tag{38}$$

$$S_2=T_{21}=T_{14}=T_{43}=T_{32}, \tag{39}$$

$$S_0=T_{13}=T_{31}=T_{24}=T_{42}. \tag{40}$$

Reference to Fig. 4 suggests that while S_1 and S_2 may

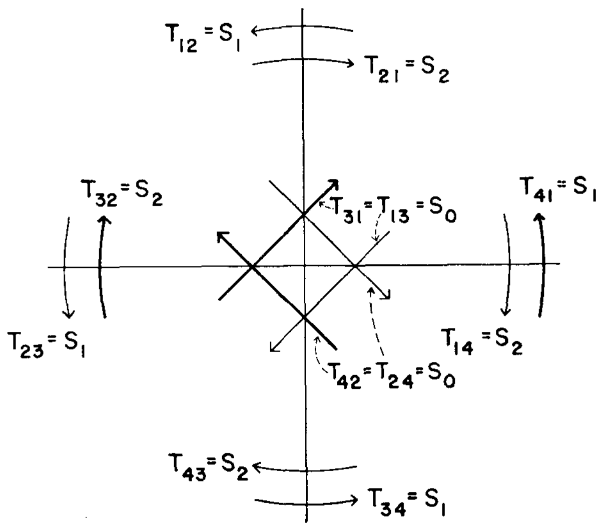

FIG. 4. The conditional discrete axis-crossing rates and their relationship to the unconditional axis-crossing rates. The thick arrows correspond to positive zero-crossing rates.

be considered to be conditional axis-crossing rates, involving rotations in the positive and negative directions, respectively, either direction of rotation may be assigned to S_0 which denotes crossings involving a jump through one quadrant. For reasons which will become clear shortly, we shall define the discrete positive and negative conditional axis crossing rates as

$$A_1 = S_1 + S_0, \tag{41}$$

$$A_2 = S_2 + S_0, \tag{42}$$

respectively. However, it should be noted that with discrete observation, it is not strictly possible to assign a sense of rotation to the vector, because complete rotations of the vector between the instants of observation cannot be ruled out.

The crossing rate S_1 is obtained by dividing the probability that $X = X(t) > 0$, $Y = Y(t) < 0$ and $X' = X(t+\tau) > 0$, $Y' = Y(t+\tau) > 0$ by the sampling interval τ; thus,

$$S_1 = \frac{1}{\tau}\int_0^\infty \int_0^\infty \int_{-\infty}^0 \int_0^\infty p(X,Y,X',Y')dX\,dY\,dX'\,dY'. \tag{43}$$

Substituting from (6) and carrying out the integration, it may be shown that

$$S_1 = (4\pi\tau)^{-1}[\arccos c + \arcsin s] - (4\pi^2\tau)^{-1}[(\arccos c)^2 - (\arcsin s)^2]. \tag{44}$$

Similarly,

$$S_2 = (4\pi\tau)^{-1}[\arccos c - \arcsin s] - (4\pi^2\tau)^{-1}[(\arccos c)^2 - (\arcsin s)^2] \tag{45}$$

and

$$S_0 = (4\pi^2\tau)^{-1}[(\arccos c)^2 - (\arcsin s)^2]. \tag{46}$$

The conditional positive and negative axis-crossing rates as defined by (41) and (42) are now given by

$$A_1 = (4\pi\tau)^{-1}[\arccos c + \arcsin s], \tag{47}$$

$$A_2 = (4\pi\tau)^{-1}[\arccos c - \arcsin s]. \tag{48}$$

The above results may be used to verify Eq. (35) for the unconditional discrete zero-crossing rate. Reference to Fig. 4 shows that

$$S_+ = T_{41} + T_{42} + T_{31} + T_{32}. \tag{49}$$

Substituting in the above from (38)–(40) and (44)–(46) yields (35).

