Weibull Distribution, Iterative Likelihood Techniques and Hydrometeorological Data

RAYMOND K. W. WONG

Atmospheric Sciences Division, Research Council of Alberta, Edmonton, Alberta, Canada T6G 2C2

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ABSTRACT

Rapidly converging maximum likelihood procedures for estimating and testing Weibull distribution parameters are presented, together with numerical examples of their applications. Goodness-of-fit comparisons based on nine sets of meteorological or hydrological data were made among the gamma, lognormal, three-parameter kappa and Weibull distributions. The Weibull distribution is shown to be a reasonable alternative to the other three distributions.

1. Introduction

The versatility of the Weibull distribution (Weibull, 1957) and its wide application in meteorology and hydrology are well known. The commonly used procedures to obtain its parameters usually do not follow the maximum likelihood (ML) approach, mainly because of its relatively complex computational aspects. An easy-to-follow and computationally efficient ML estimation technique is presented. Also presented are likelihood ratio test procedures for the evaluation of possible scale or shape differences between two Weibull populations. Goodness-of-fit comparisons based on precipitation (hailfall, rainfall and snowfall) and streamflow measurements are made among the gamma, three-parameter kappa [hereafter referred to as kappa (3)], lognormal and Weibull distributions. Numerical examples are also given.

2. The Weibull distribution

The density function $f(x)$ and the distribution function $F(x)$ of the two-parameter Weibull distribution are

$$f(x)=\begin{cases}\dfrac{\alpha}{\beta}\left(\dfrac{x}{\beta}\right)^{\alpha-1}\exp\left[-\left(\dfrac{x}{\beta}\right)^{\alpha}\right], & x\geqslant,\alpha>,\beta>0\\ 0, & \text{elsewhere}\end{cases}\tag{1}$$

$$F(x)=\begin{cases}1-\exp\left[-\left(\dfrac{x}{\beta}\right)^{\alpha}\right], & x\geqslant,\alpha>,\beta>0\\ 0, & \text{elsewhere}\end{cases}\tag{2}$$

where $\alpha>0$ is the shape parameter and $\beta>0$ is the scale parameter. The νth moment about zero of the Weibull random variable X is given by

$$E(X^{\nu})=\beta^{\nu}\Gamma(1+\nu/\alpha),\tag{3}$$

where

$$\Gamma(t)=\int_0^{\infty}\xi^{t-1}\exp(-\xi)d\xi$$

is the gamma function. In particular, the mean and variance of the Weibull distribution are $\mu=\beta\Gamma(1+1/\alpha)$ and $\sigma^2=\beta^2\{\Gamma(1+2/\alpha)-[\Gamma(1+1/\alpha)]^2\}$, respectively.

3. Estimation procedures for a given sample

a. *Method of moments estimates*

The moments method is used to provide initial estimates of α and β for the iterative ML procedure described in the next subsection. Considering the relations

$$\mu_2'/(\mu_1')^2=\Psi(\alpha)=\Gamma(1+2/\alpha)/[\Gamma(1+1/\alpha)]^2,\tag{4}$$

$$\beta=(\mu_2'/\mu_1')\chi(\alpha),\tag{5}$$

where $\chi(\alpha)=\Gamma(1+1/\alpha)/\Gamma(1+2/\alpha)$, the first and second sample moments ($\tilde{\mu}_1'$ and $\tilde{\mu}_2'$) and Table 1 may then be used to obtain the method of moments estimates $\tilde{\alpha}$ and $\tilde{\beta}$.

b. *Maximum likelihood* (ML) *estimation and Newton-Raphson iteration*

The ML estimates of α and β are the solutions of

$$\left.\begin{aligned}\frac{\partial \ln L}{\partial\alpha}&=0\\ \frac{\partial \ln L}{\partial\beta}&=0\end{aligned}\right\},\tag{6}$$

where

$$\ln L=N\ln\alpha-N\alpha\ln\beta+(\alpha-1)\sum_{i=1}^{N}\ln x_i-\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}$$

for the Weibull distribution and a sample of size N. The Newton-Raphson method for solving simultaneous equations is used. Given appropriate initial values $\alpha_{(0)}$ and $\beta_{(0)}$, the rth iterative solution yields

$$\left.\begin{aligned}\alpha_{(r)}&=\alpha_{(r-1)}+h_{(r)}\\ \beta_{(r)}&=\beta_{(r-1)}+k_{(r)}\end{aligned}\right\}, \tag{7}$$

where $h_{(r)}$ and $k_{(r)}$ are correction terms obtained in the rth step of iteration. Generally, the values of the correction terms (h,k) for each step may be represented, after Taylor expansion of (6), by

