Application of the Theory of Adjustment to Double Theodolite Observations

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ABSTRACT

The position of a pilot balloon or any other airborne target which is followed by two theodolites is usually computed by means of the known baselength (i.e., the distance between the two theodolite stations) and three observed angle readings. It is to be noted, however, that four angle readings are available. In order to obtain an optimum estimation of the target's location the theory of adjustment of observations should be applied. With this method the estimation of the location is based on making the magnitude of the (root mean square) error a minimum. The method is particularly suitable for high speed electronic computers. The error analysis makes it possible to give some valuable hints in planning specific experiments with two optical or radio theodolites.

1. Introduction

The method of double theodolite observations finds application in quite a number of meteorological experiments. The method is considered to be one of the most accurate means to determine details of the upper wind structure, especially in the lower levels of the atmosphere. Furthermore the method is used for checking new radiosonde equipment (Dowsky, 1961), for tracking constant level balloons and rockets and for stereographic pictures of noctilucent clouds and other cloud systems (Witt, 1962). It is not essential to the theory which type of equipment is used when the target is tracked simultaneously from different sites. It may consist of optical theodolites, photo-theodolites, radio goniometers and even radar sets.

The method involves the determination of the three-dimensional position of the target by simultaneous readings of the azimuth and elevation angles from two theodolites located at different sites. The position of the target is specified by three coordinates. The baselength, i.e., the distance between the two theodolite stations, is known. The measurement gives four readings: two elevation angles and two azimuth angles. The amount of measured data is therefore greater than necessary, since a certain relationship must exist between the four variables. A quantity like the target height may therefore be evaluated using different combinations of three angles out of a set of four. As there are four combinations possible, four ways of evaluation of the target height can be realized and the results will be equal if the observed data incorporate no errors. To derive formulae like the target height we consider first the arrangement that the ground elevation of both theodolite sites is the same and the baselength $b=\overline{P_1P_2}$ is known with negligible inaccuracy. Denoting the azimuth angles in the points of observation P_1 and P_2 by α_1 and α_2 and the elevation angles by ϵ_1 and ϵ_2 (cf. Fig. 1) the standard formulae for the height h may easily be derived by application of some elementary trigonometric relations. For the four combinations of three angles one obtains successively:

$$h_1(\alpha_1,\alpha_2,\epsilon_1):\quad h_1=b\sin\alpha_2\sin^{-1}(\alpha_1+\alpha_2)\tan\epsilon_1, \tag{1}$$

$$h_2(\alpha_1,\alpha_2,\epsilon_2):\quad h_2=b\sin\alpha_1\sin^{-1}(\alpha_1+\alpha_2)\tan\epsilon_2, \tag{2}$$

$$h_3(\epsilon_1,\epsilon_2,\alpha_1):\quad h_3^2(\cot^2\epsilon_1-\cot^2\epsilon_2)-2h_3b\cos\alpha_1\cot\epsilon_1+b^2=0, \tag{3}$$

$$h_4(\epsilon_1,\epsilon_2,\alpha_2):\quad h_4^2(\cot^2\epsilon_2-\cot^2\epsilon_1)-2h_4b\cos\alpha_2\cot\epsilon_2+b^2=0, \tag{4}$$

h_3 and h_4 are found by applying the cosine rule in triangle ABC. It is observed that for the combinations $(\epsilon_1,\epsilon_2,\alpha_1)$ and $(\epsilon_1,\epsilon_2,\alpha_2)$ there are two solutions of h_3 resp. h_4. In fact, the angles (ϵ_1,α_1) determine a "ray" through P_1 which intersects the cone $\epsilon_2=$const., with vertex P_2, in two points.

Experiments have shown that the values for the target height, when calculated by means of the above expressions, may differ considerably due to the errors inherent in the measurement. Some inaccuracy in the measurement of the angles is unavoidable, since, e.g., the measurements may not be properly synchronized, errors may be made in reading the scales, one of the theodolites or both may not be properly levelled and oriented to a fixed direction, etc. On the other hand the non-linear expressions for h introduce a magnification of the errors in certain parts of the working area. Especially in the vicinity of the base, h_1 and h_2 are critical, and when the target is passing the base plane itself h_1 and h_2 become even indeterminate. Similar effects are observed in terms of h_3 and h_4, when the target happens to pass a plane through P_1 or P_2 which is orthogonal to the base plane. See Section 3.

The question arises whether a method can be developed which eliminates these inconsistencies and reduces the inaccuracy to a minimum. A first attempt has been made by Thyer (1962). In brief, his method involves the determination of the shortest distance between two "rays" which are defined by the sets of angles (ϵ_1,α_1) and (ϵ_2,α_2) and the location of a point on the shortest line which divides it in the ratio of the radial distances towards both observation points. A solution was found by using a vector method which was programmed for the IBM 709 computer.

This paper describes another method which in principle is based on the theory of adjustment of observations, commonly used by astronomers and geometers and to be found in any textbook dealing with planning and analyses of experiments and theories of observation, e.g., Linnik (1961). Problems of this kind come to the fore when between the variables to be observed certain relationships exist. In our case the conditional relationship involves an expression between the four angles. Referring to (1) and (2) this takes the form:

$$\varphi=\sin\alpha_1\tan\epsilon_2-\sin\alpha_2\tan\epsilon_1=0,\qquad \alpha_1+\alpha_2\neq 0 \quad\text{or}\quad \neq\pi. \tag{5}$$

In the base plane, where $\alpha_1+\alpha_2=0$ or π, the condition happens to be an identity.

