Gamma Size Distribution and Stochastic Sampling Errors

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ABSTRACT

A maximum likelihood approach to the application of the gamma size distribution is described and compared with the method of moments approach suggested by Ulbrich. Estimation of distribution parameters based on the maximum likelihood principle and Ulbrich's estimation method have different weighting characteristics, which are illustrated through the use of quantile-quantile plots. The ability of the gamma size distribution to describe curvature on a semilogarithmic diagram, and the mathematical simplicity of incorporating it in the sampling error model based on the Poisson process make it possible to derive a sampling error model with consideration given to changes in size distribution shape. It is also shown that variations in size distribution shape can have significant effects on the estimation of sampling errors.

1. Introduction

In studies of the variability of meteorological measurements, one may consider two major components: statistical errors and natural physical variability. For a single measuring device, the statistical sampling errors can be reduced by appropriate improvements of the instrument or sampling method. The importance of studying sampling errors is related to various areas of meteorology, ranging from cloud and precipitation physics to weather modification evaluation. Previous efforts in this area include Cornford (1967, 1968), Mueller and Sims (1966), Joss and Waldvogel (1969) and a very comprehensive study by Gertzman and Atlas (1977).

Cornford (1967) used the Poisson probability model to study raindrop and cloud droplet sampling, and concluded that in order to have the observed concentration lying within 50 percent of the true mean 95 percent of the time, the measuring device must be able to sample at least 23 drops. After extending the calculation from the integrated size spectrum to that for individual size ranges, Cornford (1968) concluded that for a 10 percent accuracy 95 percent of the time, a sample must contain 300–500 drops for a 0.25 mm size range. Mueller and Sims (1966) studied the necessary sampling volume required to determine the rainfall rate and the radar reflectivity from the drop size distribution and concluded that a sample of 44 m^3 is required to determine the rate to within 10 percent accuracy for 95 percent of the time. Gertzman and Atlas (1977) extended and generalized the work of Joss and Waldvogel (1969). Based on the Poisson process, they derived the expression for the fractional standard deviation (FSD) using the exponential size distribution (e.g. Marshall and Palmer, 1948). Various relationships were also derived.

The present study is directed at two problems in sampling error modeling. First, there are often observed departures from the Marshall–Palmer type size distribution, particularly in terms of curvature of the distribution on the usual semilogarithmic size distribution plot (i.e. logarithm of particle number versus size). Second, it has been observed that the shape of the size distribution changes with the length of sampling time. For example, Joss and Gori (1978) showed that the average distribution shape of raindrops changes with sampling duration, and only with increasing sampling time is the exponential shape approached. This suggests the need to incorporate both the shape of size distribution in the estimate of sampling errors and a more flexible size distribution function than the usual Marshall–Palmer type for the description of particle size spectra. Recently Ulbrich (1983a, 1983b) has shown that distribution shape can be reasonably modeled by the gamma size distribution having the form

$$N(D) = N_0 D^{\mu} \exp - \lambda D. \tag{1}$$

The shape of the size distribution is determined by μ, and λ is related to the scale of the size distribution. A relationship between two of the parameters, N_0 and μ, was obtained, both empirically and theoretically, and estimation procedures using remote measurables or distribution moments were described. In the present study, an alternative approach to the gamma size distribution is adopted together with its application to the computation of stochastic sampling errors of size distribution-related variables. The effects

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of distribution shape on sampling error computation are also examined.

In the subsequent sections, the Poisson process, which is the basis for the sampling error model, is briefly described, followed by a section on the gamma size distribution using the probabilistic approach. A procedure by which the gamma size distribution parameters are obtained is presented in Section 4. In Section 5, the relationship between the gamma size distribution shape parameter and the Joss–Gori shape factors is discussed. In Section 6, the equations for the sampling error model incorporating the gamma size distribution are derived. The assessment of shape effects on sampling error estimates is the subject of Section 7. The effects of shape on particle number estimates are discussed in Sections 8 and 9. Conclusions are listed in Section 10.

2. The Poisson process

The Poisson process is the basis of the present sampling error model, which describes the fluctuation of a size distribution-related variable due to the stochastic fluctuation of particle number in a given size range. Following Gertzman and Atlas (1977), Cornford (1967) and others, the fluctuation of particle number for a given size range in a specified particle size distribution is assumed to be adequately modeled by the Poisson probability law, i.e.

$$P_n = \frac{\theta^n}{n!} e^{-\theta}, \quad \theta \geqslant 0 \tag{2}$$

where P_n is the probability of n particles occurring, and θ is the Poisson parameter.

The Poisson distribution has been widely used in cloud models to describe the random fluctuation of droplet numbers in simulating the collection growth of cloud drops. (See, e.g., Telford, 1955; Gillespie, 1975; Pruppacher and Klett, 1980). It is well known in statistical theory that the Poisson distribution can be considered as a large sample approximation to the binomial distribution when the probability of an event occurring is small. Also, for samples reasonably large, the Poisson distribution can be approximated by the Gaussian distribution. The most relevant property of the Poisson distribution described by (2), however, is that its mean and variance are the same (θ).

Theoretical considerations relating to the use of the Poisson distribution are discussed in Sasyo (1965), and its application in sampling error studies is described in detail in Cornford (1967).

3. The gamma size distribution: a probabilistic approach

The gamma size distribution is the specified particle size distribution in the present sampling error model application. One may use any probability density function for describing particle size distribution provided that reasonable goodness-of-fit is obtained and an additional distribution parameter representing total particle number concentration is defined. The gamma size distribution is an appropriate choice in the present case because of its ability to describe curvature on a semilogarithmic size distribution plot and the mathematical simplicity of incorporating it in the sampling error model.

The probabilistic approach to particle size distribution is described in this section by first considering the usual Marshall–Palmer-type exponential size distribution, which is represented by

$$N(D)dD = N_0 \exp(-\lambda D)dD \tag{3}$$

and $N(D)dD$ is the number of particles found between sizes D and $D + dD$. N_0 and λ are the size distribution parameters to be estimated. Ordinarily N_0 and λ are obtained as the intercept and slope of a straight line after a logarithmic transformation of (3). Here $N(D)$ represents the number of particles per unit volume of air per unit size interval. Integrating (3) over the entire size spectrum will give the total number concentration of particles per unit volume $N^* = N_0/\lambda$. Such a size distribution can be related to a statistical probability density function.

Let $f(D)$ be any continuous probability density function to be used in describing the particle size distribution. Let $f(D)dD$ represent the proportion containing particles of sizes between the range D, $D + dD$; then $N(D)dD$, which is the number of particles found in the same size range, can be written as $N^*f(D)dD$.