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 \ln L}{\partial\alpha^2} & \dfrac{\partial^2 \ln L}{\partial\alpha\partial\beta} \\ \dfrac{\partial^2 \ln L}{\partial\beta\partial\alpha} & \dfrac{\partial^2 \ln L}{\partial\beta^2} \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{\partial \ln L}{\partial\alpha} \\ -\dfrac{\partial \ln L}{\partial\beta} \end{bmatrix}. \tag{8}$$

The list of derivatives involved are given in Appendix A. The termination of iteration occurs when the correction terms reach a certain small value, for example, <0.0001. Extension to the three-equation case is obvious.

4. Likelihood ratio tests

Consider two samples, designated as sample 1 and sample 2, with sample sizes n_1 and n_2, respectively.

TABLE 1. The values of α, $\Psi(\alpha)$ and $\chi(\alpha)$.

α	$\Psi(\alpha)$	$\chi(\alpha)$
0.1	184752.38	0.00
0.2	252.00	0.00
0.3	30.24	0.00
0.4	10.87	0.03
0.5	6.00	0.08
0.6	4.09	0.16
0.7	3.14	0.25
0.8	2.59	0.34
0.9	2.24	0.42
1.0	2.00	0.50
1.1	1.83	0.57
1.2	1.70	0.63
1.3	1.60	0.68
1.4	1.52	0.72
1.5	1.46	0.76
1.6	1.41	0.79
1.7	1.37	0.82
1.8	1.33	0.85
1.9	1.30	0.87
2.0	1.27	0.89
2.5	1.18	0.95
3.0	1.13	0.99
3.5	1.10	1.01
4.0	1.08	1.02
4.5	1.06	1.03
5.0	1.05	1.03
5.5	1.04	1.04
6.0	1.04	1.04

TABLE 2. Null and alternative hypotheses (H_0 and H_1) of likelihood ratio tests.

Test 1	$H_0: \alpha_1=\alpha_2=\alpha;\ \beta_1=\beta_2=\beta$ $H_1: \alpha_1=\alpha_2=\alpha;\ \beta_1, \beta_2$
Test 2	$H_0: \alpha_1=\alpha_2=\alpha;\ \beta_1, \beta_2$ $H_1: \alpha_1, \alpha_2;\ \beta_1, \beta_2$
Test 3	$H_0: \alpha_1=\alpha_2=\alpha;\ \beta_1=\beta_2=\beta$ $H_1: \alpha_1, \alpha_2;\ \beta_1=\beta_2=\beta$
Test 4	$H_0: \alpha_1, \alpha_2;\ \beta_1=\beta_2=\beta$ $H_1: \alpha_1, \alpha_2;\ \beta_1, \beta_2$

Likelihood ratio tests for the scale or shape parameter differences between the samples may be made. Two alternative pairs of independent tests are possible. The null and alternative hypotheses for each test are listed in Table 2. These tests require the estimation of parameters using the method described in the previous section and the estimation of parameters involving the hypotheses $\alpha_1, \alpha_2; \beta_1=\beta_2=\beta$ and $\alpha_1=\alpha_2=\alpha; \beta_1, \beta_2$. The latter estimation procedures are also approached using the Newton-Raphson algorithm. The list of derivatives is shown in Appendix B. Reasonable initial values for these procedures are the ML estimates for the pooled sample (samples 1 and 2 combined).