2. Adjustment procedure

In general an experiment is performed to evaluate a quantity f, which is a function of say n variables $x_1\cdots x_n$. The obtained value of f will not be exact, due to errors in the measurement of $x_1\cdots x_n$. Taking the root mean square errors $\sigma_1\cdots\sigma_n$ as a measure of precision for the (stochastic) variables $x_1\cdots x_n$, the measure of precision for f is given by the well-known law of propagation of errors:

$$\sigma_f=\left\{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\sigma_i\right)^2\right\}^{\frac{1}{2}}. \tag{6}$$

The resulting error cannot be reduced, unless the measurement is repeated. However, problems of the sort under consideration here with some conditions between $x_1\cdots x_n$ are suitable for a real reduction of the error when applying an adjustment procedure. When the conditions are given by $\varphi_1=0,\ \cdots\varphi_M=0$ $(M\leq n)$ then the principle underlying such a procedure involves the evaluation of an estimator $S(f)$ in terms of f for which an optimal reduction of the variance is sought. In modern analyses of experiments this is achieved by a process specified by the following requirements:

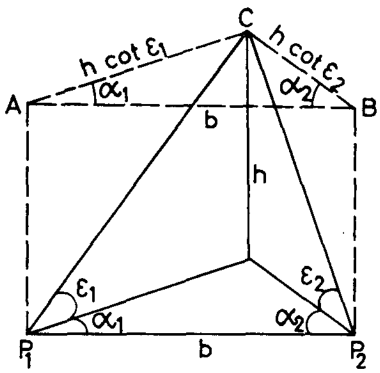

FIG. 1. Special double theodolite arrangement.

a) If the measured data $x_1\cdots x_n$ satisfy the conditions $\varphi_1=0,\ \cdots\varphi_M=0$ the estimator $S(f)$ equals $f(x_1\cdots x_n)$;
b) $S(f)$ is such that its variance reaches a minimum value.

Estimators fulfilling these requirements are not uniquely defined but in the common mathematical approach the admissable estimators are restricted to the class of functions:

$$S(f)=f+\lambda_1\varphi_1+\cdots\lambda_M\varphi_M$$

where the coefficients λ_i may be freely chosen. In order that these functions satisfy both a) and b) the values of λ_i have to be specified further, so as to obtain a minimum variance σ_S^2. The scheme of computation for the proper set λ_i is not reproduced here, since it can be found elsewhere (Bouman and De Jong, 1964). We will confine ourselves to presenting a survey of the scheme and rules which govern the adjustment procedure in matrix terminology.

Let the set of conditions $\varphi_1=0,\ \cdots\varphi_M=0$ be assembled in a $(M,1)$ column matrix:

$$\phi=\begin{vmatrix}\varphi_1\\ \vdots\\ \varphi_M\end{vmatrix}.$$

Let further S be the (n,n) covariance matrix:

$$S=\begin{vmatrix}\sigma_1^2 & 0 & \cdots & 0\\ 0 & \cdot & & \vdots\\ \vdots & & \cdot & \\ & & \cdot & 0\\ 0 & \cdots & 0 & \sigma_n^2\end{vmatrix}.$$

The Jacobian ϕ_x of ϕ is:

$$\phi_x=\begin{vmatrix}\dfrac{\partial\varphi_1}{\partial x_1} & \cdots & \dfrac{\partial\varphi_1}{\partial x_n}\\ \vdots & & \vdots\\ \dfrac{\partial\varphi_M}{\partial x_1} & \cdots & \dfrac{\partial\varphi_M}{\partial x_n}\end{vmatrix}$$

and the $(1,n)$ row matrix representing the gradient vector of f:

$$f_x=\left|\frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n}\right|.$$

The covariance matrix S involves only diagonal entries, expressing the assumption that the measurement of x_i is (stochastically) independent. In our problem for example the measurement of an elevation angle is supposed not to be influenced by that of an azimuth angle.

If $\phi_x{}^T$ denotes the transposed matrix of ϕ_x and if we introduce the covariance matrix Γ:

$$\Gamma=\phi_x S\phi_x{}^T,$$

then the estimator $S(f)$ we are interested in becomes:

$$S(f)=f(x)-f_xS\phi_x{}^T\Gamma^{-1}\phi(x). \tag{7}$$

Here the symbol within parenthesis indicates substitution of the measured data. All factors without explicit indication of the variables within brackets involve exact or true values of x_i.

In addition the (minimum) variance $\sigma_S{}^2$ of $S(f)$ becomes:

$$\sigma_S{}^1=f_xSf_x{}^T-f_xS\phi_x{}^T\Gamma^{-1}\phi_xSf_x{}^T. \tag{8}$$

The formulae (7) and (8) are the basic formulae in the theory of adjustment.

