A Mathematical Model for the Generation of Hourly Temperatures

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ABSTRACT

A stochastic model for hourly temperatures for Big Spring, Tex., has been developed. The governing parameters were deduced from an 11-year developmental sample, and give hourly temperatures as a function of harmonics representing annual and diurnal variations, and a first-order Markov chain process. The latter incorporates adjustments for the seasonal variation of the serial (hour-to-hour) correlation coefficient, and for the seasonal and diurnal variations of the variability and non-normality of frequency distributions of hourly temperatures. Each of the characteristics is given explicitly as a function of hour of the year.

Two 10-year samples were generated and compared to the developmental sample. Criteria were established to determine how well the model duplicates nature. The variability of mean monthly temperature and the frequency of occurrence of low diurnal ranges are underestimated. However, the model gives good estimates of the duration of temperatures below 32°F, and above 65° and 90°F, and of the frequency distribution of monthly 3, 6, 12, 24, 72 and 144 h maximum and minimum temperatures.

The general applicability of the model and its utility are discussed. The model could be used to determine the effects of climatic trends, e.g., a gradual cooling, on the average length of the growing season, the mean number of heating/cooling degree days, and other temperature-related parameters.

1. Introduction

A major concern of applied meteorology is the determination of the characteristics of hourly temperatures. These include the duration of temperatures above or below specified threshold values, the highest (lowest) temperature to be expected in a given duration, and the times of the first (last) occasion of specified temperatures over annual or diurnal cycles. These characteristics can be deduced, usually with considerable labor, from the climatic record or approximated by regression techniques or mathematical models.

Since the annual and diurnal variations of extraterrestrial radiation can be described adequately, and because the free-air temperature near the ground reflects this forcing function, at least in the mean, it should be possible to model the time course of mean hourly temperature by Fourier (harmonic) analysis. The irregular and aperiodic fluctuations which characterize weather, as opposed to climate, can be simulated by a Markov chain process.

Gringorten (1966) used the Markov chain process in developing a stochastic model which gives monthly values of the frequency and duration of a number of weather events, including temperature. Sharon (1967) showed that Gringorten's model has two shortcomings. It requires a vast amount of data and tends to overestimate the duration of weather events. He suggest that a short-period cycle, accounting for diurnal variations, be introduced into the model. In other relevant studies it has been shown that, when Fourier analysis is applied to mean monthly temperatures, the first harmonic accounts for about 98% of the variance (Craddock, 1956; Carson, 1963; Roden, 1965). Roden also found that all harmonics other than the first were statistically insignificant.

Bailey (1968) showed that the mean annual range of temperature can furnish a close estimate of the standard deviation of a complete population of hourly temperatures at a station. Polowchak and Panofsky (1968) subjected daily mean temperatures at 17 North American stations to spectrum analysis. After the annual variation was removed, most of the variance was in periods of one to two weeks, depending upon time of year and location.

In this study we attempt to develop a mathematical model of hourly temperatures for Webb Air Force Base, Big Spring, Tex. The objective is to use an 11-year developmental sample (1950–60) to derive parameters incorporable into an equation which can be used to generate many years of hourly temperatures. These experimental samples can be compared with the developmental sample to determine how successfully the model duplicates nature. Such a model, if properly

TABLE 1. Values of the parameters used to generate the annual and diurnal harmonics and the percent of variance accounted for.

Parameter	Value	Percent of variance accounted for Singly	Cumulative
$\bar{T}$	59.72		
A_1	−5.40	74.26	74.26
B_1	−19.86		
A_{365}	−10.01	21.74	96.00
B_{365}	−4.88		
A_{730}	2.19	0.89	96.89
B_{730}	0.52		
A_{1095}	0.56	0.08	96.97
B_{1095}	−0.34		

constructed, would have great utility. In addition to providing user-oriented characteristics of hourly temperature such as frequencies and durations, on a relatively short-term basis, one could determine the effect of prolonged warming or cooling on the mean length of the growing season, or the number of cooling or heating degree days. In this time of heightened concern with climatic changes, such a method would provide valuable insights.

2. Procedure

A plot of hourly temperatures for a year at a midlatitude station reveals two distinct periodicities, as well as irregular and aperiodic fluctuations. The former are the diurnal and annual variations, which are closely related to receipts of global radiation. Global radiation is in turn approximated by extraterrestrial radiation, receipts of which are easily quantified. The temporal variation of the latter, when accumulated over periods of a week or month, and plotted for a year, is clearly sinusoidal; receipts for 24 h periods are truncated sine curves.

Thus, the course of *mean* hourly temperatures can be approximated by harmonics which represent both the diurnal and annual cycles. To determine the parameters of these harmonics, the means of the 11 temperatures for each of the 8760 h of the year were calculated from the developmental sample, and Fourier analysis applied to these 8760 means. The first and 365th harmonics, corresponding to the annual and diurnal cycles, account for 96.0% of the variance of the original time series.

The diurnal cycle is not well represented by the 365th harmonic alone. The generalized diurnal variation of temperature is asymmetrical, the time from minimum to maximum being less than that from maximum to minimum. To make the times of highest and lowest daily temperatures agree with those of the developmental sample it was necessary to add the 730th and 1095th harmonics (corresponding to 12 and 8 h variations). The equation used to generate mean hourly temperatures is

$$\begin{aligned}T_t=\bar{T}&+\{A_1\sin[(360/N)t]+B_1\cos[(360/N)t]\}\\&+\{A_{365}\sin[(360/N)365t]+B_{365}\cos[(360/N)365t]\\&+A_{730}\sin[(360/N)730t]+B_{730}\cos[(360/N)730t]\\&+A_{1095}\sin[(360/N)1095t]\\&+B_{1095}\cos[(360/N)1095t]\},\end{aligned}\quad (1)$$

where T_t is the temperature at hour t, $\bar{T}$ the mean annual hourly temperature, A_i and B_i are amplitude coefficients determined from the developmental sample, the harmonic numbers are shown as subscripts, and N is the number of observations in the fundamental period (8760). The annual and diurnal terms are set apart by braces. The values of the constants and of the percent of the variance accounted for by each harmonic, are shown in Table 1. Note that close to 97% of the variance of mean hourly temperatures has been accounted for by the mathematical representation.

The model at this point gives only the periodic course of mean hourly temperatures. To simulate the irregular and aperiodic variations, i.e., the weather as opposed to climate, a first-order Markov chain process can be used. This is the process Gringorten (1966) employed, but his model did not incorporate annual and diurnal cycles.