In the case of the exponential size distribution, an intuitive reexpression will then be in terms of N^* and the probability density function. Since

$$N(D)dD = N_0 \exp(-\lambda D)dD = N^*f(D)dD,$$
$$N^* = N_0/\lambda, \tag{4}$$

therefore $f(D) = \lambda \exp(-\lambda D)$, which is the exponential probability density function, and particle number density N^* appears as a separate parameter. There is a certain intuitive appeal in using this probability density function approach to describe particle size distribution.

Smith (1982) discussed various ways of presenting this distribution graphically. The usual display of particle size distribution is that of a semilogarithmic plot. However, for any size distribution presented this way, using the exponential size distribution means fitting a straight line through the observed points. Therefore exponential size distributions of the Marshall–Palmer type cannot adequately describe certain observed curvature of the distribution on a semilogarithmic plot.

Other forms of particle size distribution have been introduced to describe distribution shape. Best (1951), for example, introduced

$$1 - F = \exp[-(D/C)^k] \quad (5)$$

where F is the proportion of liquid water comprised of cloud drops having diameters less than D. This is in effect a mass-weighted variant of the Weibull distribution (Weibull, 1951) which has been widely used in wind energy, life-testing and other applications. The Khrgian-Mazin distribution (see e.g. Pruppacher and Klett (1980), p. 11), on the other hand, is a function of the form:

$$N(D) = AD^2 \exp(-BD) \quad (6)$$

where the parameters A and B are related to the moments of the distribution. This can be related to the gamma size distribution used recently by Ulbrich and Atlas (1982) and Ulbrich (1983a,b). The latter references describe the use of the gamma size distribution with a free shape parameter μ, which can be estimated from observations. The size distribution function used by Ulbrich (1983b) is represented by Eq. (1).

This form of the gamma size distribution can alternatively be written in terms of the gamma probability-density function in a manner similar to that described earlier for the exponential size distribution. Specifically, this involves the introduction of a density parameter N^*, which is then multiplied by the usual two-parameter gamma probability-density function. This alternative form of the gamma size distribution is represented by

$$N(D) = N^*f(D) = \frac{N^*}{\Gamma(\alpha)\beta^\alpha} \exp(-D/\beta) \cdot D^{\alpha-1} \quad (7)$$

where

$$\alpha > 0, \quad \beta > 0, \quad D \geqslant 0,$$

$$\Gamma(\alpha) = \int_0^\infty \exp(t) \cdot t^{\alpha-1} dt. \quad (8)$$

Here, α is the shape parameter, β the scale parameter and N^* the particle number density. The gamma density function $f(D)$ is related to the Weibull probability distribution in that they are special cases of the generalized gamma-Stacey distribution (Stacey 1962). But while the gamma probability density function approaches a symmetric bell-shape as its shape parameter becomes large, the Weibull distribution does not. For the special case of $\alpha = 3$, $N^*/2\beta^3 = A$ and $1/\beta = B$, the gamma size distribution is reduced to the Khrgian–Mazin drop size distribution of Eq. (6). For the special case of $\alpha = 1$ and $N^* = N_0/\lambda$, the gamma size distribution is identical to the usual exponential size distribution of the Marshall–Palmer type. Berry and Reinhardt (1973) have considered using (7) in cloud models. There is a direct correspondence between (7) and the gamma size distribution function used in Ulbrich (1983b). When both are used in describing the particle size distribution per unit volume, the following relationships hold:

$$\left.\begin{aligned} N_0 &= \frac{N^*}{\Gamma(\alpha)\beta^\alpha} \\ \mu &= \alpha - 1 \\ \lambda &= 1/\beta \end{aligned}\right\} . \quad (9)$$

By definition of the gamma probability density function, the above relations linking (1) and (7) hold only for $\mu > -1$. This restriction arises because of the use of a probability density function and the maximum likelihood principle. Since there is a significant proportion of observed size distributions having shape parameter $\mu > -1$ (Ulbrich 1983b), it is not expected that this restriction will seriously compromise the general usefulness of the model. Also, this probability density approach means that more flexible density functions can be used in cases where the size distribution is heavily skewed, i.e., $\mu < -1$. A possible candidate is the three-parameter kappa distribution (Mielke and Johnson, 1973). The gamma size distribution having different values of the parameters is shown in the semilogarithmic plot of Fig. 1. These are simulated graphs of different shape and scale parameters with the variance of the probability

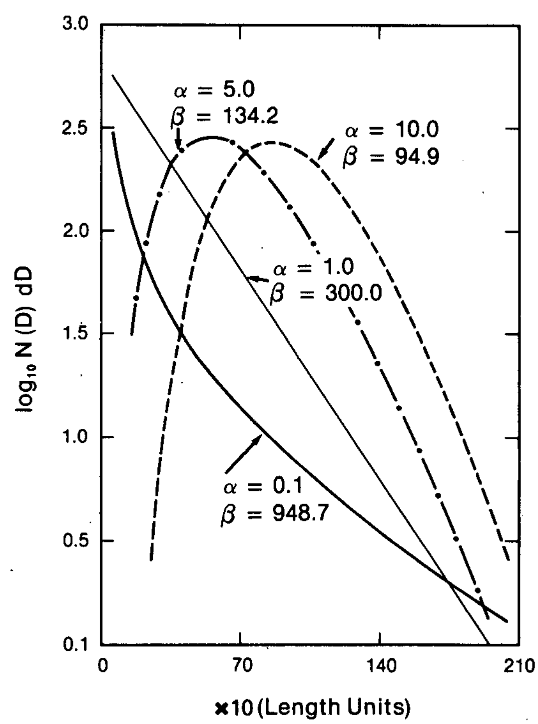

FIG. 1. The gamma size distribution having different values of the parameters α and β.

distribution $\alpha\beta^2$ and the density parameter N^* held constant. The aim is to show the curvature characteristics of the gamma size distribution. The units of the axes are not important for this purpose. The ordinate represents the logarithm of the particle number per unit volume, while the abscissa represents particle sizes in length units. For Fig. 1, N^* is set at 10^4 [L^{-3}], $\alpha\beta^2$ is 9.0×10^4 [L^{-2}] and dD is 20.0 [L].

4. Estimation of parameters

There are three parameters to be estimated in the gamma size distribution. Ulbrich (1983b) presents an effective estimation procedure based on the higher moments of the size distribution (1), for the parameters N_0, μ and λ. An alternative procedure based on the maximum likelihood principle is used in the present study to obtain estimates for the parameters N^*, α and β in the gamma size distribution represented by (7). Such a procedure weights more on the drop sizes having higher frequency of occurrence, while Ulbrich's moments method weights more on the drop sizes having the larger values.