The test statistic for each of the four tests described in Table 2 is evaluated from the logarithm of the joint likelihood function, i.e.,

$$\begin{aligned}\ln L_J={}&n_1\ln\alpha_1-n_1\alpha_1\ln\beta_1+(\alpha_1-1)\sum_{i=1}^{n_1}\ln x_{1i}\\ &-\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha_1}+n_2\ln\alpha_2-n_2\alpha_2\ln\beta_2\\ &+(\alpha_2-1)\sum_{j=1}^{n_2}\ln x_{2j}-\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha_2},\end{aligned} \tag{9}$$

where the subscripts 1 and 2 correspond to samples 1 and 2, respectively. After obtaining the ML estimates of the parameters for the null and alternative hypothesis of a particular test, the value of $\ln L_J(\hat{\omega})$ and $\ln L_J(\hat{\Omega})$ [the logarithm of the joint likelihood function relating to the null and alternative parameter space (ω and Ω), respectively] can be evaluated using (9). If the null hypothesis is in fact true, the likelihood ratio test statistic given by

$$T=2[\ln L_J(\hat{\Omega})-\ln L_J(\hat{\omega})] \tag{10}$$

has an approximate large sample distribution which is chi-square with one degree of freedom (Wilks, 1962).

5. Comparisons and numerical examples

Five sets of Alberta hailfall data were used in goodness-of-fit comparisons among the gamma, the log-

TABLE 3. The logarithm of likelihood function for comparison. Ranks are in parentheses.

Data	Lognormal	Gamma	Kappa (3)	Weibull
	Distribution			
Set A	−36.61 (2)	−38.04 (3)	−36.42 (1)	−38.23 (4)
Set B	−116.15 (1)	−117.88 (4)	−116.24 (2)	−117.56 (3)
Set C	−324.42 (4)	−324.08 (3)	−320.43 (1)	−322.71 (2)
Set D	−337.18 (4)	−334.39 (2)	−335.67 (3)	−334.28 (1)
Set E	−410.62 (4)	−406.78 (2)	−407.52 (3)	−406.06 (1)
Set F	−179.90 (1)	−180.72 (3)	−180.01 (2)	−182.22 (4)
Set G	−131.10 (4)	−130.46 (1)	−130.78 (3)	−130.55 (2)
Set H	−182.08 (3)	−182.31 (4)	−181.67 (1)	−181.81 (2)
Set I	41.64 (1)	40.26 (4)	40.81 (2)	40.36 (3)
Total rank	(24)	(26)	(18)	(22)

normal, the kappa (3) and the Weibull distributions. The ML estimation procedures relating to the gamma, the lognormal and the kappa (3) distributions are given elsewhere (Mielke, 1976; Baughman *et al.*, 1976; Meilke and Johnson, 1973). The data sets selected consist of different measures and a variety of sampling situations (e.g., storm intensity, network density, network location, etc.). These are measurements obtained from hail-dented 1 ft^2 styrofoam pads (Strong, 1974). Data sets A and B, are respectively, the maximum hail mass and the maximum impact energy from 17 haildays in 1973. Data sets C, D and E are hail impact energy values obtained during the 22 July 1973, 2 August 1973 and 16 August 1973 storms. Also used for comparison are the four sets of streamflow and precipitation data considered by Mielke and Johnson (1974). These are relabeled as data sets F, G, H and I, respectively. The results are summarized in Table 3.

The logarithm of the likelihood function ($\ln L$) is the criterion used in the goodness-of-fit comparisons. The

TABLE 4. Two samples of hail mass measurements.*

Sample 1	Sample 2
677	1753
457	692
378	914
998	623
1630	579
609	1592
10	815
35	433
101	43
421	15
601	75
279	30
29	192
135	12
148	905
665	749
	2860
	1193
	1475

* Data from Strong (1974). Mass measurements are in grams.

TABLE 5. Estimates of Weibull distribution parameters.

Moments method (only approximate values are necssary)	$\Psi(\alpha)$	$\tilde{\alpha}$	$\tilde{\mu}_2'/\tilde{\mu}_1'$	$\chi(\alpha)$	$\tilde{\beta}$
Sample 1	1.85	1.1	828.78	0.57	472.41
Sample 2	1.84	1.1	1451.57	0.57	827.39

Maximum likelihood method	α	β	$\ln L$
Sample 1	0.98	444.13	−113.68
Sample 2	0.86	737.73	−145.37

best-fit distribution for a given data set maximizes $\ln L$. The results show that the Weibull distribution compares favorably with the lognormal and the kappa (3) distributions in describing the data selected. The commonly used gamma distribution appears to be inferior in this respect.