In its general form a Markov chain process postulates that for a persistent time series

$$\left.\begin{aligned}X_1, X_2, X_3, \ldots, X_n\\X_i=aX_{i-1}+b\epsilon_i\end{aligned}\right\}, \quad (2)$$

where ϵ_i is a random variable independent of X_{i-1} (Feller, 1950). It is desirable that all X and ϵ_i be normally distributed with mean zero and variance one $(N/0,1)$. With that constraint

$$E(X_i)=E(X_{i-1})=E(\epsilon_i)=0,^1 \quad (3)$$

and, since by definition (Mood and Graybill, 1963),

$$\left.\begin{aligned}E(X_i^2)&\equiv\text{means of } X_i+\text{variance of } X_i\\E(X_i^2)&=E(X_{i-1}^2)=E(\epsilon_i^2)=1\end{aligned}\right\} \quad (4)$$

and because X_{i-1} and ϵ_i are independent,

$$E(X_{i-1}\epsilon_i)=E(X_{i-1})E(\epsilon_i)=0. \quad (5)$$

From (2)

$$X_i^2=a^2X_{i-1}^2+2abX_{i-1}\epsilon_i+b^2\epsilon_i^2,$$

$$E(X_i^2)=a^2E(X_{i-1}^2)+2abE(X_{i-1}\epsilon_i)+b^2E(\epsilon_i^2).$$

In view of (4) and (5), this becomes

$$1=a^2+b^2. \quad (6)$$

If ρ is the correlation between successive values of X, then by definition (Dixon and Massey, 1969),

$$\rho\equiv E(X_iX_{i-1})/\sigma_{X_i}^2\sigma_{Xi-1}^2,$$

[1] For $E(X_i)$ read expected value of X_i.

where the terms in the denominator are the variances of X_i and X_{i-1}; or

$$\rho = E(X_i X_{i-1}). \quad (7)$$

From (2)

$$X_i X_{i-1} = a X_{i-1} X_{i-1} + b \epsilon_i X_{i-1},$$
$$E(X_i X_{i-1}) = aE(X_{i-1} X_{i-1}) + bE(\epsilon_i X_{i-1}).$$

In view of (4), (5) and (7)

$$a = \rho, \quad (8)$$

and in view of (6) and (8)

$$b = (1-\rho^2)^{\frac{1}{2}}. \quad (9)$$

Therefore,

$$X_i = \rho X_{i-1} + (1-\rho^2)^{\frac{1}{2}} \epsilon_i, \quad (10)$$

where the X_i, X_{i-1} and ϵ_i are $N/0$, 1, and the ϵ_i can be picked at random by the computer. The model that results when the X_i of (10) are added to the T_i of (1) may be used to generate temperatures which will have irregular fluctuations, but the frequency distributions of these will of course also be $N/0$, 1.

An examination of the characteristics of hourly temperatures, as deduced from the developmental sample, reveals that the model at this point fails to duplicate nature in the following ways. These are listed in decreasing order of their significance, i.e., the model will be grossly in error if the first item below is not accounted for, whereas the omission of allowance for fifth item will produce negligible differences between the model and nature.

1) The standard deviation of hourly temperatures (how much the temperature at any hour of the year varies over many years) is much greater than unity, and varies with hour of the day and day of the year. In general, temperatures are more variable in winter and during the warmest part of the day.

2) Although the frequency distributions of hourly temperatures are approximately normal in winter, those in summer are negatively skewed. This variation appears to be periodic, and changes gradually with the seasons.

3) The serial correlation coefficient ρ also varies seasonally. Although the differences are small, ρ is highest in winter and lowest in summer.

4) The time at which maximum and minimum diurnal temperatures are observed varies seasonally. On the average, minima occur just about sunrise, and maxima at 1500 h in the winter and 1600 h in the summer.

5) The mean monthly diurnal range of temperature varies among months. This variation is not large, however, being about 4°F between extremes.

Characteristics 1)–3) can be incorporated into the model by making them explicit functions of hour of the year. The manner in which this is accomplished is explained next. Because 4) and 5) are minor, and because allowance for them would make the model more complex than is necessary at this point, they are disregarded.

TABLE 2. Values of the parameters used to generate the standard deviation of hourly temperatures and the percent variance accounted for.

Parameter	Value	Percent of variance accounted for: Singly	Cumulative
$\bar{S}$	8.39		
C_1	1.78	64.17	64.17
D_1	2.53		
C_{36}	−1.56	22.03	86.20
D_{36}	−0.92		

Since, according to the developmental sample, the standard deviation of hourly temperatures changes only gradually with the seasons, this parameter was calculated from the developmental sample in 10-day periods. Each month was divided into three periods: first 10 days, second 10 days, and the remaining days. Then the standard deviation of all the 0100 hour, 0200 hour, . . . , 2400 hour temperatures for each period for all 11 years, was calculated and analyzed. There were, therefore, exactly or approximately 110 values of hourly temperatures in each calculation of each of 864 values of standard deviation.

This analysis shows that the standard deviation is 180° out of phase with the annual variation of temperature, i.e., the highest values occur at times of lowest temperatures and vice versa, and that this parameter is in phase with the diurnal variation of temperature. This time dependence again suggests Fourier analysis. This was performed on the standard deviations of the 864 periods specified above. The first harmonic (annual variation) and the 36th (diurnal variation) account for 86.2% of the variance. These two harmonics were thus used to approximate the time-dependent course of the standard deviation of hourly temperatures, and an equation similar to (1) was incorporated into the model.

Table 2 gives the values associated with this Fourier analysis. Fig. 1 shows both the values of standard deviation calculated from the developmental sample, and those generated by the analysis. Perhaps surprisingly, very little smoothing was necessary in drawing the values from the developmental sample.

Expression (10) is a function of the serial correlation coefficient ρ of successive hourly temperatures. Sharon (1967) has discussed the choice of ρ and its influence on the output of Gringorten's model. He notes that when cyclical changes are not in a model the value or values of ρ which produce the best results are not those derived from the serial correlation in a natural series. The implication here, it would seem, is that this discrepancy might be overcome in the model developed here because this model does incorporate annual and diurnal cycles. In addition, since the *modus operandi* of

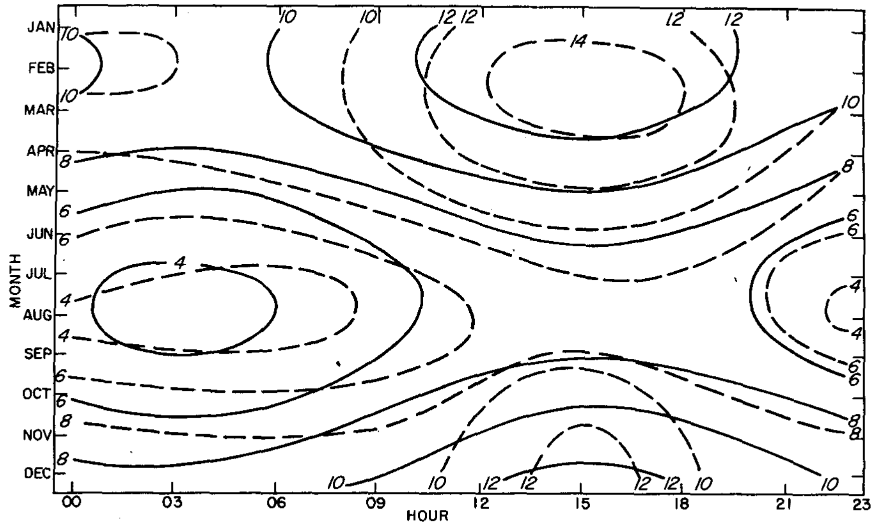

FIG. 1. Time variation of the standard deviation of hourly temperatures. The dotted lines are isopleths (°F) as determined from the developmental sample, the solid lines isopleths of the generated values.

this research has been to let nature "tell" us the values of the model's parameters, it seems reasonable to insert into (10) the values of ρ derived from the developmental sample.