For practical purposes, it is useful to consider the parameter estimation problem from actual observations of particle size distributions. This means that the data obtained are from the entire sampling volume while the unit volume is used when (7) is defined. It is necessary, in such cases, to include the size dependence of particle terminal-fall speed in the parameter estimation procedure. Assuming that $v(D) = kD^b$ is an adequate representation of the size dependence of fall speed, then the size dependence of sampling volume can be represented by $V(D) = AtkD^b$, where A is the sampling area and t is the sampling time. Alternatively, one can write $V(D) = V_0D^b$. To obtain the gamma size distribution parameters N^*, α and β in (7) from a set of observations, consider first the following definition of N_t, which is the total particle number observed for the entire size spectrum:

$$N_t = \int_0^\infty N(D)V(D)dD. \tag{10}$$

After substituting the gamma size distribution (7) for $N(D)$, V_0D^b for $V(D)$, and multiplying both the numerator and denominator by $\Gamma(b + \alpha)\beta^b$, one obtains

$$N_t = \frac{N^*V_0\beta^b\Gamma(b+\alpha)}{\Gamma(\alpha)} \times \int_0^\infty \frac{1}{\Gamma(b+\alpha)\beta^{b+\alpha}} D^{b+\alpha-1}e^{-D/\beta}dD. \tag{11}$$

The integrand is actually a gamma probability density function with $b + \alpha$ as shape parameter and β as scale parameter. Therefore, the size dependence of the sampling volume has an effect on the size distribution shape. In fact, the integrand describes the frequency distribution actually observed. Since it is the probability density function describing the observed particle size distribution, the shape parameter $b + \alpha$ and the scale parameter β are obtainable from the observations using established methods. In the present study, the iterative maximum likelihood (ML) estimation procedure of Mielke (1976) was used. The equations were modified to handle grouped data, and the method of Greenwood and Durand (1960) was used to provide initial values which further increased the speed of convergence in the iterative scheme. By definition of a probability density function, the integral is equal to unity. The total particle number for the complete spectrum is therefore

$$N_t = \frac{N^*V_0\beta^b\Gamma(b+\alpha)}{\Gamma(\alpha)}, \tag{12}$$

which is the coefficient outside the integral in (11).

When there is spectrum truncation due to instrument, sampling method or otherwise, the integration limits are D_1 and D_2, the minimum and maximum detectable sizes respectively. In this case, the integral gives the proportion of N_t that falls in the size range (D_1, D_2). The parameter

$$N_{12} = \frac{N^*V_0\beta^b\Gamma(b+\alpha)}{\Gamma(\alpha)} \times \int_{D_1}^{D_2} \frac{1}{\Gamma(b+\alpha)\beta^{b+\alpha}} D^{b+\alpha-1}e^{-D/\beta}dD \tag{13}$$

is therefore the number of particles observed in the size range $[D_1, D_2]$, and the parameter N^* can be obtained from (13) as

$$N^* = \frac{N_{12}\Gamma(\alpha)}{V_0\beta^b}[\gamma(\alpha + b, D_2/\beta) - \gamma(\alpha + b, D_1/\beta)]^{-1} \tag{14}$$

where $\gamma(a, x)$ is the incomplete gamma function, $\gamma(a, x) = \int_0^x e^{-t}t^{a-1}dt$ and $\Gamma(a) = \gamma(a, \infty)$. Similarly, from (16),

$$N^* = \frac{N_t\Gamma(\alpha)}{V_0\beta^b\Gamma(b+\alpha)}. \tag{15}$$

The use of (14) or (15) for calculating N^* depends on whether the actual particle number observed can be considered as approximating N_t or N_{12}. Incidentally, (12) also yields $N^* = N_t/V_0$ when $b = 0$.

For the purpose of illustration, a raindrop size distribution (Waldvogel, 1974, spectrum a) and a cloud droplet size distribution (English and Marwitz, 1981, placebo cloud) were fitted using Eqs. (1) and (7). The semilogarithmic plots of the observed and fitted size distributions are shown in Figs. 2 and 3. The instrument truncation limits are D_1 = 0.3 mm and D_2 = 5.5 mm for the raindrop size distribution, and D_1 = 2 μm and D_2 = 32 μm for the cloud droplet size distribution. Note that the jagged appear-

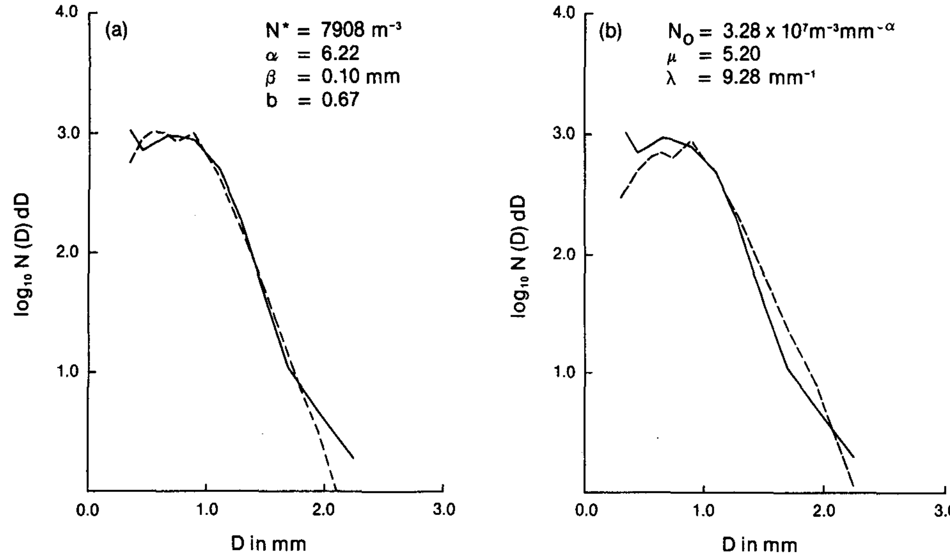

FIG. 2. Size distribution plot of Waldvogel's rain drop data fitted with the gamma size distribution (a) ML method (b) Ulbrich (1983b) method.

ance of the fitted curves of Fig. 2 is a result of the nonuniform class interval dD used. To illustrate the weighting characteristics, the quantile–quantile (q–q) relationship is plotted in each case. (See, e.g., Morris and Rolph, 1981). In essence, the q–q relationship between the observed quantiles and the fitted quantiles is linear with slope equal to unity for a perfect fit. A q–q plot (or probability plot) thus gives a visual measure of how consistent the data set is with the assumed probability distribution. The q–q plots for both the Waldvogel raindrop sample and the English and Marwitz cloud droplet sample are shown in Fig. 4 and Fig. 5 respectively. For more detail, the departure from the perfect fit line was also plotted for the two examples in Fig. 6. The root mean square departure (r.m.s.d.) is used as an overall measure of the goodness-of-fit. The different weighting characteristics mentioned earlier are discernible.

