NOTES

On the Integral of the Surface Layer Profile-Gradient Functions

ROBERT BENOIT

Division de Recherche en Prévision Numérique, Service de l'Environnement Atmosphérique, Dorval, Canada

15 March 1977 and 31 May 1977

ABSTRACT

New forms of the exact primitives (Nickerson and Smiley, 1975) for the Businger *et al.* (1971) surface layer gradients are described for the unstable case. In addition to all the theoretical advantages of the Nickerson and Smiley solution, the new formulas do not suffer from a numerical divergence problem which so far has hindered the free use of the exact primitives for the slightly unstable case.

---

1. Introduction

In dealing with the stability-dependent surface layer (SL) one has to integrate the SL dimensionless gradients

$$\varphi_m = \frac{kz}{u_*}\frac{dU}{dz}, \tag{1}$$

where k is the von Kármán constant, u_* the friction velocity and U the mean wind speed at height z. This function occurs, in particular, when the vertical turbulent fluxes have to be estimated from the knowledge of mean properties (wind speed, temperature, specific humidity) at one "anemometer" level in the SL and at the boundary surface itself. This calculation often constitutes an important element in the more comprehensive parameterizations of the planetary boundary layer, like those of Deardorff (1972) and Benoit (1976).

High quality determinations of φ_m and φ_T, the equivalent function related to the mean potential temperature (Θ) profile, have been available since the 1970's. Our result is concerned with the widely used forms of φ_m and φ_T obtained by Businger *et al.* (1971). The primitives of these functions are put in the form

$$F = \int_{z_0}^{z} \frac{\varphi(z'/L)}{kz'}dz' = F\left(\frac{z}{L}, \frac{z_0}{L}\right). \tag{2}$$

With φ_m in (2), the F function, F_m, gives $U(z)/u_*$. In Eq. (2), L and z_0 are, respectively, the Monin-Obukhov length scale and the roughness length. We shall denote by F_T the integral corresponding to φ_T.

No problems are encountered for the stable case ($L>0$) where F is log-linear; the neutral case is approached smoothly both in the analytical and numerical sense, i.e., F tends toward its neutral form $F_N = k^{-1}\ln(z/z_0)$ as $L\rightarrow\infty$. Paulson (1970) has obtained an approximate form of F for the unstable case ($L<0$); since z_0 is generally much smaller than $|L|$, he replaced z_0 by zero in the calculation of the diabatic part of F to obtain

$$F_D = \int_{z_0}^{z} \frac{(\varphi-1)}{kz'}dz' \approx \int_{0}^{z} \frac{(\varphi-1)}{kz'}dz'. \tag{3}$$

This simplification is generally very tolerable. However, for the estimation of surface fluxes from wind and temperature measurements at one anemometer level and at the surface, it is very convenient to perform an "exchange" of stability parameter between its internal (or implicit) form z/L and an external form. The bulk Richardson number

$$\mathrm{Ri}_B = \frac{g}{\Theta_0}\frac{[\Theta(z_a)-\Theta(z_0)]}{U^2(z_a)}z_a \tag{4}$$

is a widely used (e.g., Deardorff, 1972) example of the latter form. Since F_T is, by definition, given by

$$F_T = \frac{-u_*}{(\overline{w\theta})_0}[\Theta(z)-\Theta(z_0)], \tag{5}$$

where $(\overline{w\theta})_0$ is the flux of potential temperature at the surface, it is easy to find that Ri_B is proportional to the product $(z_a/L)\cdot(F_T/F_m^2)$; if this expression does not vary monotonically with z_a/L, the usefulness of Ri_B is removed since a given Ri_B then corresponds with more than one value of z_a/L. This is exactly what occurs with the Paulson approximation for which we give an example in Table 1. For the chosen value of z_a/z_0, the invalid range begins around $-z_a/L\approx 300$.

More recently, Nickerson and Smiley (1975) have reported the exact expressions for F_m and F_T (their F and G). They present evidence that Ri_B now remains a smooth, monotonic function of z_a/L for any value of z_a/z_0 and as far as desired on the z_a/L axis. Thus their result is most welcome from the point of view of the stability parameter exchange problem.

There is only one slight difficulty with the use of their Eqs. (5) and (6) for F_m and F_T. Although these are valid for any unstable case (i.e., any negative value for z_a/L) and have the correct *analytical* limit as z_a/L tends to zero (i.e., they tend toward $F_{mN}=F_N$ and $F_{TN}=0.74\,F_N$), they are not well-behaved *numerically* when $|z_a/L|$ is too small. This is due to the appearance of 0/0 type fractions in their Eqs. (5) and (6).