Accordingly, the serial correlation of successive hourly temperatures was calculated for each month of each of the 11 years (resulting in a series of $24\times31 = 744$ for any January, for example), and these values averaged for each month. As Table 3 indicates, there is very little difference among these; they average 0.964, with a minimum of 0.962 in August and a maximum of 0.971 in March. There is a systematic variation among the months, however; for this area of Texas, and unpublished research has shown this to apply to the north central and eastern United States, ρ is highest at times when frontal passages and air mass changes are most frequent, and lowest in their absence. This observation is contrary to Sharon's assertion that series in which variations are most prominent would yield lower serial correlation coefficients.

Although these monthly mean values of ρ vary in a regular manner, they were not subjected to harmonic analysis because the annual variation is so small. For consistency, the values of ρ used in (10) were interpolated from the above analysis for each of the 36 ten-day periods of the year.

With the above refinements the time dependence of both the standard deviation and the serial correlation coefficient have been accounted for. However, at this point the model will produce frequency distributions for each of the 8760 hourly temperatures which are normal, or nearly normal, with variance as described above. As previously noted, hourly temperatures at Big Spring are nearly normally distributed during the winter, but are markedly negatively skewed during the summer.

To investigate this skewness each month was divided into three periods, as before, and the skewness of the frequency distributions of each of the 864 periods was calculated. The skewness parameter used was the ratio

TABLE 3. Serial correlation coefficients ($\times10^3$) between successive hourly temperatures for each month of each year of the 11-year developmental sample and the annual and monthly means.

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Annual mean
1	963	959	963	969	954	965	940	942	948	946	967	969	957
2	972	974	971	972	968	963	957	952	967	967	968	975	967
3	968	960	969	962	971	947	954	941	946	959	977	961	960
4	964	970	965	965	978	957	954	957	953	964	963	966	963
5	967	963	977	963	974	963	957	954	955	974	955	969	964
6	973	971	976	965	967	956	959	955	959	961	970	967	965
7	965	983	967	972	956	949	950	950	951	963	963	962	961
8	974	967	971	977	951	972	945	948	957	976	973	949	963
9	962	969	972	975	968	954	954	951	968	971	967	967	965
10	969	977	965	979	969	956	958	957	965	963	973	954	965
11	974	973	981	968	965	962	965	961	959	969	966	969	968
Mean	968	970	971	970	966	958	954	952	958	965	967	964	964

of the third moment of the frequency distribution to the cube of its standard deviation.

This analysis shows that the skewness of frequency distributions of hourly temperatures is time dependent, varying with day of the year and hour of the day. Representative values are −1.40 for the hottest time of the day (about 1600 hours) in the summer, and around zero (no skewness) for the hours just before sunrise in the winter.

Because this skewness is time dependent and shows relatively large variations throughout the year, the calculated values were subjected to Fourier analysis. The first harmonic (annual variation) and the 36th (diurnal variation) account for 40.1% of the variance (Table 4). These harmonics were used to approximate the time-dependent course of the skewness of frequency distributions of hourly temperatures, and an equation similar to (1), for skewness, was incorporated into the model. The parameters of this equation are also given in Table 4.

Fig. 2 compares the values of skewness calculated from the developmental sample with those derived from the Fourier expression. Considerable smoothing was required in drawing the isopleths for the developmental sample. Since a smaller proportion of the variance is accounted for than was the case with standard deviation, the correspondence between observed and predicted values is not as good (cf. Figs. 1 and 2). This could be anticipated because the third moment of a frequency distribution is, in general, subject to more nonsystematic variability than is the second moment. In part for this reason, we believe that the remaining variance is not systematically ordered, and that the calculation of other harmonics would not be worthwhile.

Now we need a function which will transform variates which are normally distributed into variates which will be distributed with specified skewness. Brooks and Carruthers (1953) show that the adjusted normal distribution transformation accomplishes the reverse of this, i.e., transforms a non-normal distribution to normality. According to them, this transformation corrects for the skewness by adding the quantity Sk(B) to the area under the normal frequency curve to the left of a given value of $Z[Z=(X-\bar{X})/\sigma_X]$. Sk is the skewness of the frequency distribution, as previously defined, and $B=(y/6)(1-Z^2)$, where y is the ordinate of the normal curve. This adjusted normal transformation thus becomes

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x'} \exp(-Z'^2/2)dZ' = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} \exp(-Z^2/2)dZ, \quad (4)$$

where Z, Sk, and B are as above, X is the original variate from the skewed distribution, X' the transformed variate, which will be normally distributed, and $Z'=(X'-\bar{X})/\sigma_X$.

TABLE 4. Values of the parameters used to generate the skewness of frequency distributions of hourly temperatures, and the percent variance accounted for.

Parameter	Value	Percent of variance accounted for: Singly	Cumulative
$\overline{Sk}$	−0.4125		
E_1	0.0910	13.18	13.18
F_1	0.1556		
E_{36}	0.2538	26.91	40.09
F_{36}	0.0400		

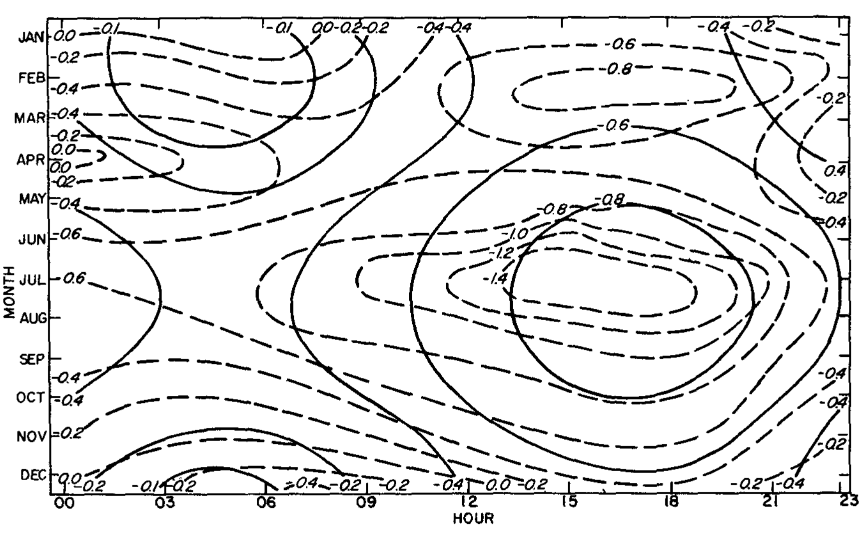

FIG. 2. Time variation of the skewness of the frequency distribution of hourly temperatures. The dotted lines are isopleths determined from the developmental sample, the solid lines isopleths of the generated values.