The first term on the right of (8) is nothing but the variance $\sigma_f{}^2$ of f. In fact $f_xSf_x{}^T$ is the matrix form of (6). The second term therefore represents the real variance reduction with respect to $\sigma_f{}^2$. The presence of conditional equations between the basic variables x_i indicates the existence of "equivalent" functions $\tilde{f}$ for the quantity to be investigated in the sense that, e.g., the height formulae $h_1, \cdots h_4$ yield the same value for true values of the angles. Then it may appear that in (7) and (8) the use of an alternative formula f would lead to a new estimator and another variance reduction. However, it can be shown that (7) and (8) are invariant with respect to a substitution of an equivalent function. This feature is closely related to another approach of the adjustment procedure, which is based on the set of equivalent functions proper. In fact one could prefer an estimation of f by "pooling" the numerical results of a certain number N of equivalent functions $\tilde{f}_i$. For example in our problem a pooling of the values $h_1, \cdots h_4$:

$$S(h)=g_1h_1+g_2h_2+g_3h_3+g_4h_4$$

with

$$\sum_{i=1}^{4} g_i=1$$

and computing the weight factors g_i in such a way that the variance becomes a minimum. However, it can be shown (Bouman and De Jong, 1964) that this process leads again to the same estimator (7) and minimum variance (8), provided that the number N of equivalent functions equals the total number of conditions plus one:

$$N=M+1.$$

When this number is exceeded the problem of pooling the values f_i is unsolvable and if the number is less than $M+1$, the estimation yields a minimum variance which is in excess of that for $N=M+1$.

3. Adjustment procedure applied to the theodolite problem

The calculation associated with the adjustment procedure is often tedious especially when the number of conditional equations is increased. But in our problem there is one condition only ($M=1$) so that here the scheme of adjustment is particularly adequate for application. To this aim the expressions (7) and (8) can be presented in another form, which is more suitable for practical computation.

Let f_1 and f_2 be two equivalent functions, then these will suffice to find an "optimal" pooling of values, since $M=1$. Let further f_1 and f_2 be chosen in such a way that

$$\varphi=\rho(f_1-f_2)$$

where ρ is a proportionality factor. Since here the "matrix" ϕ involves only one element, ϕ becomes a scalar. Then

$$\phi^T=\Phi=\varphi$$

$$\phi_x=\varphi_x.$$

In virtue of (7) one has:

$$S(f)=f_1(x)-f_{1,x}S\varphi_x{}^T\Gamma^{-1}\varphi(x). \tag{9}$$

The covariance matrix $\Gamma=\varphi_xS\varphi_x{}^T$ is a scalar and

$$\varphi_x=\rho(f_{1,x}-f_{2,x})+\rho_x(f_1-f_2).$$

In φ_x true values have to be substituted, but then the second term on the right vanishes. Substituting for φ and φ_x in (9) and rearranging terms leads to

$$S(f)=\frac{\rho}{\varphi_xS\varphi_x{}^T}\Big\{-(f_{2,x}S\varphi_x{}^T)f_1(x) + (f_{1,x}S\varphi_x{}^T)f_2(x)\Big\}, \tag{10}$$

which expresses $S(f)$ explicitly as a linear combination of $f_1(x)$ and $f_2(x)$.

Proceeding with (8) a substitution of φ and φ_x gives:

$$\sigma_S{}^2=\rho^2\frac{(f_{1,x}S\,f_{1,x}{}^T)(f_{2,x}S\,f_{2,x}{}^T)-(f_{1,x}S\,f_{2,x}{}^T)^2}{\varphi_xS\varphi_x{}^T}. \tag{11}$$

We will apply these formulae to the most general arrangement of the theodolites, i.e., the ground elevations of the points of observation P_1 and P_2 are supposed to be different. The height difference is denoted by a. Then the formulae for the target height and the conditional equation are slightly changed. Comparing Fig. 2 one derives:

$$h_1 = b \sin\alpha_2 \sin^{-1}(\alpha_1+\alpha_2) \tan\epsilon_1 + a \qquad (12)$$

resp.

$$h_2 = b \sin\alpha_1 \sin^{-1}(\alpha_1+\alpha_2) \tan\epsilon_2 \qquad (13)$$

and the condition becomes after equating (12) and (13):

$$\varphi = b(\sin\alpha_1 \tan\epsilon_2 - \sin\alpha_2 \tan\epsilon_1) - a \sin(\alpha_1+\alpha_2) = 0, \quad \alpha_1+\alpha_2 \neq 0 \quad \text{or} \quad \neq \pi. \qquad (14)$$

In the base plane the condition is an identity. A system of equivalent height formulae is trivial there. The height formula reduces there to the simple form

$$h = (b + a \cot\epsilon_1)(\cot\epsilon_1 \pm \cot\epsilon_2)^{-1}.$$

Let the matrix S be:

$$S = \begin{vmatrix} \sigma_{\alpha_1}{}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{\alpha_2}{}^2 & & \vdots \\ \vdots & & \sigma_{\epsilon_1}{}^2 & 0 \\ 0 & \cdots & 0 & \sigma_{\epsilon_2}{}^2 \end{vmatrix}.$$

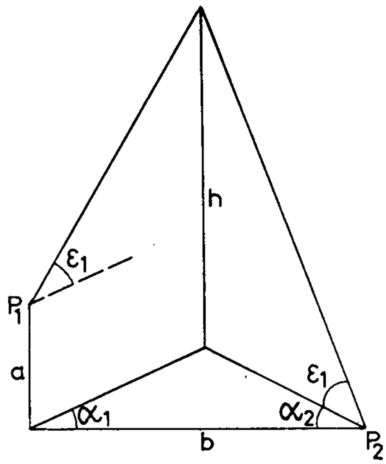

Fig. 2. General arrangement for double theodolite observation.

where the diagonal entries represent the variances of the angles α_1 and α_2 resp. ϵ_1 and ϵ_2.