For illustration, two samples of hail mass values and the estimates of their Weibull parameters are shown in Tables 4 and 5. The results of the tests are listed in Table 6. The four test statistics T_1, T_2, T_3 and T_4 give p (probability of type *I* error) values of 0.15, 0.65, 0.42 and 0.20, respectively.

These test statistics correspond respectively to the four tests listed in Table 2. Test 1 assumes a common shape parameter for the two samples considered and tests for a difference in the scale parameter, while Test 2 examines the common shape assumption. Alternatively, Test 3 assumes a common scale parameter for the two samples and tests for a difference in the shape parameter, while Test 4 examines the common scale assumption. In the example given, the test statistics indicate greater difference in the scale parameter than in the shape parameter between the two samples, though none of them is significant at the 5% level. The relative importance of these tests varies with the situation under which they are used. For example, when evaluating a cloud-seeding experiment where rain enhancement is the goal, one may expect to see a treatment-induced scale change when the rainfall data of a seeded and an unseeded sample are compared, the major interest will then be in Test 1 or Test 4.

For the cases used in testing the technique described in Appendix A, convergence is very rapid. It is observed that the estimation procedure of the parameters relating to the hypothesis α_1, α_2; $\beta_1=\beta_2=\beta$ described in Appendix B converges much faster than the one for the hypothesis $\alpha_1=\alpha_2=\alpha$; β_1, β_2. Preference is hence given to the second pair of tests in Table 2. However, for the example cited, the difference is between 0.70 s and 1.22 s of CPU time on the University of Alberta Amdahl 470V/6 computer. The program uses a Gaussian elimination subroutine in solving for the correction terms, with a convergence criterion of <0.0001.

TABLE 6. Summary statistics.

Test 1	$H_0: \alpha_1=\alpha_2=\alpha;\ \beta_1=\beta_2=\beta$	
	$H_1: \alpha_1=\alpha_2=\alpha;\ \beta_1, \beta_2$	
Under H_0	$\alpha = 0.87$	
	$\beta = 594.49$	
The logarithm of the likelihood function is -260.21.		
Under H_1	$\alpha = 0.91$	
	$\beta_1 = 432.01$	
	$\beta_2 = 756.15$	
The logarithm of the likelihood function is -259.15.		
The test statistic $T_1 = 2.12$.		
Test 2	$H_0: \alpha_1=\alpha_2=\alpha;\ \beta_1, \beta_2$	
	$H_1: \alpha_1, \alpha_2;\ \beta_1, \beta_2$	
Under H_0	$\alpha = 0.91$	
	$\beta_1 = 432.01$	
	$\beta_2 = 756.15$	
The logarithm of the likelihood function is -259.15.		
Under H_1	$\alpha_1 = 0.98$	
	$\alpha_2 = 0.86$	
	$\beta_1 = 444.13$	
	$\beta_2 = 737.73$	
The logarithm of the likelihood function is -259.05.		
The test statistic $T_2 = 0.2$.		
Test 3	$H_0: \alpha_1=\alpha_2=\alpha;\ \beta_1=\beta_2=\beta$	
	$H_1: \alpha_1, \alpha_2;\ \beta_1=\beta_2=\beta$	
Under H_0	$\alpha = 0.87$	
	$\beta = 594.49$	
The logarithm of the likelihood function is -260.21.		
Under H_1	$\alpha_1 = 1.00$	
	$\alpha_2 = 0.79$	
	$\beta = 559.32$	
The logarithm of the likelihood function is -259.88.		
The test statistic $T_3 = 0.66$.		
Test 4	$H_0: \alpha_1, \alpha_2;\ \beta_1=\beta_2=\beta$	
	$H_1: \alpha_1, \alpha_2;\ \beta_1, \beta_2$	
Under H_0	$\alpha_1 = 1.00$	
	$\alpha_2 = 0.79$	
	$\beta = 559.32$	
The logarithm of the likelihood function is -259.88.		
Under H_1	$\alpha_1 = 0.98$	
	$\alpha_2 = 0.86$	
	$\beta_1 = 444.13$	
	$\beta_2 = 737.73$	
The logarithm of the likelihood function is -259.05.		
The test statistic $T_4 = 1.66$.		