By substitution for B, Eq. (4) becomes

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x'} \exp(-Z'^2/2)dZ' = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} \exp(-Z^2/2)dZ + \mathrm{Sk}(y/6)(1-Z^2).$$

By definition (Panofsky and Brier, 1958)

$$y = 0.3989 \exp(-Z^2/2).$$

Therefore,

$$(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x'} \exp(-Z'^2/2)dZ' = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} \exp(-Z^2/2)dZ + 0.0665\,\mathrm{Sk}(1-Z^2)\exp(-Z^2/2). \quad (5)$$

When (5) is solved for X' and Z' (i.e., when X, Z and Sk are known) the adjusted normal transformation is obtained and a skewed distribution is transformed into normality. When (5) is solved for Z and X (meaning X', Z' and Sk are known) the reverse adjusted normal transformation is obtained, and a normal distribution is transformed into one with specified skewness.

Expression (5) must be solved numerically. It is a transcendental equation, and there are three classical methods used to approximate its roots (Hovanessian and Pipe, 1969)—bisection method, secant method and Newton's method. In testing the validity of the reverse adjusted normal distribution it was determined that (5) should be solved for Z using (i) Newton's method when $-2<Z<2$; (ii) the secant method when $\mathrm{Sk}\leqslant 0$ and $Z'\leqslant -2$, or when $\mathrm{Sk}>0$ and $Z'\geqslant 2$; and (iii) the bisection method when $\mathrm{Sk}>0$ and $Z'\geqslant 2$, or when $\mathrm{Sk}>0$ and $Z'\geqslant 2$.

The model now can generate normally distributed temperatures and transform these into appropriately skewed distributions. The model accounts for the time dependence of the standard deviation, of the serial correlation coefficient, and of the skewness of frequency distributions of temperature. Because the Markov chain process is a function of ϵ_i, a randomly selected variate, any number of dissimilar samples of hourly temperatures for a year can be generated. The model was run on the computer, and two 10-year experimental samples were generated.

3. Verification

Ideally, the experimental samples generated by the model should be compared with hourly temperatures observed at a period of time other than that of the developmental sample, e.g., the succeeding 10-year period. Since one objective of any numerical simulation is to predict future occurrences, verification against such an independent sample would provide better estimates of the utility of the model than verification against the sample from which the model was developed. However, since it was thought necessary to use all of the available 11 years of record to develop reasonably stable statistics, no such independent sample could be utilized. And, since this research constitutes a first step in the construction of a model of hourly temperatures, at least in terms of incorporating annual and diurnal cycles, it seems reasonable in this case to verify the model against the sample from which it was developed.

The verification criteria employed include two of those suggested by Gringorten (1966). These represent user-oriented needs, and are identified as 3) and 4) below. In addition, and to go beyond the utilitarian aspects and investigate, in a more general way, how well the model duplicates natures, criteria 1) and 2) are included:

1) Mean monthly, and mean annual temperatures and their variability.
2) Diurnal ranges of temperature.
3) The duration of hourly temperatures 90°F and above, 65°F and above, and 32°F and below, for specified lengths of consecutive hours, by months.
4) The frequency distributions of the monthly M-hour maxima and minima.

With respect to the first criterion, summarized in Table 5, monthly mean hourly temperatures compare well. This is expected, of course, since the model must, in the long run, match the temperatures of the developmental sample. The standard deviations of monthly mean hourly temperatures do not compare favorably. The model underestimates the variability; in 21 of the 24 comparisons the variability of the model is less than that of nature. Similarly, the annual mean hourly tem-

TABLE 5. Comparison of the means and the standard deviations of monthly mean hourly temperatures for the developmental sample and the two experimental samples.

	Developmental sample		Experimental sample 1		Experimental sample 2	
Month	Mean	Std dev	Mean	Std dev	Mean	Std dev
Jan	40.67	3.70	39.48	1.83	40.72	2.52
Feb	43.29	4.45	39.48	4.12	41.06	4.20
Mar	49.24	3.79	49.17	3.06	49.96	2.54
Apr	59.14	2.80	60.66	2.53	59.14	3.16
May	68.11	2.40	69.37	3.30	69.43	2.40
Jun	77.73	2.54	76.87	2.30	77.25	1.45
Jul	79.05	2.48	79.92	1.37	79.84	0.83
Aug	78.19	1.50	78.04	1.00	78.01	0.80
Sep	71.35	2.11	70.55	1.00	70.19	1.32
Oct	60.79	2.44	59.63	1.91	59.99	2.35
Nov	47.26	2.75	49.20	2.12	49.53	1.44
Dec	41.20	2.56	42.07	2.45	41.48	3.29
Annual	59.78	1.24	59.63	0.63	59.80	0.56

TABLE 6. Comparison of the percent occurrence (first column) and cumulative frequency (second column) of the indicated diurnal temperature ranges.

Month	Diurnal temperature range (°F) 0–5		6–10		11–15		16–20		21–25		26–30		31–35		35		Mean
	00%*	00%	00%	00%	00%	00%	3%	3%	23%	26%	30%	56%	22%	78%	22%	100%	29.41
January	00*	00	00	00	1	1	3	4	19	23	22	45	29	74	26	100	30.13
	1*	1	3	4	10	14	13	27	16	43	22	65	15	80	20	100	25.90
	00	00	1	1	2	3	11	14	23	37	24	61	23	84	16	100	27.84
April	00	00	00	00	2	2	9	11	25	36	27	63	24	87	13	100	27.92
	1	1	4	5	9	14	7	21	13	34	28	62	15	77	23	100	27.07
	00	00	00	00	1	1	13	14	30	44	33	77	21	98	2	100	26.28
July	00	00	1	1	2	3	12	15	30	45	36	81	18	99	1	100	25.79
	00	00	2	2	3	5	19	24	29	53	35	88	11	99	1	100	24.44
	00	00	00	00	1	1	6	7	27	34	41	75	16	96	4	100	25.66
October	00	00	00	00	3	3	10	13	28	41	34	75	20	95	5	100	26.60
	00	00	5	5	8	13	11	24	20	44	15	59	21	80	20	100	26.55

* The first experimental sample, the second experimental sample and the developmental sample.

peratures compare favorably, but the model underestimates the variability.

Table 6 compares the percent occurrence and cumulative frequency of the diurnal temperature ranges for four months (January, April, July and October). In general, the model underestimates low diurnal ranges; this is compensated by an overestimation of high diurnal ranges. January and July are the months in which this discrepency is most apparent. The departures from the developmental sample in April are mixed, while in October overestimation occurs in the 21–30°F range and lower and higher ranges are underestimated.