Under the assumption (36), we may again expand c, s and the inverse trigonometric functions in power series. This yields

$$S_1 = \tfrac{1}{2}(M_2^{\frac{1}{2}} + M_1) - \pi\tau(M_2 - M_1^2) + O(\tau^2), \tag{50}$$

$$S_2 = \tfrac{1}{2}(M_2^{\frac{1}{2}} - M_1) - \pi\tau(M_2 - M_1^2) + O(\tau^2), \tag{51}$$

$$S_0 = \pi\tau(M_2 - M_1^2) + O(\tau^2). \tag{52}$$

The series expansions for A_1 and A_2 are

$$A_1 = \tfrac{1}{2}(M_2^{\frac{1}{2}} + M_1) + O(\tau^2), \tag{53}$$

$$A_2 = \tfrac{1}{2}(M_2^{\frac{1}{2}} - M_1) + O(\tau^2). \tag{54}$$

It may be recalled that $M_2^{\frac{1}{2}}$, M_1 and $M_2 - M_1^2$ are equal to the root-mean-square frequency, mean frequency, and variance of hte Doppler spectrum, respectively. It is also seen that as $\tau \rightarrow 0$, $A_1, S_1 \rightarrow W_1$ and $A_2, S_2 \rightarrow W_2$. However, S_1 and S_2 differ from W_1 and W_2 by terms of order τ while A_1 and A_2 differ from W_1 and W_2 by terms of order τ^2. This is the reason why A_1 and A_2, rather than S_1 and S_2, were defined as the discrete analogs of the continuous conditional axis-crossing rates in the positive and negative directions. Eqs. (53) and (54) suggest that estimates of both f_{rms} and $\bar{f}$ denoted by $\tilde{f}_{\mathrm{rms}}$ and $\tilde{\bar{f}}$ may be obtained by measuring the conditional positive and negative axis-crossing rates, and computing their following combinations:

$$\tilde{f}_{\mathrm{rms}} = A_1 + A_2 = S_1 + S_2 = (2\pi\tau)^{-1}\arccos c, \tag{55}$$

$$\tilde{\bar{f}} = A_1 - A_2 = S_1 - S_2 = (2\pi\tau)^{-1}\arcsin s. \tag{56}$$

The estimate $\tilde{f}_{\mathrm{rms}}$ is identical to the zero-crossing rate estimate already discussed. An alternative estimator for $\bar{f}$ which is preferable to $\tilde{\bar{f}}$ is

$$\mathring{f} = \arctan\left\{\frac{\sin[2\pi\tau(A_1 - A_2)]}{\cos[2\pi\tau(A_1 + A_2)]}\right\}. \tag{57}$$

On substituting from (55) and (56), it is seen that

$$\mathring{f} = \arctan\left(\frac{s}{c}\right), \tag{58}$$

so that f is analogous to the pulse-pair estimator. For symmetrical spectra, the expected value of $\mathring{f}$ is equal to $\bar{f}$, while there is a bias in the case of unsymmetrical spectra. These points have been discussed by Benham *et al.* (1972) in connection with the pulse-pair estimator, and therefore will not be considered here. In meteorological practice, $\sigma_f\tau$ is generally less than 0.1 so that $\mathring{f}$ provides a very good approximation to $\bar{f}$.

6. Summary and concluding remarks

In this paper we have considered crossing rate methods of estimating Doppler spectral parameters of signals returned by meteorological targets (these signals are characterized by Gaussian statistics). It has been shown that with discrete sampling, as in the case of pulsed Doppler radar employing box-carring, the bipolar video crossing rate is a good estimator of the root-mean-square frequency of the spectrum if the standard deviation and mean of the spectrum are small compared to the sampling rate. On the basis of computations for model spectra, it has been suggested that an approximate correction may be applied to the discrete zero-crossing rate to recover a more accurate rms frequency if the spectrum variance is known.

The zero-crossing rate method has been extended to the complex plane, and a method of measuring both the spectrum mean and variance has been suggested. The conditional positive and negative axis crossing rates have been defined as the rates at which the signal vector crosses any given phase angle in the positive (counterclockwise) and negative (clockwise) directions, respectively. It has been shown that with continuous sampling the conditional positive axis crossing rate is equal to half the sum of the rms frequency and mean frequency of the spectrum, while the conditional negative axis-crossing rate is equal to half the difference of the two frequencies. The effects of discrete sampling of the signal on the axis-crossing rates have been considered. Again, the discrete crossing rates provide good approximations to the corresponding continuous crossing rates provided the standard deviation of the spectrum is small compared to the sampling rate. A combination of the conditional positive and negative axis-crossing rates (Eq. 57) has been suggested as an unbiased estimator of the spectrum mean for symmetrical spectra. This estimator for the spectrum mean is similar to the pulse-pair estimator.

Throughout this paper we have dealt only with expected values of crossing rate parameters. Future investigations should be directed toward computing the variances of the crossing rate parameters, so that the accuracy of this method of measurement may be assessed.

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