The values of N_0, μ and λ obtained using (9) and the ML estimates, as well as the values of N^*, α and β obtained using (9) and the estimates based on Ulbrich's approach, are listed in Table 1. It can be seen that the major difference lies in the values of N^* and N_0. For the assessment of how well the fitted distribution predicts values of rainfall parameters based on the drop size distribution, the ratios of the estimated parameter to the observed parameter for the two different estimation methods are computed. The results are shown in Table 2. A ratio of unity means perfect prediction. For the examples given, the estimates based on the ML method tend to be closer to the observations for small values of n, whereas the estimates based on Ulbrich's (1983b) method are most accurate for the parameter that is used to obtain N_0. In the present application, the observed liquid water content is used in computing N_0, after μ and

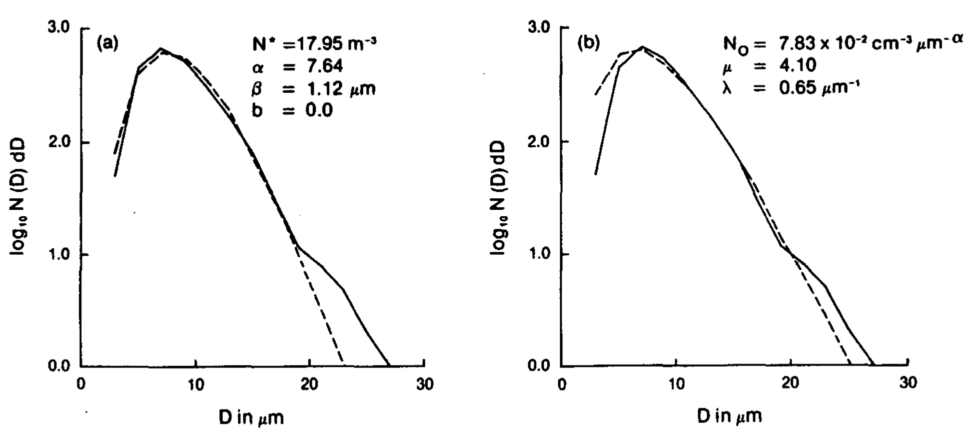

FIG. 3. Size distribution plot of English and Marwitz cloud droplet data fitted with the gamma size distribution (a) ML method (b) Ulbrich (1983b) method.

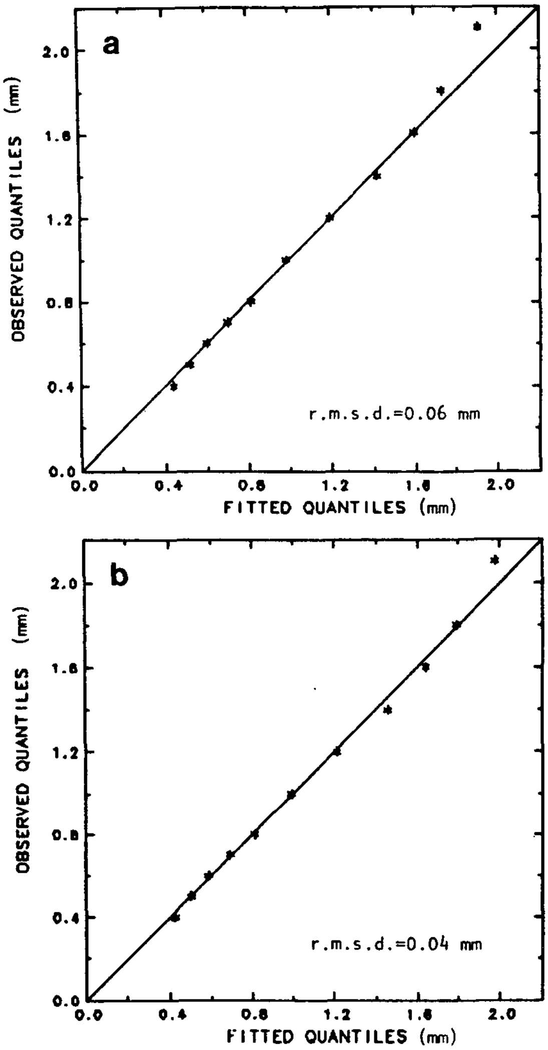

FIG. 4. The q–q plots of Waldvogel's rain drop data using (a) ML method (b) Ulbrich (1983b) method.

λ have been obtained from the moments of the gamma size distribution (1). Considering the differences in the size distribution parameters shown in Table 1, the accuracy of the estimated rainfall parameters shown in Table 2 is quite reasonable.

5. The shape parameter

The shape parameter α of the gamma size distribution can be related to other measures for the description of size distribution shape. Joss and Gori (1978), for example, have proposed shape factors defined in terms of integrals $D(P)$ and $D(Q)$. The shape factor $S(PQ)$ can be represented by

$$S(PQ) = \frac{\left|\dfrac{D(P) - D(Q)}{D(P) + D(Q)}\right|_{\text{observed}}}{\left|\dfrac{D(P) - D(Q)}{D(P) + D(Q)}\right|_{\text{exponential}}} \tag{16}$$

with

$$D(P) = \frac{\int D^p N(D) dD}{\int D^{p-1} N(D) dD}, \quad D(Q) = \frac{\int D^q N(D) dD}{\int D^{q-1} N(D) dD}.$$

The integration limits are from zero to infinity for the exponential case and instrument-dependent for the observed case. For example, when $P = Z$, the reflectivity factor, and $Q = \sigma$, the optical extinction, one obtains $p = 6$ and $q = 2$. The shape factor indicates the curvature of a size distribution on a semilogarithmic plot. The choice of p and q determines the drop size region in which curvature is described; hence the shape factor $S(Z\sigma)$ describes the curvature over the drop size regions which contribute most to the optical extinction coefficient σ and the radar