To calculate $S(h)$ and its minimum variance we have to evaluate the matrix products $h_{1,x}Sh_{1,x}{}^T$, $h_{2,x}Sh_{2,x}{}^T$, $h_{1,x}Sh_{2,x}{}^T$, $h_{1,x}S\varphi_x{}^T$, $h_{2,x}S\varphi_x{}^T$ and $\varphi_x S\varphi_x{}^T$. Here the x_i have to be identified with α_1, α_2, ϵ_1 and ϵ_2. We have:

$$h_{1,x} = \left| \frac{\partial h_1}{\partial\alpha_1} \; \frac{\partial h_1}{\partial\alpha_2} \; \frac{\partial h_1}{\partial\epsilon_1} \; \frac{\partial h_1}{\partial\epsilon_2} \right|,$$

$$h_{1,x} = b \sin^{-2}(\alpha_1+\alpha_2) \left| -\sin\alpha_2 \cos(\alpha_1+\alpha_2) \tan\epsilon_1,\ \sin\alpha_1, \tan\epsilon_1,\ \sin\alpha_2 \sec^2\epsilon_1 \sin(\alpha_1+\alpha_2),\quad 0 \right|,$$
$$h_{2,x} = b \sin^{-2}(\alpha_1+\alpha_2) \left| \sin\alpha_2 \tan\epsilon_2,\ -\sin\alpha_1 \cos(\alpha_1+\alpha_2) \tan\epsilon_2,\ 0,\ \sin\alpha_1 \sec^2\epsilon_2 \sin(\alpha_1+\alpha_2) \right|,$$
$$\varphi_x = \left| b \cos\alpha_1 \tan\epsilon_2 - a\cos(\alpha_1+\alpha_2),\ -b\cos\alpha_2 \tan\epsilon_1 - a\cos(\alpha_1+\alpha_2),\ -b \sin\alpha_2 \sec^2\epsilon_1,\ b \sin\alpha_1 \sec^2\epsilon_2 \right|. \qquad (15)$$

Next, the variance of h_1 becomes:

$$h_{1x}Sh_{1,x}{}^T = b^2 \sin^{-4}(\alpha_1+\alpha_2)\{\sin^2\alpha_2 \tan^2\epsilon_1 \cos^2(\alpha_1+\alpha_2)\sigma_{\alpha_1}{}^2 + \sin^2\alpha_1 \tan^2\epsilon_1 \sigma_{\alpha_2}{}^2 + \sin^2\alpha_2 \sec^4\epsilon_1 \sin^2(\alpha_1+\alpha_2)\sigma_{\epsilon_1}{}^2\}, \qquad (16)$$

$h_{2,x}Sh_{2,x}{}^T$ is obtained by merely interchanging α_1 and α_2, respectively, ϵ_1 and ϵ_2.

Further

$$h_{1,x}Sh_{2,x}{}^T = -b^2 \sin^{-4}(\alpha_1+\alpha_2) \cos(\alpha_1+\alpha_2) \tan\epsilon_1 \tan\epsilon_2 (\sin^2\alpha_2 \sigma_{\alpha_1}{}^2 + \sin^2\alpha_1 \sigma_{\alpha_2}{}^2).$$

Next

$$h_{1x}S\varphi_x{}^T = b \sin^{-2}(\alpha_1+\alpha_2)[-\sin\alpha_2 \cos(\alpha_1+\alpha_2) \tan\epsilon_1\{b\cos\alpha_1 \tan\epsilon_2 - a\cos(\alpha_1+\alpha_2)\}\sigma_{\alpha_1}{}^2$$
$$-\sin\alpha_1 \tan\epsilon_1\{b \cos\alpha_2 \tan\epsilon_1 + a\cos(\alpha_1+\alpha_2)\}\sigma_{\alpha_2}{}^2 - b\sin^2\alpha_2 \sec^4\epsilon_1 \sin(\alpha_1+\alpha_2)\sigma_{\epsilon_1}{}^2],$$

$$h_{2,x}S\varphi_x = b \sin^{-2}(\alpha_1+\alpha_2)[\sin\alpha_2 \tan\epsilon_2\{b\cos\alpha_1 \tan\epsilon_2 - a\cos(\alpha_1+\alpha_2)\}\sigma_{\alpha_1}{}^2$$
$$+ \sin\alpha_1 \cos(\alpha_1+\alpha_2) \tan\epsilon_2\{b\cos\alpha_2 \tan\epsilon_1 + a\cos(\alpha_1+\alpha_2)\}\sigma_{\alpha_2}{}^2 + b\sin^2\alpha_1 \sec^4\epsilon_2 \sin(\alpha_1+\alpha_2)\sigma_{\epsilon_2}{}^2]$$

and

$$\varphi_x S\varphi_x{}^T = \{b\cos\alpha_1 \tan\epsilon_2 - a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_1}{}^2 + \{b\cos\alpha_2 \tan\epsilon_1 + a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_2}{}^2$$
$$+ \sin^2\alpha_2 \sec^4\epsilon_1 \sigma_{\epsilon_1}{}^2 + \sin^2\alpha_1 \sec^4\epsilon_2 \sigma_{\epsilon_2}{}^2. \qquad (17)$$