6. Conclusions

The Weibull distribution is a reasonable alternative to the lognormal and kappa (3) distributions for describing precipitation and streamflow data. It probably provides better fit than the commonly used gamma distribution in many meteorological or hydrological applications. Maximum likelihood estimation and likelihood ratio test procedures based on the Weibull distribution and using the Newton-Raphson approach are presented. Two alternative pairs of tests are described and the second pair is preferred for higher computational efficiency.

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APPENDIX A

Derivatives Involved in the Newton-Raphson Method to Obtain ML Estimates of Weibull Distribution Parameters

$$\frac{\partial \ln L}{\partial \alpha}=\frac{N}{\alpha}-N\ln\beta+\sum_{i=1}^{N}\ln x_i-\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln\left(\frac{x_i}{\beta}\right)$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{\alpha}{\beta}\left[\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}-N\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=-\frac{N}{\alpha^2}-\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}\left[\ln\left(\frac{x_i}{\beta}\right)\right]^2$$

$$\frac{\partial^2 \ln L}{\partial \beta^2}=\frac{\alpha}{\beta^2}\left[N-(\alpha+1)\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha\partial \beta}=\frac{\partial^2 \ln L}{\partial \beta\partial \alpha}=\frac{1}{\beta}\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}+\frac{\alpha}{\beta}\sum_{i=1}^{N}\left(\frac{x_i}{\beta}\right)^{\alpha}\ln\left(\frac{x_i}{\beta}\right)-\frac{N}{\beta}.$$

APPENDIX B

Derivatives for the Newton-Raphson Method to Obtain ML Estimates of Weibull Parameters for the Likelihood Ratio Tests

To evaluate $\ln L_J$ under the hypothesis $\alpha_1=\alpha_2=\alpha$; β_1, β_2, it is necessary to obtain $\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2$ by solving

$$\frac{\partial \ln L_J}{\partial \alpha}=0,$$

$$\frac{\partial \ln L_J}{\partial \beta_1}=0,$$

$$\frac{\partial \ln L_J}{\partial \beta_2}=0.$$

The derivatives involved are as follows:

$$\frac{\partial \ln L_J}{\partial \alpha}=\frac{n_1}{\alpha}-n_1\ln\beta_1+\sum_{i=1}^{n_1}\ln x_{1i}-\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha}\ln\left(\frac{x_{1i}}{\beta_1}\right)+\frac{n_2}{\alpha}-n_2\ln\beta_2+\sum_{j=1}^{n_2}\ln x_{2j}-\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha}\ln\left(\frac{x_{2j}}{\beta_2}\right)$$

$$\frac{\partial \ln L_J}{\partial \beta_1}=\frac{\alpha}{\beta_1}\left[\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha}-n_1\right]$$

$$\frac{\partial \ln L_J}{\partial \beta_2}=\frac{\alpha}{\beta_2}\left[\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha}-n_2\right]$$

$$\frac{\partial^2 \ln L_J}{\partial \alpha\partial \beta_1}=\frac{\partial^2 \ln L_J}{\partial \beta_1\partial \alpha_1}=\frac{1}{\beta_1}\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha}+\frac{\alpha}{\beta_1}\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha}\ln\left(\frac{x_{1i}}{\beta_1}\right)-\frac{n_1}{\beta_1}$$

$$\frac{\partial^2 \ln L_J}{\partial \alpha\partial \beta_2}=\frac{\partial^2 \ln L_J}{\partial \beta_2\partial \alpha}=\frac{1}{\beta_2}\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha}+\frac{\alpha}{\beta_2}\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha}\ln\left(\frac{x_{2j}}{\beta_2}\right)-\frac{n_2}{\beta_2}$$

$$\frac{\partial^2 \ln L_J}{\partial \beta_1\partial \beta_2}=\frac{\partial^2 \ln L_J}{\partial \beta_2\partial \beta_1}=0$$

$$\frac{\partial^2 \ln L_J}{\partial \alpha^2}=-\frac{(n_1+n_2)}{\alpha^2}-\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha}\left[\ln\left(\frac{x_{1i}}{\beta_1}\right)\right]^2-\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha}\left[\ln\left(\frac{x_{2j}}{\beta_2}\right)\right]^2$$

$$\frac{\partial^2 \ln L_J}{\partial \beta_1^2}=\frac{\alpha}{\beta_1^2}\left[n_1-(\alpha+1)\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta_1}\right)^{\alpha}\right]$$

$$\frac{\partial^2 \ln L_J}{\partial \beta_2^2}=\frac{\alpha}{\beta_2^2}\left[n_2-(\alpha+1)\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta_2}\right)^{\alpha}\right].$$