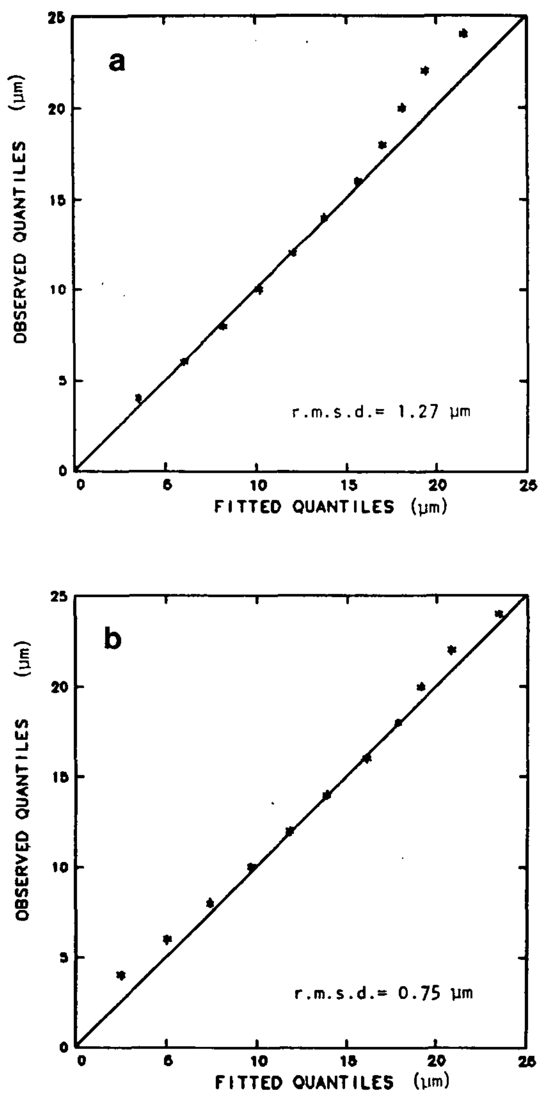

FIG. 5. The q–q plots of English and Marwitz's cloud droplet data using (a) ML method (b) Ulbrich (1983b) method.

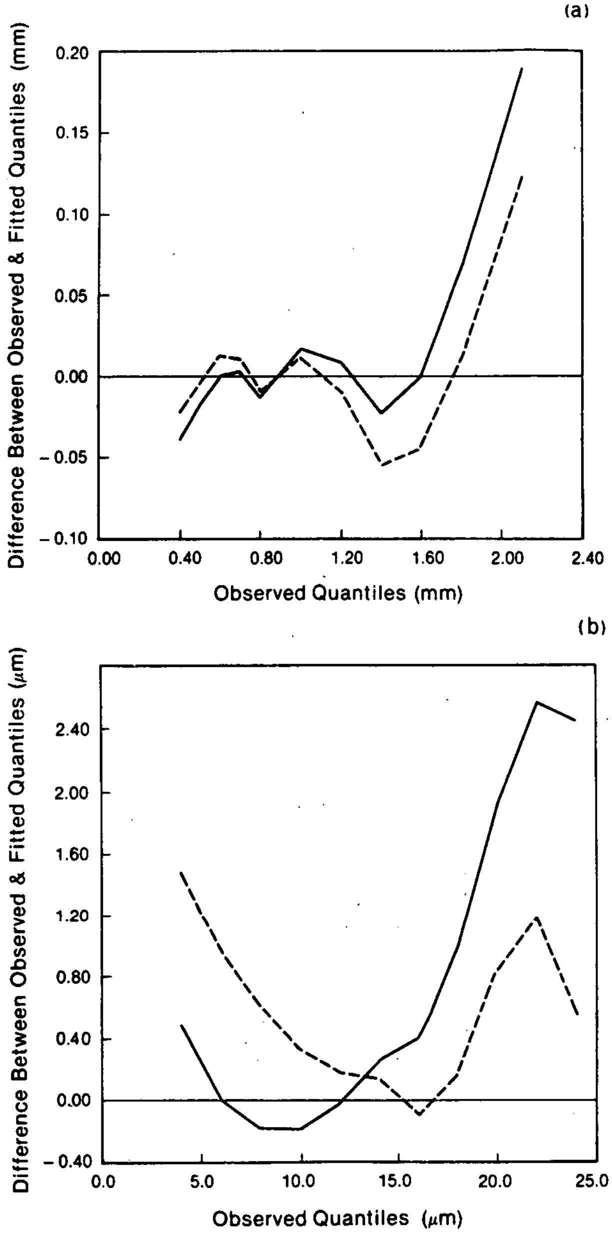

FIG. 6. Difference between observed and fitted quantiles for (a) Waldvogel's rain drop data and (b) English and Marwitz's cloud drop data using the ML method (dashed line) and Ulbrich (1983b) method (solid line).

reflectivity Z. This is considered as representative of the average shape of the size distribution, whereas other shape factors $S(W\sigma)$ with $p = 3$, $q = 2$, $S(R^*W)$ with $p = 4$, $q = 3$ and $S(ZR^*)$ with $p = 6$, $q = 4$ provide more details about the size distribution shape (Joss and Gori, 1978).

Ulbrich (1983b) derived the relationship between the Joss–Gori shape factors and the gamma size distribution shape μ. The result is

$$S(PQ) = \frac{p+q}{p+q+2\mu}. \tag{17}$$

From the definition of $S(PQ)$, one can see that the Joss–Gori type shape factors depend on the observed maximum and minimum sizes, whereas the gamma shape parameter does not. Therefore, the above relationship holds only with the assumption that an ideal measuring device is used with no truncation on the observable drop size spectrum, i.e. the integration limits can be taken to cover the interval $[0, \infty)$ when evaluating the numerator in (16).

Without spectrum truncation, the gamma shape parameter α bears a very simple relationship to $S(Z\sigma)$, i.e.

$$S(Z\sigma) = 4/(\alpha + 3). \tag{18}$$

Hence $S(Z\sigma)$ is less than, equal to or greater than unity respectively for α being greater than, equal to or less than unity. The $S(Z\sigma) - \alpha$ relationship in this case is shown in Fig. 7. Incidentally, α greater than unity represents a downward facing concavity of the size distribution curve on a semilogarithmic plot.

6. The fractional standard deviation (FSD)

In the present study, FSD is used as a measure for the statistical sampling error due to the Poisson fluctuation of particle size number. In statistics FSD is also called the coefficient of variation. The gamma size distribution (7) will be used in subsequent derivations for FSD computation.

Assuming a Poisson process for the number of particles observed by a measuring device over a size range $[D, D + dD]$, Gertzman and Atlas (1977) derived the expressions for the FSD of variables $X^n(D)$, which are related to particle size D by relationships such as

$$X^n(D) = CD^n \tag{19}$$

where C is a constant dependent upon the particular variable.

Examples include $n = 2$ for optical extinction, $n = 3$ for liquid water, $n = 6$ for radar reflectivity factor and $n = 0$ for particle number concentration. The FSD for a variable X was then shown to be

$$\mathrm{FSD} = \frac{\sigma_X}{\langle X \rangle} = \frac{\left\{\int [X^n(D)]^2 \cdot V(D) \cdot N(D) \cdot dD\right\}^{1/2}}{\int X^n(D) \cdot V(D) \cdot N(D) \cdot dD} \tag{20}$$

where $\langle X \rangle$ is the ensemble average for the contribution by all particles to the variable X, σ_X is the standard deviation of X, $V(D) = V_0 D^b$ represents the volume distribution and $N(D)$ is the mean particle size distribution. The integration limits are to be specified dependent upon the individual cases. Note that the numerator in (20) represents the standard deviation of X. Although it resembles the square root of the

TABLE 1. Comparison of values of parameters obtained by the ML method and the method of Ulbrich (1983b). The values in parentheses are computed using Eq. (9).