The multiplicative factor ρ is in view of (14):

$$\rho = -\sin^{-1}(\alpha_1+\alpha_2)$$

After these preliminaries the final results for $S(h)$ and $\sigma_S{}^2$ are obtained by substitution of these products in (10) and (11). It requires a lengthy computation to arrive at expressions for $S(h)$ and $\sigma_S{}^2$. They involve measured data as well as true values. By rearranging terms and intermediate use of the conditional equation (14) the expressions can be simplified, although their final form is still somewhat complicated. For practical use the remaining true values have to be replaced in the final form by measured data. Without giving all mathematical details the final expressions become:

$$\begin{aligned}S(h)=&\left[\{b\cos\alpha_1\tan\epsilon_2-a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_1}^2+b^2\sin^2\alpha_1\sec^4\epsilon_2\sigma_{\epsilon_2}^2\right]h_1\\&+\left[\{b\cos\alpha_2\tan\epsilon_1+a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_2}^2+b^2\sin^2\alpha_2\sec^4\epsilon_1\sigma_{\epsilon_1}^2\right]h_2\left[\{b\cos\alpha_1\tan\epsilon_2-a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_1}^2\right.\\&\left.+\{b\cos\alpha_2\tan\epsilon_1+a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_2}^2+b^2\sin^2\alpha_2\sec^4\epsilon_1\sigma_{\epsilon_1}^2+b^2\sin^2\alpha_1\sec^4\epsilon_2\sigma_{\epsilon_2}^2\right]^{-1}.\end{aligned}\tag{18}$$

Respectively:

$$\begin{aligned}\sigma_S^2=&b^4\sin^{-4}(\alpha_1+\alpha_2)\{\sin^2\alpha_1\sin^2\alpha_2\tan^2\epsilon_1\tan^2\epsilon_2\sin^2(\alpha_1+\alpha_2)\sigma_{\alpha_1}^2\sigma_{\alpha_2}^2+\sin^4\alpha_2\tan^2\epsilon_2\sec^4\epsilon_1\sigma_{\alpha_1}^2\sigma_{\epsilon_1}^2\\&+\sin^2\alpha_1\sin^2\alpha_2\tan^2\epsilon_1\sec^4\epsilon_2\cos^2(\alpha_1+\alpha_2)\sigma_{\alpha_1}^2\sigma_{\epsilon_1}^2+\sin^2\alpha_1\sin^2\alpha_2\tan^2\epsilon_2\sec^4\epsilon_1\cos^2(\alpha_1+\alpha_2)\sigma_{\alpha_2}^2\sigma_{\epsilon_1}^2\\&+\sin^4\alpha_1\tan^2\epsilon_1\sec^4\epsilon_2\sigma_{\alpha_2}^2\sigma_{\epsilon_2}^2+\sin^2\alpha_1\sin^2\alpha_2\sec^4\epsilon_1\sec^4\epsilon_2\sin^2(\alpha_1+\alpha_2)\sigma_{\epsilon_1}^2\sigma_{\epsilon_2}^2\}\\&\times\left[\{b\cos\alpha_1\tan\epsilon_2-a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_1}^2+\{b\cos\alpha_2\tan\epsilon_1+a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_2}^2\right.\\&\left.+b^2\sin^2\alpha_2\sec^4\epsilon_1\sigma_{\epsilon_1}^2+b^2\sin^2\alpha_1\sec^4\epsilon_2\sigma_{\epsilon_2}^2\right]^{-1}\end{aligned}\tag{19}$$

which solves the problem of adjustment for the target height. Their interpretation cannot in general be adapted to manual and graphical techniques. Unless use can be made of a high speed electronic computer, quick results are not to be expected.

In order to have an impression of the overall variance reduction, the special case of the arrangement which has been described in Section 1 will be investigated in more detail, because it reveals some interesting and valuable points which justify the application of the method.

Taking $a=0$ and $\sigma_{\alpha_1}^2=\sigma_{\alpha_2}^2=\cdots\sigma_{\epsilon_2}^2=\sigma^2$ the formulae (18) and (19) reduce and, respectively, to

$$\begin{aligned}S(h)=&(\cos^2\alpha_1\tan^2\epsilon_2+\sin^2\alpha_1\sec^4\epsilon_2)h_1+(\cos^2\alpha_2\tan^2\epsilon_1+\sin^2\alpha_2\sec^4\epsilon_1)h_2\\&\times(\cos^2\alpha_1\tan^2\epsilon_2+\cos^2\alpha_2\tan^2\epsilon_1+\sin^2\alpha_1\sec^4\epsilon_2+\sin^2\alpha_2\sec^4\epsilon_1)^{-1},\end{aligned}\tag{20}$$

$$\begin{aligned}\sigma_S^2=&b^2\sin^{-4}(\alpha_1+\alpha_2)\{(\sin^2\alpha_1+\sin^2\alpha_2)(\sin^2\alpha_1\tan^2\epsilon_1\sec^4\epsilon_2+\sin^2\alpha_2\tan^2\epsilon_2\sec^4\epsilon_1)\\&+\sin^2(\alpha_1+\alpha_2)\sin^2\alpha_1\sin^2\alpha_2\sec^4\epsilon_1\sec^4\epsilon_2(\tan^2\epsilon_1-\sec^4\epsilon_1)(\tan^2\epsilon_2-\sec^4\epsilon_2)\}\sigma^2\\&\times(\cos^2\alpha_1\tan^2\epsilon_2+\cos^2\alpha_2\tan^2\epsilon_1+\sin^2\alpha_1\sec^4\epsilon_2+\sin^2\alpha_2\sec^4\epsilon_1)^{-1}.\end{aligned}\tag{21}$$