To evaluate $\ln L_J$ under the hypothesis α_1, α_2; $\beta_1=\beta_2=\beta$, it is necessary to obtain $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}$ by solving

$$\frac{\partial \ln L_J}{\partial \alpha_1}=0,$$

$$\frac{\partial \ln L_J}{\partial \alpha_2}=0,$$

$$\frac{\partial \ln L_J}{\partial \beta}=0.$$

The derivatives involved are as follows:

$$\frac{\partial \ln L_J}{\partial \alpha_1}=\frac{n_1}{\alpha_1}-n_1\ln\beta+\sum_{i=1}^{n_1}\ln x_{1i}-\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta}\right)^{\alpha_1}\ln\left(\frac{x_{1i}}{\beta}\right)$$

$$\frac{\partial \ln L_J}{\partial \alpha_2}=\frac{n_2}{\alpha_2}-n_2\ln\beta+\sum_{j=1}^{n_2}\ln x_{2j}-\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta}\right)^{\alpha_2}\ln\left(\frac{x_{2j}}{\beta}\right)$$

$$\frac{\partial \ln L_J}{\partial \beta}=\frac{\alpha_1}{\beta}\left[\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta}\right)^{\alpha_1}-n_1\right]+\frac{\alpha_2}{\beta}\left[\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta}\right)^{\alpha_2}-n_2\right]$$

$$\frac{\partial^2 \ln L_J}{\partial \beta\partial \alpha_1}=\frac{\partial^2 \ln L_J}{\partial \alpha_1\partial \beta}=\frac{1}{\beta}\left[\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta}\right)^{\alpha_1}+\alpha_1\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta}\right)^{\alpha_1}\ln\left(\frac{x_{1i}}{\beta}\right)-n_1\right]$$

$$\frac{\partial^2 \ln L_J}{\partial \beta\partial \alpha_2}=\frac{\partial^2 \ln L_J}{\partial \alpha_2\partial \beta}=\frac{1}{\beta}\left[\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta}\right)^{\alpha_2}+\alpha_2\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta}\right)^{\alpha_2}\ln\left(\frac{x_{2j}}{\beta}\right)-n_2\right]$$

$$\frac{\partial^2 \ln L_J}{\partial \alpha_1\partial \alpha_2}=\frac{\partial^2 \ln L_J}{\partial \alpha_2\partial \alpha_1}=0$$

$$\frac{\partial^2 \ln L_J}{\partial \alpha_1^2}=-\frac{n_1}{\alpha_1^2}-\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta}\right)^{\alpha_1}\left[\ln\left(\frac{x_{1i}}{\beta}\right)\right]^2$$

$$\frac{\partial^2 \ln L_J}{\partial \alpha_2^2}=-\frac{n_2}{\alpha_2^2}-\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta}\right)^{\alpha_2}\left[\ln\left(\frac{x_{2j}}{\beta}\right)\right]^2$$

$$\frac{\partial^2 \ln L_J}{\partial \beta^2}=\frac{n_1\alpha_1+n_2\alpha_2}{\beta^2}-\frac{\alpha_1(\alpha_1+1)}{\beta^2}\sum_{i=1}^{n_1}\left(\frac{x_{1i}}{\beta}\right)^{\alpha_1}-\frac{\alpha_2(\alpha_2+1)}{\beta^2}\sum_{j=1}^{n_2}\left(\frac{x_{2j}}{\beta}\right)^{\alpha_2}.$$

To evaluate $\ln L_J$ under the hypotheses α_1, α_2; β_1, β_2 and $\alpha_1=\alpha_2=\alpha$; $\beta_1=\beta_2=\beta$, the method in Appendix A is employed. The pooled sample is used for the latter, while in the former case, $\ln L_J$ is the sum of the $\ln L$ values obtained for the individual samples.

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