Sample	Parameter	ML method	Ulbrich (1983b) method	Units
Waldvogel (1974) raindrop data	N^*	7908.44	(5574.87)	m^{-3}
	α	6.22	(6.20)	dimensionless
	β	0.10	(0.11)	mm
	N_0	(8.39×10^7)	3.28×10^7	$m^{-3}\ mm^{-\alpha}$
	μ	(5.22)	5.20	dimensionless
	λ	(10.18)	9.28	mm^{-1}
English and Marwitz (1981) cloud droplet data	N^*	17.95	(20.20)	cm^{-3}
	α	7.64	(5.10)	dimensionless
	β	1.12	(1.55)	μm
	N_0	(3.13×10^{-3})	7.83×10^{-2}	$cm^{-3}\ \mu m^{-\alpha}$
	μ	(6.64)	4.10	dimensionless
	λ	(0.89)	0.65	μm^{-1}

second raw moment, the expression was obtained as a standard deviation with the assumptions that the particle number for each size class fluctuates as a Poisson process and that the fluctuations are independent between size classes.

Substituting the gamma size distributions (7) into (20) and integrating over the range $D = [0, \infty)$, one obtains

$$\langle X \rangle = \int_0^\infty X^n(D) \cdot V(D) \cdot N(D) \cdot dD$$

$$= \frac{CV_0N^*}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty D^{n+b+\alpha-1} e^{-D/\beta} dD \qquad (21)$$

$$\sigma_x^2 = \int_0^\infty [X^n(D)]^2 \cdot V(D) \cdot N(D) \cdot dD$$

$$= \frac{C^2V_0N^*}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty D^{2n+b+\alpha-1} e^{-D/\beta} dD$$

$$\mathrm{FSD}_c = \left\{\frac{\Gamma(\alpha)\Gamma(2n+b+\alpha)}{V_0N^*\beta^b}\right\}^{1/2} \Big/ \Gamma(n+b+\alpha) \qquad (22)$$

where the subscript c represents the complete spectrum. For the partial size spectrum over the range $D = D_1$ to $D = D_2$,

$$\mathrm{FSD}_p = \frac{\left\{\dfrac{\Gamma(\alpha)}{V_0N^*\beta^b}[\gamma(2n+b+\alpha, D_2/\beta) - \gamma(2n+b+\alpha, D_1/\beta)]\right\}^{1/2}}{\gamma(n+b+\alpha, D_2/\beta) - \gamma(n+b+\alpha, D_1/\beta)}. \qquad (23)$$

For the case with $N^* = N_0/\lambda$, $\alpha = 1$ and $\lambda = 1/\beta$, the above equations are equivalent to the corresponding expressions for the exponential type size distribution, viz.

TABLE 2. Ratios of estimated rainfall parameter to observed rainfall parameter from the two different estimation methods.

Sample	Parameter	n	ML/observed	Ulbrich/observed
Waldvogel (1974) raindrop data	particle number	0	1.0000	0.7589
	extinction coefficient	2	1.0293	0.9251
	liquid water content	3	1.0174	1.0000
	rainfall rate	3.67	1.0047	1.0493
	X-band microwave attenuation	4	0.9975	1.0734
	reflectivity factor	6	0.9388	1.2126
English and Marwitz (1981) cloud droplet data	particle number	0	0.9997	1.1161
	extinction coefficient	2	0.9920	1.0082
	liquid water content	3	0.9648	1.0000
	rainfall rate	3.67	0.9340	0.9992
	X-band microwave attenuation	4	0.9151	0.9991
	reflectivity factor	6	0.7634	0.9925

$$\mathrm{FSD}_c = \left\{\frac{\lambda^{b+1}\Gamma(2n+b+1)}{V_0 N_0}\right\}^{1/2} \Big/ \Gamma(n+b+1) \quad (24)$$

$$\mathrm{FSD}_p = \frac{[\gamma(2n+b+1, \lambda D_2) - \gamma(2n+b+1, \lambda D_1)]^{1/2}}{[\gamma(n+b+1, \lambda D_2) - \gamma(n+b+1, \lambda D_1)]} \times \left[\frac{\lambda^{1+b}}{V_0 N_0}\right]^{1/2} \quad (25)$$

presented in Gertzman and Atlas (1977).

7. Shape effects on FSD

Equations (22) and (23) can be used in computing the stochastic sampling error for a large selection of variables based on size distributions with considerations of changes in distribution shape. This is particularly important regarding the observed relationships between the sampling period and the size distribution shape such as those found for raindrops by Joss and Gori (1978). Such relationships will likely exist in other particle sampling environments. The use of the gamma size distribution implies that such relationships can be incorporated in the computation of FSD. Conversely, for a given desired FSD, the appropriate sampling period for a given sampling area and a given distribution shape can be obtained.

For the purpose of evaluating the effects of size distribution shape on FSD, the ratio R_c is defined as the ratio of FSD_c using the gamma size distribution to that using the exponential size distribution. Similarly, R_p is defined as the ratio of the two FSD_p.

From (22), (23), (24) and (25), it is easily seen that

$$R_c = \left[\frac{N_0}{\lambda N^*(\lambda\beta)^b}\cdot\frac{\Gamma(\alpha)\Gamma(2n+b+\alpha)}{\Gamma(2n+b+1)}\right]^{1/2} \times \frac{\Gamma(n+b+1)}{\Gamma(n+b+\alpha)} \quad (26)$$

$$R_p = \left\{\frac{N_0}{\lambda N^*(\lambda\beta)^b}\cdot\frac{\Gamma(\alpha)[\gamma(2n+b+\alpha, D_2/\beta) - \gamma(2n+b+\alpha, D_1/\beta)]}{\gamma(2n+b+1, \beta D_2) - \gamma(2n+b+1, \lambda D_1)}\right\}^{1/2} \times \frac{\gamma(n+b+1, \lambda D_2) - \gamma(n+b+1, \lambda D_1)}{\gamma(n+b+\alpha, D_2/\beta) - \gamma(n+b+\alpha, D_1/\beta)}. \quad (27)$$