These expressions have been programmed for the ZEBRA computer, developed by the Postal and Telegraphic Service in the Netherlands. By way of example an error analysis has been performed for a theodolite set up with $a=0$, $b=5$ km and $\sigma_{\alpha_1}=\cdots\sigma_{\epsilon_2}=\sigma=1$ milliradian, which is a reasonable value for modern theodolite equipment including radio goniometers. First, in Figs. 3a, b and c the scalar fields are shown for the "error" σ_{h_1}, which pertains to the conventional height formula h_1, at levels of 1, 5 and 20 km. $\sigma_{h_1}=h_{1,x}Sh_{1_1x}{}^T$ has already been evaluated, see expression (16). The figures reveal the phenomenon already explained in the introduction, that the error increases rapidly in the vicinity of the base plane, except in the proximity of the point P in a direction normal to the base. There is a point where the error reaches a minimum value. The locus of this point has been computed and is represented by the dotted curve in Fig. 3c. The curve represents the horizontal projection of a balloon track for which the height determination using h_1 is most accurate. The graphs for h_2 are identical with those in Fig. 3 but symmetric with respect to the line bisecting the base P_1P_2 perpendicularly. A similar error analysis has been made for h_3 and the results are shown in Figs. 4a, b and c. They originate from a calculation of σ_{h_3} by means of the law of propagation of errors (6). The result is reproduced here without comment:

$$\begin{aligned}\sigma_{h_3}^2=&h^2\{b^2\sin^2\alpha_1\cot^2\epsilon_1\sigma_{\alpha_1}^2\\&+b^2\sin^2\alpha_1\cot^2(\alpha_1+\alpha_2)\operatorname{cosec}^4\epsilon_1\sigma_{\epsilon_1}^2\\&+\cos^2\epsilon_2\operatorname{cosec}^6\epsilon_2\sigma_{\epsilon_2}^2\}\{h(\cot^2\epsilon_1-\cot^2\epsilon_2)\\&-b\cos\alpha_1\cot\epsilon_1\}^{-2}.\end{aligned}$$

It is observed that the error increases rapidly here when the target approaches a plane through P_2 which is orthogonal to the baseplane. Again the graphs for h_4 are identical with those for h_3, but symmetric with respect to the plane bisecting P_1P_2 perpendicularly. In the vicinity of the base plane h_3 and h_4 give rather accurate results. In the base plane itself $\sigma_{h_3}=\sigma_{h_4}$.

Both series of graphs have to be compared with those for the estimation (20). The figures for the minimum error $\sigma_{\hat{S}}$ based on formula (21) are represented in Figs. 5a, b and c. The error field is now regular in the whole working area, with exception of the points P_1 and P_2, being singular points. It is remarkable that in Figs. 5b and c a minimum appears on the line bisecting the base itself. This minimum moves outwards when the height increases. It can be shown that the minimum merges into the midpoint of $\overline{P_1P_2}$ itself when

$$h=\tfrac{1}{4}\sqrt{2+2\sqrt{5}}\,b=0.63601b.$$

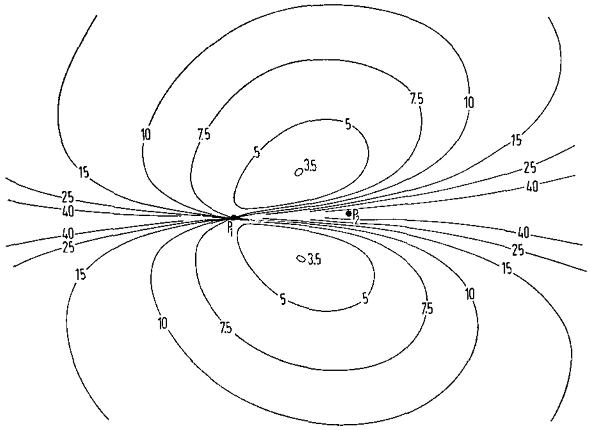

FIG. 3a. Error analysis for $\sigma_{h_1}(m)$ associated with the height formula $h=h_1(\alpha_1,\alpha_2,\epsilon_1)$ at a level of 1 km. Base length 5 km, rms error for the angles 1 millirad.