The mean diurnal range for each sample also is given in Table 6. The three samples compare well in April and October, but the mean diurnal range is somewhat overestimated in January and July. An interesting feature of the model is evident in these numbers. No explicit allowance for diurnal range variation among months was included in the model. But the random element was modified to incorporate seasonal and diurnal changes in variability. Thus, when hourly temperatures are relatively highly variable, as in January, the diurnal range increases. Conversely, it is less in the other three months.

How well the model duplicates nature with regard to the duration of hourly temperatures above or below specified threshold values is shown in Tables 7, 8 and 9. If N is the number of hours in a month (e.g., 744 in January) and M the number of consecutive hours, there are $N-M+1$ separate but overlapping M-hour periods in the month. Each M-hour period (where $M=3$, 6, 12 and 24 h) in each month of the two 10-year experimental samples and the 11-year developmental sample was examined. Each occurrence of these durations was determined, and the mean and standard deviation of the number of occurrences for each threshold temperature-month-duration were calculated. Tables 7, 8 and 9 show the mean occurrences for temperatures of 90°F and above, 65°F and above, and 32°F and below, respectively. Also shown are percentage departures from the developmental sample mean for each experimental sample mean.

Since the mean, the standard deviation and the number of observations in each of two samples are known, Student's t-statistic can be used to determine the probability that both samples were drawn from the

TABLE 7. Comparison of the average occurrence of temperatures of 90°F and above for 3 and 6 consecutive hours.

Month	Number of consective hours 3 h		6 h	
	4.7*	213.3**	1.0	233.3
April	3.2*	113.3	0.5	66.7
	1.5*		0.3	
	28.1	47.9	9.2	37.3
May	20.2	6.3	5.7	−14.9
	19.0		6.7	
	72.9	−15.6	30.9	−15.1
June	70.9	−17.9	28.1	−22.8
	86.4		36.4	
	105.4	20.0	47.7	32.5
July	100.8	14.8	45.0	25.0
	87.8		36.0	
	78.8	10.8	31.2	28.4
August	77.1	8.4	30.0	23.5
	71.1		24.3	
	20.3	18.0	5.5	61.8
September	18.6	8.1	3.0	−11.8
	17.2		3.4	
	1.3	62.5	0.1	—
October	0.6	−25.0	—	—
	0.8		—	—

* The first experimental sample, the second experimental sample and the developmental sample.
** Percent change from the developmental sample.

TABLE 8. Comparison of the average occurrence of temperatures of 65°F and above for 3, 6, 12 and 24 consecutive hours.

Month	Number of consecutive hours							
	3 h		6 h		12 h		24 h	
	26.5*	12.3**	12.7	122.8	1.2	—	—	—
January	22.3*	−5.5	8.3	45.6	—	—	—	—
	23.6*		5.7		—	—	—	
	21.0	−40.8	9.0	−38.8	0.4	300.0	—	—
February	25.1	−29.3	10.1	−22.3	—	—	—	—
	35.5		13.0		0.1	—	—	—
	74.8	−19.3	36.2	−23.3	1.9	−32.1	—	—
March	75.0	−18.1	38.7	−18.0	5.1	82.1	—	—
	91.6		47.2		2.8		—	
	216.2	10.4	147.8	12.1	48.7	24.6	14.3	217.8
April	186.8	−4.6	123.8	−6.1	33.2	−15.1	1.7	−62.2
	195.9		131.8		39.1		4.5	
	410.8	8.6	341.6	12.6	218.5	28.4	112.0	76.1
May	417.7	10.4	344.6	13.6	212.0	24.6	98.0	54.1
	378.3		303.3		170.1		63.6	
	607.4	−2.0	569.2	−2.7	498.9	−4.4	396.0	−8.4
June	624.3	0.8	583.9	−0.2	508.1	−2.6	399.0	−7.7
	619.6		585.2		521.9		432.5	
	704.3	−1.6	683.8	−2.2	645.8	−3.3	582.8	−5.0
July	703.2	−1.8	682.8	−2.4	642.9	−3.7	571.5	−6.9
	715.9		699.4		667.7		613.7	
	681.2	−2.1	652.3	−2.4	596.5	−2.9	509.9	−3.0
August	681.8	−2.0	652.0	−2.5	595.8	−3.0	504.2	−4.1
	695.6		668.4		614.4		525.7	
	431.2	−8.0	356.9	−9.2	222.4	−12.4	106.6	0.5
September	[419.3	−10.5]	[343.0	−12.8]	205.7	−19.0	85.0	−19.9
	468.5		393.2		253.8		106.1	
	185.5	−13.8	116.9	−19.0	25.0	−42.7	4.3	−60.2
October	192.3	−10.7	125.0	−13.4	30.4	−30.3	5.1	−52.8
	215.3		144.4		43.6		10.8	
	53.4	−2.0	23.2	26.8	0.6	50.0	—	—
November	57.8	6.0	22.5	23.0	0.6	50.0	—	—
	54.5		18.3		0.4		—	—
	28.4	61.4	13.3	146.3	3.6	1100.0	—	—
December	20.5	16.5	8.3	53.7	0.4	33.3	—	—
	17.6		5.4		0.3		—	

* The first experimental sample, the second experimental sample and the developmental sample.
** Percent change from the developmental sample.
Values in brackets are those which the t-test indicates that the experimental sample and the developmental sample could not have come from the same population.

same population. This statistic was calculated for each threshold temperature-month-duration pair. At the 5% significance level, the hypothesis that the two samples (one developmental, one experimental) could have been drawn from the same population was rejected in only four of the 152 pairings. These four are shown in brackets in the tables.

Thus, the differences between the model and nature are not statistically significant. Note, however, that the signs of the departures in Tables 7–9 tend to be similar across both months and durations. For example, in Table 9, the departures for March, April, November and December, for all durations, are negative. The negative departures across durations can be explained, for the most part, because these values reflect the strong persistence of hourly temperatures, and thus are not independent. If a period of temperatures is below (above) the threshold value for 3 h, it is likely to be below (above) for longer durations.