2. Revised solutions

In his own derivation of the exact result, the author came to an expression which is fully equivalent to the Nickerson and Smiley equations but does not suffer from the above-mentioned numerical defect. The new proposed exact expressions are

$$F_m=\frac{1}{k}\left\{\ln(z/z_0)+\ln\left[\frac{(\zeta_0^2+1)(\zeta_0+1)^2}{(\zeta^2+1)(\zeta+1)^2}\right]+2[\tan^{-1}(\zeta)-\tan^{-1}(\zeta_0)]\right\}, \quad (6)$$

where

$$\zeta\equiv(1-15z/L)^{\frac{1}{4}},\quad \zeta_0\equiv(1-15z_0/L)^{\frac{1}{4}}.$$

For the temperature function, we have

$$F_T=\frac{0.74}{k}\left[\ln(z/z_0)+2\ln\left(\frac{\lambda_0+1}{\lambda+1}\right)\right], \quad (7)$$

where

$$\lambda\equiv(1-9z/L)^{\frac{1}{2}},\quad \lambda_0\equiv(1-9z_0/L)^{\frac{1}{2}}.$$

[Incidentally, the logarithmic function has been left out of their Eq. (6) due to a typographical error which was later reported in the Corrigendum on p. 979 of Vol. 14 of the *Journal of Applied Meteorology*.] It is evident that (6) and (7) have the proper limit as $-L$ goes to infinity.

Table 2 illustrates the *numerical* (only) divergence (from F_N and $0.74\,F_N$) of the Nickerson and Smiley version of F_m and F_T in the neutral limit, together with the restored convergence of F_m and F_T when these are computed according to our Eqs. (5) and (6). These calculations were made on a machine having a 10-digit accuracy. In the example shown, z/z_0 is 10^3 and the Nickerson and Smiley version of F_m starts to diverge around $-z/L=10^{-4}$; for a smoother surface, the difficulty begins to show at a smaller $|L|$, e.g., at $-z/L=10^{-2}$ for $z/z_0=10^5$.

TABLE 1. Example of a turning point (or non-monotonicity with z/L) for an external stability parameter $\mathrm{Ri}_B \propto (z/L)(F_T/F_m^2)$ when Paulson's (1970) approximation to F_m and F_T is used. The z/z_0 parameter is set at 10^3 (1E3)*.

$-z/L$	F_m	F_T	$\frac{z}{L}\frac{F_T}{F_m^2}$
0	19.74	14.61	0
1E−2	19.63	14.50	− 3.76E−4
1E−1	18.97	13.87	− 3.85E−3
1E0	16.63	11.50	− 4.16E−2
1E1	12.57	7.57	− 4.79E−1
1E2	7.43	3.02	− 5.47
2E2	5.75	1.59	− 9.62
3E2	4.74	0.75	−10.0
4E2	4.01	0.15	− 3.73
4.31E2	3.82	−2.94E−3	8.68E−2
1E3	1.66	−1.76	6.39E2

* Tabular entries, where appropriate, are multiplied by 10 to an exponent, e.g., $E2=10^2$.

TABLE 2. Numerical behavior of two exact versions for F_m and F_T in the neutral limit ($z/L\to 0$). The z/z_0 parameter is set at 10^3 (1E3).

	F_m-F_{mN}		F_T-F_{TN}	
$-(z/L)$	Nickerson and Smiley	Eq. (6)	Nickerson and Smiley	Eq. (7)
1	−3.09E0	−3.09E0	−3.09E0	−3.09E0
1E−1	−7.7E−1	−7.7E−1	−7.317E−1	−7.317E−1
1E−2	−1.02E−1	−1.02E−1	−9.199E−2	−9.199E−2
1E−3	−1.07E−2	−1.07E−2	−9.478E−3	−9.472E−3
1E−4	−1.07E−3	−1.07E−3	−9.497E−4	−9.502E−4
1E−5	3.83E−2	−1.07E−4	−9.451E−5	−9.504E−5
1E−6	6.37E−1	−1.07E−5	2.491E−1	−9.506E−6

3. Conclusion

New forms of the exact primitives for the Businger *et al.* (1971) dimensionless gradients of the SL are described for the unstable case. They are free from a numerical divergence occurring in the neutral limit. These functions should be useful to numerical modelers of boundary layer processes whose models are likely to produce, by mere randomness, values of the stability parameter (z/L) close enough to zero to get a computer overflow or at least a very large error in the calculation of F_m and F_T.

REFERENCES

Benoit, R., 1976: A comprehensive parameterization of the atmospheric boundary layer for general circulation models. Ph.D. thesis. McGill University, Montreal, Canada, 270 pp.

Businger, J. A., J. C. Wyngaard, Y. Izumi and E. F. Bradley, 1971: Flux-profile relationships in the atmospheric surface layer. *J. Atmos. Sci.*, **28**, 181–189.

Deardorff, J. W., 1972: Parameterization of the planetary boundary layer for use in general circulation models. *Mon. Wea. Rev.*, **100**, 93–106.

Nickerson, E. C., and V. E. Smiley, 1975: Surface layer and energy budget parameterizations for mesoscale models. *J. Appl. Meteor.*, **14**, 297–300.

Paulson, C. A., 1970: The mathematical representation of wind speed and temperature in the unstable atmospheric surface layer. *J. Appl. Meteor.*, **9**, 857–861.