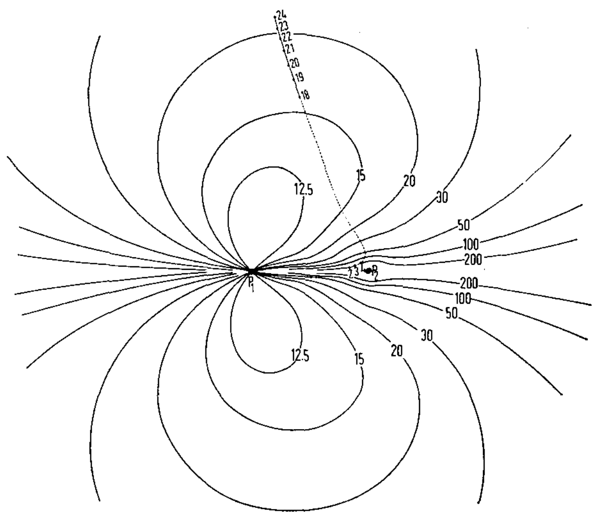

FIG. 3b. Error analysis for $\sigma_{h_1}(m)$ at a level of 5 km. The dotted curve represents the projection of a pilot balloon's trajectory.

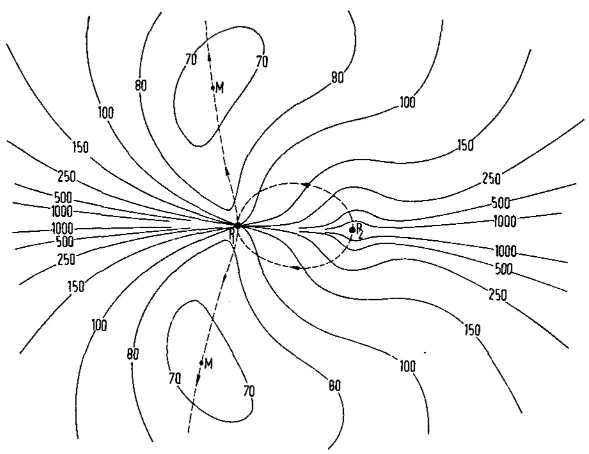

FIG. 3c. Error analysis for $\sigma_{h_1}(m)$ at a level of 20 km. The dashed curve represents the locus of the minimum M.

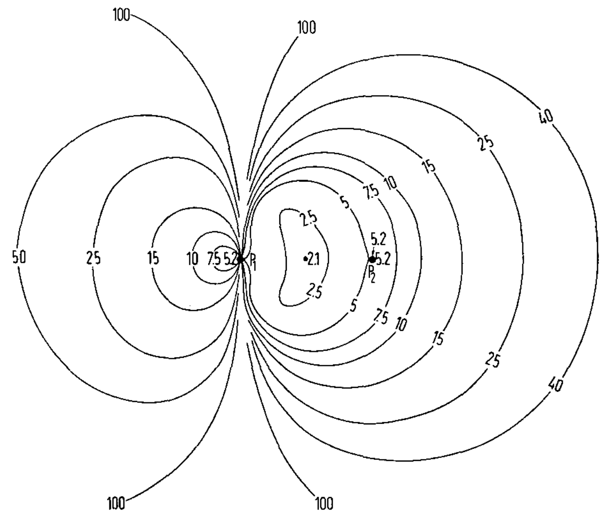

FIG. 4a. Error analysis for $\sigma_{h_3}(m)$ associated with the height formula $h=h_3(\epsilon_1,\epsilon_2,\alpha_1)$ at a level of 1 km. Base length 5 km, rms error for the angles 1 millirad.

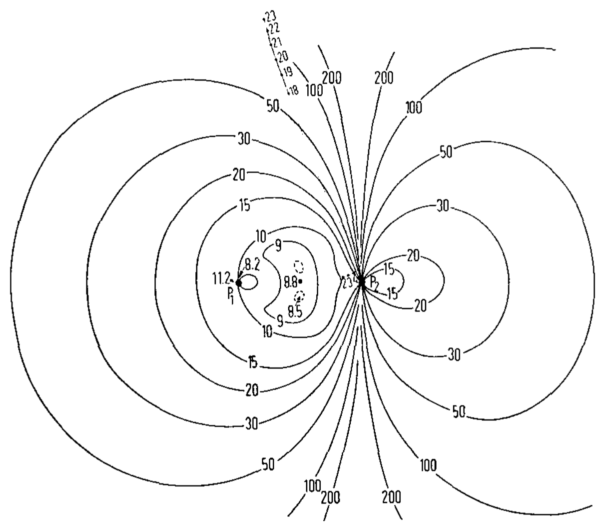

FIG. 4b. Error analysis for $\sigma_{h_3}(m)$ at a level of 5 km. The dotted curve represents the projection of the pilot balloon's trajectory.

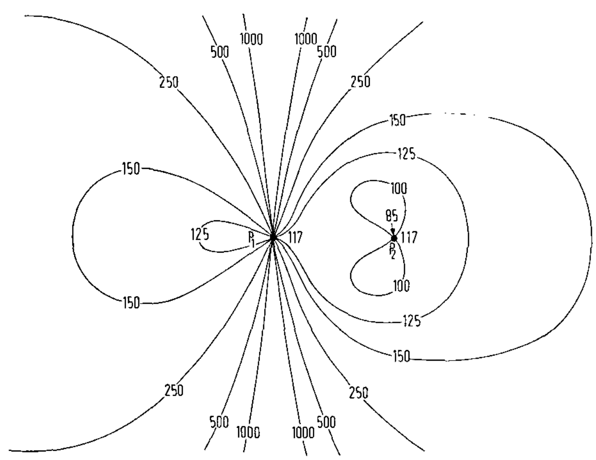

FIG. 4c. Error analysis for $\sigma_{h_3}(m)$ at a level of 20 km.