If the signs of departures are viewed across months an association with Table 6 is apparent. When the mean temperature for an experimental month is different from that of the developmental sample, the average occurrence of temperatures above or below the threshold temperatures follows predictably. For example, the mean February temperatures of both experimental samples are below that of the developmental sample. Therefore the average occurrences of

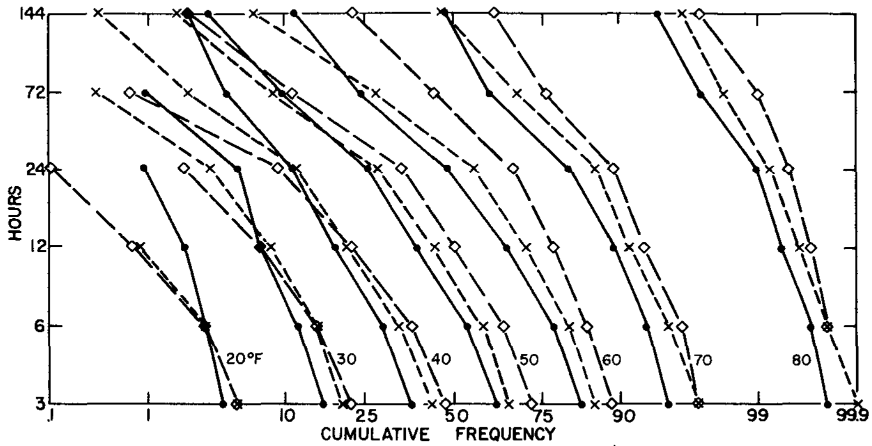

FIG. 3. Cumulative frequency distributions for the duration in hours of the specified temperatures. The points for the first experimental sample are designated by a diamond, for the second experimental sample by an X, and for the developmental sample by a dot. February maximum.

temperatures 32°F and below for 3 h, and of 65°F and above for 3 h, are greater and less, respectively, than the average occurrence for the developmental sample. Similarly, the mean temperatures of both experimental samples for May are above that of the developmental sample, and both the average occurrence of temperatures 65°F and above, and of 90°F and above, are higher than those of the developmental sample.

Thus, the discrepencies between the model and nature in Tables 7–9 are the result of the models' mean monthly temperatures being different from those of the developmental sample. Of the 47 comparisons in these tables (excluding three occasions when an experimental mean monthly temperature was exactly the same as that of the developmental sample), 34 are correctly indicated by mean monthly temperature differences. In the remaining 13 comparisons the differences in mean monthly temperatures, experimental versus developmental, are to small, generally, to produce appreciable differences in the mean occurrence of temperatures above or below specified temperature thresholds.

The maxima and minima of each separate but overlapping 3, 6, 12, 24, 72 and 144 h period, for each month, were determined for the two experimental

TABLE 9. Comparison of the average occurrence of temperatures of 32°F and below for 3, 6, 12 and 24 consecutive hours.

	Number of consecutive hours							
Month	3 h		6 h		12 h		24 h	
	199.6*	9.8**	143.5	5.5	66.9	−8.4	19.1	−45.9
January	177.4*	−2.4	125.4	−7.8	56.7	−22.3	19.7	−44.2
	181.8*							
	177.2	34.0	127.1	27.7	59.6	8.2	16.0	−45.4
February	164.3	24.3	123.5	24.1	69.3	25.8	35.3	20.5
	132.2		99.5		55.1		29.3	
	65.5	−7.9	44.6	−9.2	22.7	−5.0	9.3	−7.0
March	45.1	−36.6	24.6	−49.9	5.2	−78.2	0.2	−98.0
	71.1		49.1		23.9		10.0	
	3.1	−43.6	1.1	−59.3	—	—	—	—
April	3.2	−41.8	1.2	−55.6	—	—	—	—
	5.5		2.7		1.1			
	36.2	−48.9	[22.5	−54.2]	8.7	−53.4	2.1	−48.8
November	[32.6	−54.0]	21.1	−57.0	9.2	−50.8	0.9	−78.0
	70.8		49.1		18.7		4.1	
	146.3	−8.0	102.4	−7.5	46.0	−8.0	18.0	−20.4
December	136.4	−14.3	91.3	−17.5	36.3	−27.4	8.3	−63.3
	159.1		110.7		50.0		22.6	

* The first experimental sample, the second experimental sample and the developmental sample.
** Percent change from the developmental sample.
Values in brackets are those which the t-test indicates that the experimental sample and the developmental sample could not have come from the same population.

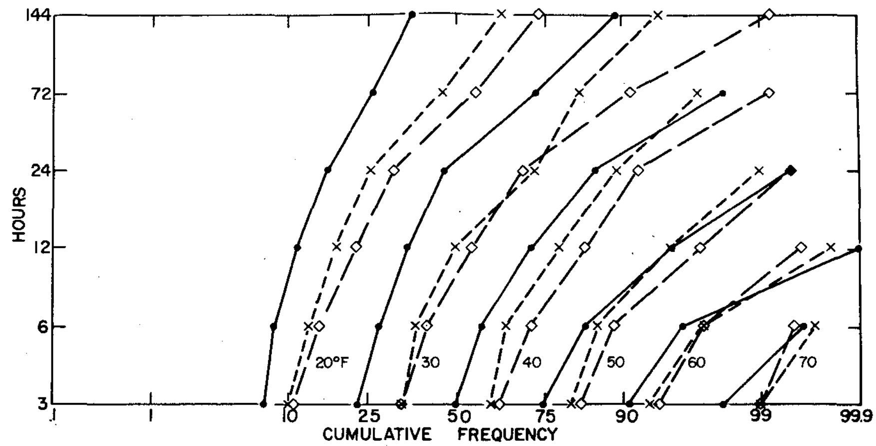

FIG. 4. As in Fig. 3 except for February minimum.

samples and the developmental sample. The results are given in Figs. 3–10, which show maxima and minima for February, May, August and November. All 12 months were examined, but for the purposes of this analysis, these four months will suffice.

Fig. 3, which shows February maxima, is interpreted as follows. For all three samples there is about a 7% probability that the maximum temperature in any 12 h period in February will be less than 30°F, and probabilities of 7, 11 and 23% (for the second experimental sample, the developmental sample, and the first experimental sample, respectively), that the maximum temperature in any 144 h period will be less than 60°F. Occasionally values for one sample are missing, either because temperatures this extreme for that particular month were not observed or not generated, or because the cumulative percentage was less than 0.1% or greater than 99.9%.

In evaluating the utility of the model with respect to this criterion, the fourth, the following are the principal conclusions. The cumulative frequency estimates of the experimental samples are, in general, close to those of the developmental sample. Specifically, for 803 or 91.7%, of the 876 plotted data points for each of the three samples, the cumulative frequency of at least one of the experimental samples is within 5% of that of the developmental sample.

There are, however, systematic differences. Note that for both maximum and minimum temperatures in February (Figs. 3 and 4) the experimental samples give larger probability estimates for most temperature durations. Similarly, in Figs. 5 and 6 (May) the estimate is consistently in the other direction. These differences, as well as others in months not shown, can be associated with the differences in mean monthly temperatures in the same way as was done for the third criterion. Both experimental samples have mean monthly temperatures for February below that of the developmental sample (see Table 5). Thus, in Figs. 3 and 4 probabilities are larger than those of the developmental sample. The reverse is true for May and November (Figs. 5, 6, 9 and 10). In August the mean monthly temperature differences among the three samples are too small to produce appreciable differences (Figs. 7 and 8).