Note that in this section λ is one of the parameters of the Marshall–Palmer-type exponential size distribution (3). It does not represent the scale parameter of the gamma size distribution Eq. (1). The factor $N_0/(\lambda^{b+1}N^*\beta^b)$ in the above expressions may be considered as consisting of two ratios, i.e. $N_0/(\lambda N^*)$ and $(\lambda\beta)^{-b}$. The first is a ratio of the mean total particle number-density estimated from the exponential size distribution model to that used in the gamma size distribution model. Assuming that the two size distributions have incorporated, in an approximate sense, the correct mean total particle number-density, i.e. $N_0/(\lambda N^*) \approx 1$, and one is considering the constant sampling volume case, i.e. $b = 0$, then the ratio of the FSD_c estimates becomes independent of λ and β. The expression is simplified to

$$R_c = \left[\frac{\Gamma(\alpha)\Gamma(2n+\alpha)}{\Gamma(2n+1)}\right]^{1/2}\cdot\frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \quad (28)$$

which can be plotted with changing α for given values of n. For the cases of n equal to 0, 2, 3 and 6, the above relationship is shown in Fig. 8. This provides an estimate of the shape effect on FSD_c, based on the differences between the two sampling error models. For example, when a gamma particle size distribution having an actual shape parameter $\alpha = 7.5$ is sampled from an ideal instrument with no spectral truncation where the FSD_c for the liquid water content is subsequently computed using an exponential model, then the actual FSD_c in this case is only 0.353 of the one computed, even when the actual mean total particle number is correct. This means an overestimation by almost three times the actual FSD value, simply by neglecting the effects of the distri-

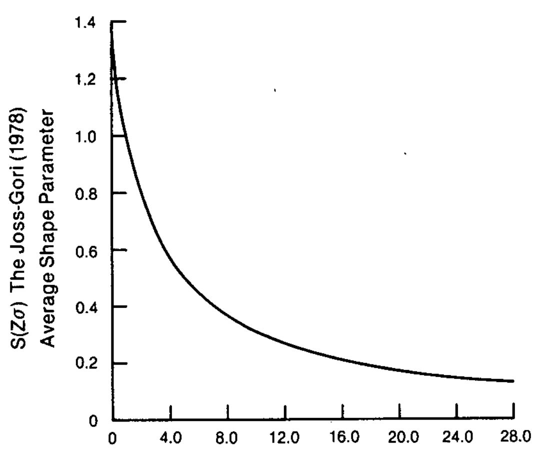

FIG. 7. The relationship between $S(Z\sigma)$ and α.

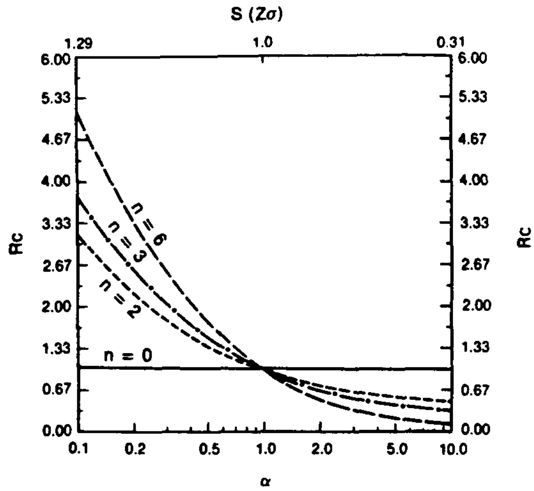

FIG. 8. Ratio R_c as a function of the gamma shape parameter.

bution shape. On the other hand, an underestimate of FSD will occur for $\alpha < 1.0$.

8. Shape effects on $N_{\rm eff}$, the effective particle number

It has been shown (Gertzman and Atlas, 1977) that the FSD of a size distribution-related variable X, for any particle size distribution $N(D)$, can be represented by

$$\mathrm{FSD}_p = \frac{\sigma_X}{\langle X_{12}\rangle} = N_{12}{}^{-1/2}[(\overline{D^{2n}})^{1/2}/(\overline{D^n})]$$

where

$$\langle X_{12}\rangle = \int_{D_1}^{D_2} X(D)\cdot V(D)\cdot N(D)dD \qquad (29)$$

$$\overline{D^m} = \frac{\int_{D_1}^{D_2} D^m\cdot V(D)\cdot N(D)\cdot dD}{\int_{D_1}^{D_2} V(D)\cdot N(D)\cdot dD}.$$

In the above equation, $\overline{D^m}$ is the mth raw moment of the size distribution and the ratio in the square brackets defines $\kappa^{-1/2}$. This equation holds for any form of $X(D)$ and requires only the assumption that the sampled particles are Poisson-distributed about the average particle size distribution $N(D)$. The FSD_p can therefore be written as

$$\mathrm{FSD}_p = (N_{12}\kappa)^{-1/2} = N_{\rm eff}^{-1/2}, \qquad (30)$$

where $N_{\rm eff}$ is the effective number of particles contributing to FSD_p. For a gamma size distribution, the parameter κ can be represented by

$$\kappa = (\overline{D^n})^2/(\overline{D^{2n}}) = \frac{[\gamma(n+b+\alpha, D_2/\beta) - \gamma(n+b+\alpha, D_1/\beta)]^2}{[\gamma(2n+b+\alpha, D_2/\beta) - \gamma(2n+b+\alpha, D_1/\beta)][\gamma(b+\alpha, D_2/\beta) - \gamma(b+\alpha, D_1/\beta)]}. \qquad (31)$$

When the integration is performed for $D_1 = 0$ to $D_2 \rightarrow \infty$, one obtains

$$\kappa = \frac{\Gamma(n+b+\alpha)^2}{\Gamma(2n+b+\alpha)\Gamma(b+\alpha)}. \qquad (32)$$

Replacing α with 1 and $1/\beta$ with λ in (31) and (32) yields the equations for κ for an exponential size distribution.

To obtain the effects of distribution shape on κ, the complete spectrum case is considered with R_κ defined as the ratio of the κ using the gamma size distribution to that using the exponential distribution, i.e.

$$R_\kappa = \frac{\Gamma(b+1)\Gamma(2n+b+1)}{\Gamma(b+\alpha)\Gamma(2n+b+\alpha)}\cdot\left[\frac{\Gamma(n+b+\alpha)}{\Gamma(n+b+1)}\right]^2. \qquad (33)$$

This ratio is a function only of n, the exponent related to the size distribution variable X; b, the exponent in the sampling volume function $V(D)$; and α, the gamma shape parameter. It is plotted for different values of n, b and α in Fig. 9.

The graphs of R_κ indicate that, with increasing value of the gamma shape parameter α, the ratio of the effective particle number to the total particle number monotonically increases with α. For the values of n and b plotted, it appears that the rate of increase is proportional to n and inversely proportional to b. With $n = 0$, the size distribution shape has no effects and R_κ is equal to unity over the entire range of α. With $n = 6$, $b = 0$, for example, R_κ can be as small as 0.039 at $\alpha = 0.1$ and as large as 85.224 at $\alpha = 10.0$. This indicates a significant increase in the proportion of total particles that are effectively contributing to the sampling error of the corresponding variable for $\alpha < 1.0$, and vice versa.