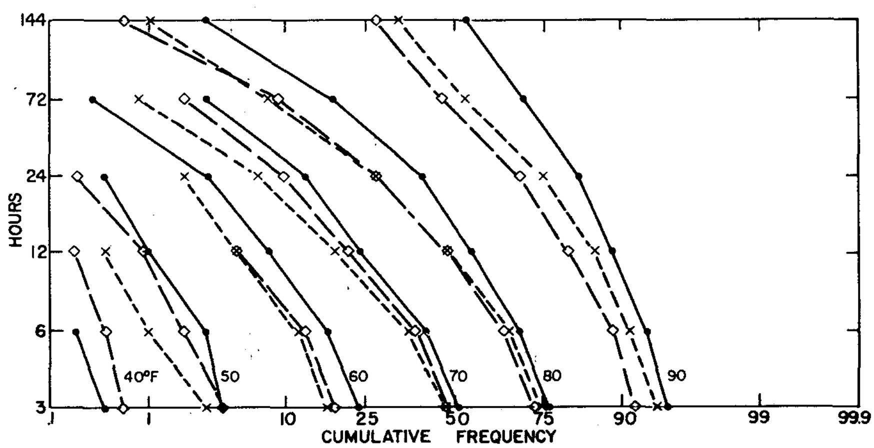

FIG. 5. As Fig. 3 except for May maximum.

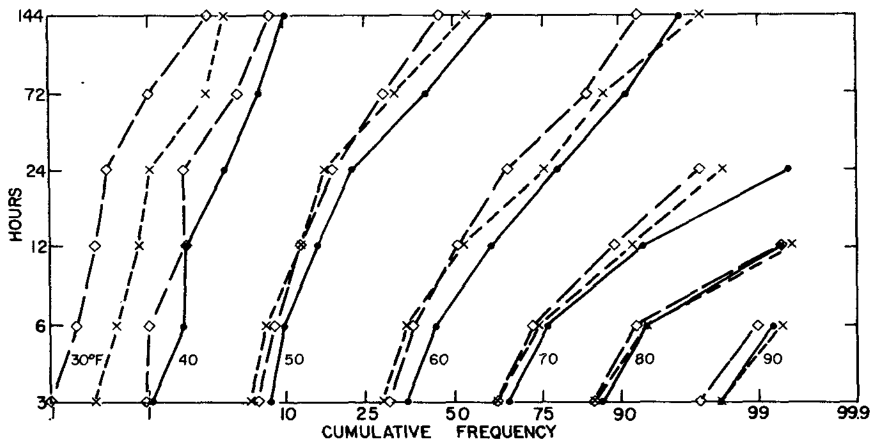

FIG. 6. As in Fig. 3 except for May minimum.

As with the third criterion, then, the systematic differences experimental versus developmental, apparent with the fourth criterion, can be ascribed to differences in mean monthly temperatures. When temperatures produced by the model for a month average above the corresponding temperature of the developmental sample the mean occurrence of temperatures 32°F and below (if the month is cold enough) was underestimated, and the mean occurrence of temperatures 65°F and above and 90°F and above (the latter only for warm months), was overestimated. Similarly, the probabilities that maximum and minimum temperatures in any M h period would be less than a specific temperature are underestimated. The reverse holds for model monthly temperatures below those of the developmental sample.

It was noted, near the beginning of the verification section, that the model must, in the long run, produce mean monthly temperatures identical to those of the developmental sample. Since the analysis of the first, third and fourth verifications criteria has shown that differences between the model and nature are due to sampling fluctuations of the former over a relatively short period of 10 years, the implication is that, given a much longer period, one in which the model's mean monthly temperatures would converge to those of the developmental sample, the duration and cumulative frequency statistics would match those of the developmental sample. It must be acknowledged, however, that the 11-year developmental period is itself a sample, and that other periods of similar length might show considerable differences from the 1950–60 period.

There appears to be no association between the discrepancies in diurnal range (Table 6) and those of the third and fourth criteria, the latter having been accounted for by mean monthly temperature differences. This is to be expected. Although there is consistent underestimation and overestimation within specified ranges in each of the experimental samples, the overall effect is compensatory, and the mean diurnal range for any month is at least approximately correct. Thus the statistics of the third and fourth criteria are, in the long run, unaffected by these diurnal range discrepancies.

4. General applicability of the model

It was noted in a previous section that one of the main criticisms of Gringorten's model was that it re-

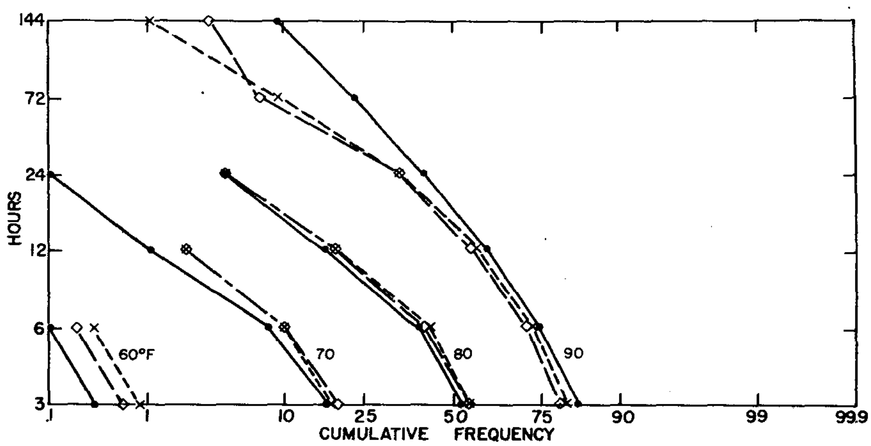

FIG. 7. As in Fig. 3 except for August maximum.

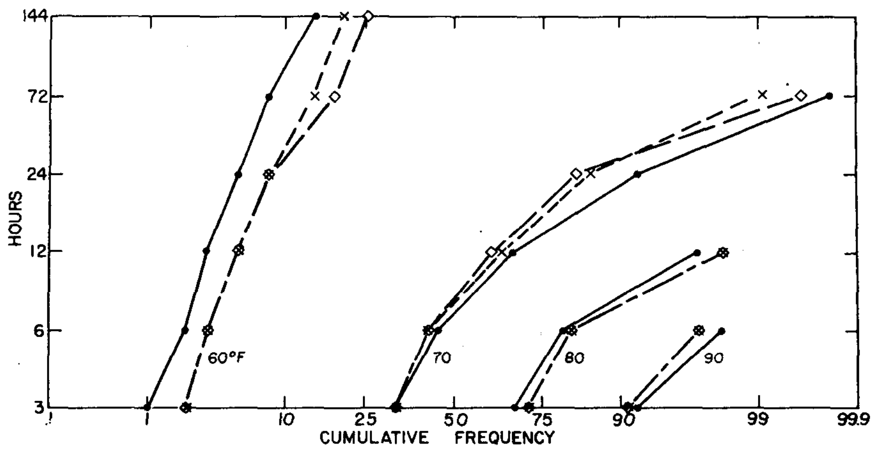

FIG. 8. As in Fig. 3 except for August minimum.

quired a vast amount of data. The present model can also be faulted in this respect. The governing parameters have been closely fitted to the developmental sample and have, of necessity, lost applicability to places other than Big Spring, Tex.