9. Shape effects on N_{12} for a given $\mathrm{FSD_p}$

It is obvious from Eq. (30) that, for a given FSD_p, N_{12} can be calculated if κ is known. It can be seen that the N_{12} obtained from (30) represents the minimum number of particles to be sampled for a sampling error smaller than or equal to the given FSD_p. For example, in the case of Waldvogel's rain data, a sampling error of 20% for total rain water ($n = 3$) using a sampling device with $D_1 = 0.3$ mm and $D_2 = 5.5$ mm, together with $\alpha + b = 6.89$, $\beta = 0.10$

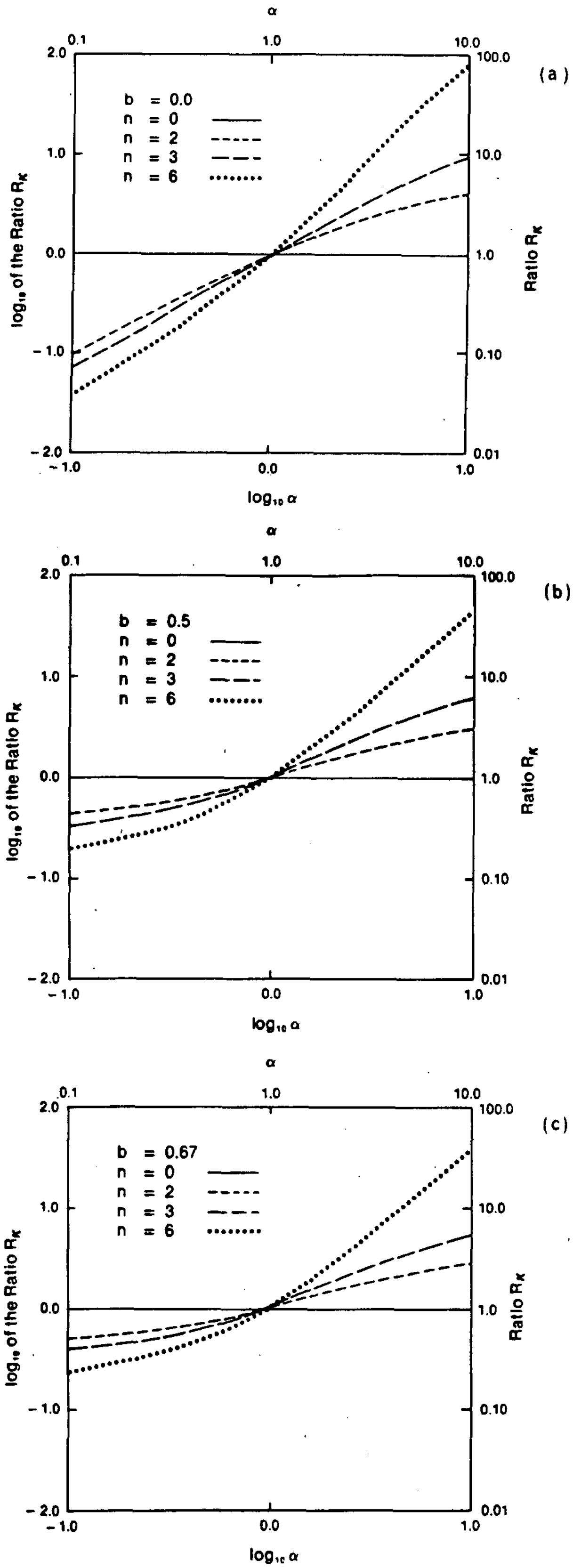

FIG. 9. Ratio R_κ as a function of the gamma shape parameter, (a) $b = 0.0$ (b) $b = 0.5$ (c) $b = 0.67$.

mm will require a minimum total of 64 drops. If needed, this can also be translated into sampling duration, given knowledge of particle number density and the receptacle area of the sampling device. The κ value computed for this case using (31) is 0.39. For a 10% error, similar calculation results in a required minimum total of 256 drops.

As before, one may also write the ratio of (30) to the corresponding expression using the exponential size distribution. Hence,

$$\left[\frac{\mathrm{FSD}_p\,(\text{gamma})}{\mathrm{FSD}_p\,(\text{exponential})}\right]^2 = \frac{N_{12}\,(\text{exponential})}{N_{12}\,(\text{gamma})}\cdot\frac{\kappa\,(\text{exponential})}{\kappa\,(\text{gamma})} = \frac{1}{R_N R_\kappa}, \quad (34)$$

where the size distribution model used is indicated in parentheses. For a specified FSD_p, the left-hand side of (34) becomes unity, and $R_N = N_{12}$ (gamma)/N_{12} (exponential) is the inverse of R_κ. One may therefore consider $1/R_\kappa$ as an indicator of the relative requirements of N_{12} to be sampled for a specified FSD_p. For example, in the measurement of optical extinction ($n = 2$), $R_N = 9.86$ for $\alpha = 0.1$ and $b = 0$, while $R_N = 0.24$ for $\alpha = 10.0$ and $b = 0$ with the same FSD_c, assuming again the complete spectrum case. This means that within a range of α likely to be encountered in various meteorological measurements, the required particle number to be sampled for a specified FSD can vary from approximately one-quarter to an order of magnitude larger than that required for an exponential size distribution, simply because of the variation in size distribution shape.

10. Conclusions

The gamma size distribution can be used in describing changes of size distribution shape, which are represented as curvature on the usual semilogarithmic plots. The probabilistic approach was used in the present study, invoking maximum likelihood estimation of the two-parameter gamma probability-density function. Quantile–quantile plots were used to compare fitting characteristics between this approach and that proposed in Ulbrich (1983b), demonstrating the fact that the maximum likelihood method weights more on the high frequency drop sizes, while Ulbrich's moments method weights more on the larger drop sizes. Reasonable goodness-of-fit can be obtained from either technique.

The gamma size distribution was then incorporated into a sampling error model of size distribution-related variables based on the Poisson process (Gertzman and Atlas 1977). The effects of size distribution shape on (a) the FSD, (b) the proportion of total particles effectively contributing to FSD, and (c) the required particle number for a given FSD were assessed. The results indicate that there are significant variations in all these parameters for the range of size distribution shape commonly encountered in meteorological measurements. For a size distribution having a downward facing concavity on a semilogarithmic plot, i.e. $\alpha > 1.0$, and for $n > 0$ there will be an overestimation of FSD and an underestimation of $\kappa = N_{\text{eff}}/N_{12}$ if the shape effects are neglected. The situation also implies significant decrease in the required number of drops for a given FSD. The reverse is true for $\alpha < 1.0$.

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