We must ask, then, what procedures must be followed to make the model applicable to other areas of, say, the conterminous United States? First, the sensitivity of the model to variations in the input parameters must be tested. Are useful results obtained only by precise specification of these parameters, as was done here, or will the model allow sufficient tolerance that these parameters can be approximated from information other than that derived from a developmental sample? The utility of the model must first be tested by varying one input parameter at a time and noting the effect on the statistics appropriate to the four criteria.

Assuming for the moment that the second option above applies, we would like to offer the following suggestions about approximating each of these parameters. The model requires one harmonic, the first, to describe the annual variation of temperature. Since the conterminous United States is well poleward of that area where the sun is overhead twice a year, and the annual course of extraterrestrial radiation is almost exactly sinusoidal, the first harmonic alone should be sufficient. To test this contention harmonic analysis was applied to the mean monthly temperatures of each of 341 state climatological divisions (U. S. Dept. of Commerce, 1973), and the percent variance accounted for by the first harmonic was calculated. This figure is over 99% for all the 48 states except Florida and the mountain west (98 to 99%), and the Pacific coast (about 97%). Thus, it appears unlikely that more than one harmonic for the annual variation would be necessary.

The annual variation of temperature can be generated by deriving either the A and B coefficients [Eq. (1)], as was done in this study, or by finding C, the amplitude, and t_{max}, the date on which the first harmonic maximizes. As an approximation to the annual harmonic, then, the annual range (warmest month minus coldest month) could be halved to obtain C, and t_{max} could be estimated by comparing the temperatures of the warmest months.

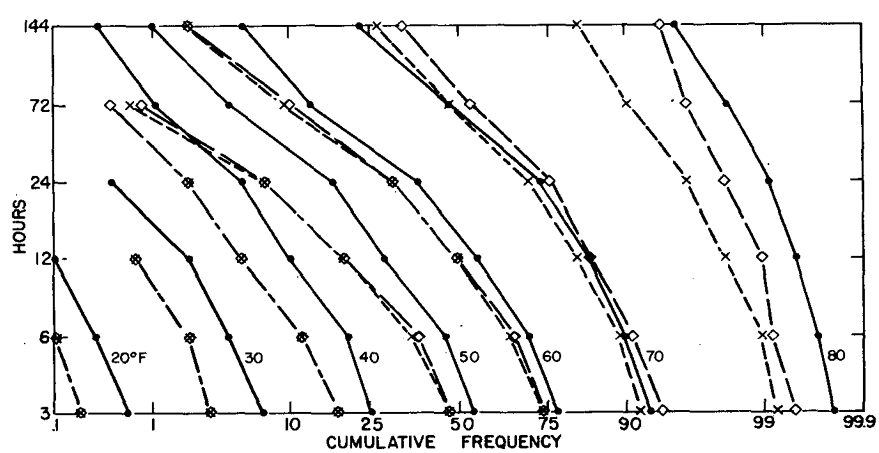

FIG. 9. As in Fig. 3 except for November maximum.

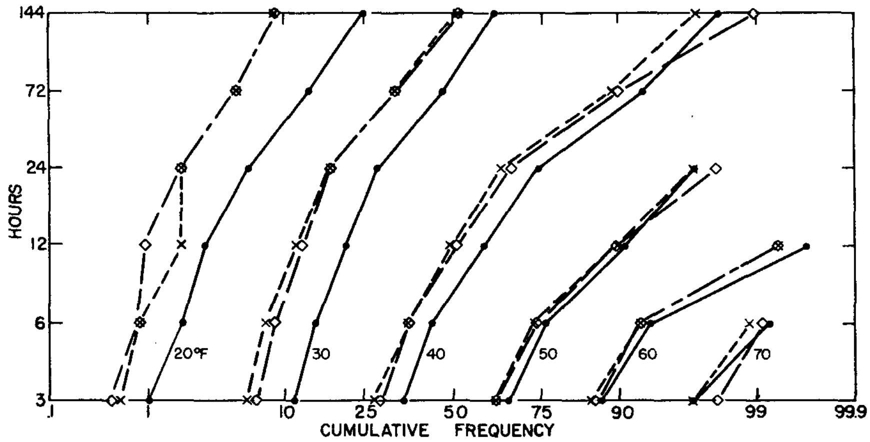

FIG. 10. As in Fig. 3 except for November minimum.

An adequate mathematical description of the diurnal variation of temperature will undoubtedly be more difficult. Three harmonics were used in this study, and it seems likely that this number would be adequate for the southern part of the United States, where differences in the length of day and night throughout the year are only moderate, and in climates lacking pronounced seasonal variations in moisture, with concomitant variations in diurnal temperature range. More complex functions may be necessary in the northern United States, and in areas like southern California, where diurnal ranges can be expected to vary from the dry summer to the wet winter.

The variability of hourly temperatures—how much the temperature at a given hour of the year varies from year to year—has been shown to have annual and diurnal components for the study area. We suspect that this association will hold for other areas, and that this variability will be amenable to mathematical description by modifications of the Fourier coefficients which describe it. A study of the geographical variation of these coefficients is thus suggested.

There should also be regional and seasonal variations in the appropriate values of the hourly serial correlation coefficient. This research suggests that where there are large day-to-day changes in temperature in response to air mass and frontal passages this parameter will be larger than in the case of more stable weather. Thus, values for the north-central United States in the winter would be higher than those for, say, the Gulf Coast states in the summer. It might be possible to approximate values of this parameter by utilizing Landsberg's (1966) values of mean interdiurnal variability of maximum and minimum temperatures.

The skewness, or non-normality, of frequency distributions of hourly temperatures is the last parameter to be considered. Although the variation with time of this parameter could be described by harmonic analysis, as was done for variability, the annual and diurnal harmonics accounted for less than half of the variance. This would probably be the case elsewhere. And it has been shown (Bryson, 1966) that frequency distributions of daily maximum temperatures may be multinormal, i.e., compound distributions which may be decomposed into two or more separate normal distributions. This complicates the problem and would require much more complex adjustments than simply varying the skewness parameter.

5. Summary and recommendations

A mathematical model of hourly temperatures for Big Spring, Tex., has been developed. These temperatures are produced by harmonics which represent annual and diurnal variations, and a Markov chain expression which incorporates adjustments for seasonal variation of the serial correlation coefficient, and for the seasonal and diurnal variations of the variability, and non-normality, of frequency distributions of hourly temperatures. Two 10-year experimental samples were generated, and their characteristics compared to those of the developmental sample.

Although the model underestimates both the variability of mean monthly temperatures and the frequency of low diurnal ranges, there is good agreement between the model and nature with respect to both durations of hourly temperatures above or below specified thresholds, and the probabilities that maximum and minimum temperatures in any m-hour period will be less than specified temperatures. A deficiency in the procedure employed in this study is that no account can be taken of how much one developmental sample differs from others, since only one 11-year period was available for analysis.

Future research should involve testing the sensitivity of the output statistics to variations in the governing parameters. This must be accomplished before the more general applicability of the model can be determined.

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