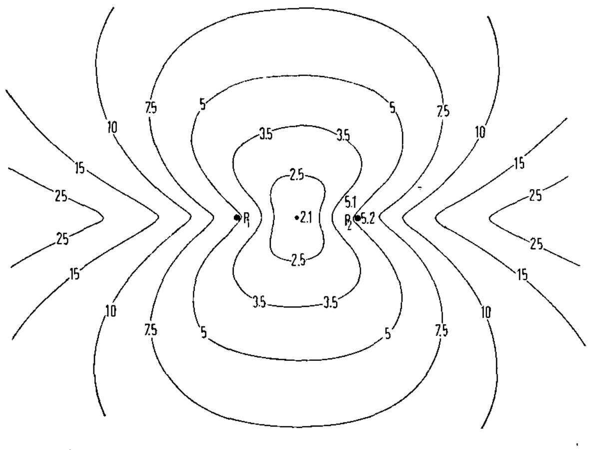

FIG. 5a. Error analysis for $\sigma_S(m)$ associated with the height formula $S(h)$ at a level of 1 km. Base length 5 km, rms error for the angles 1 millirad.

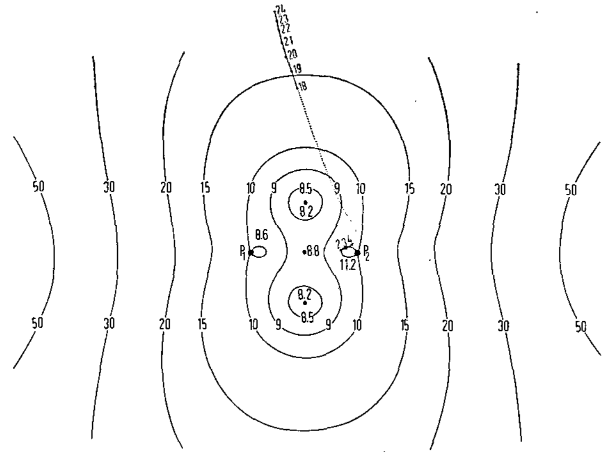

FIG. 5b. Error analysis for $\sigma_S(m)$ at a level of 5 km. Dotted curve represents the pilot balloon's trajectory.

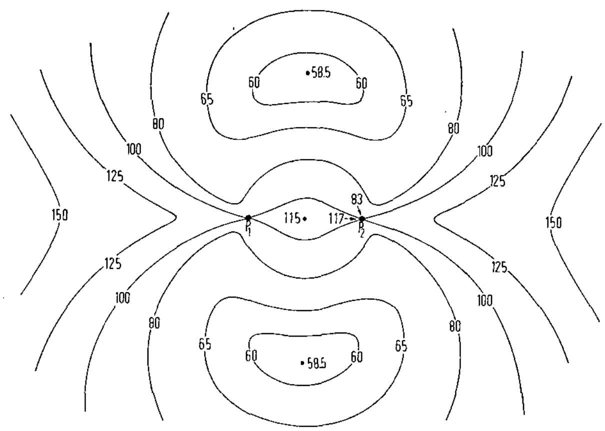

FIG. 5c. Error analysis for $\sigma_S(m)$ at a level of 20 km.

At levels below $h=0.636b$, the minimum stays in the midpoint. When y denotes the distance of the minimum to the base a diagram can be constructed showing isopleths of y as a function of the base b and height h (Fig. 6). When substituting for $\alpha_1=\alpha_2$ and $\epsilon_1=\epsilon_2$ in (21) and differentiating with respect to α_1 the equation for these isopleths in the (h,b) diagram becomes:

$$16h^4b^2-(4y^2+b^2)^2(4h^2+b^2)=0.$$

The isopleths have two asymptotes, notably the h-axis and the line $h=0.636b$, which is related to the above level where the minimum merges into the midpoint. Moreover, the isopleths are homothetic with respect to the origin. A change of scale in b, h and y has no effect on the diagram.

The diagram can be consulted for a most suitable arrangement of the theodolites in terms of a specific experiment, for instance a rocket ascent. When a comparison is made of Figs. 3, 4 and 5 it is striking that the error reduction in certain sub-domains is not only excessive but depends strongly on the formulae on which the height evaluation is based. The effect of this may be illustrated by the results of an experiment with a pilot balloon. The balloon was tracked by two theodolites 4489 m apart and two radar sets, a Decca wind-finding radar and a III Mark 7 automatic follower. The projection of the balloon based on the radar data is shown in Figs. 3b, 4b and 5b. Due to cloudiness the double theodolite observation is not complete. In the projections only those points are indicated which were simultaneously observed. For these points the balloon height was computed using h_1, h_2, h_3, h_4 and $S(h)$. Together with the radar heights the results are assembled in Table 1.

TABLE 1. Balloon height versus time for a radar- and double theodolite observation

Time (min)	h_{Decca}	$h_{\mathrm{Mark\ 7}}$	h_1	h_2	h_3	h_4	$S(h)$
2	—	—	953	679	690	690	681
3	—	—	1218	1127	1132	1146	1129
4	1561	1537	1623	1559	1561	1569	1562
18	5865	5840	5828	5862	5683	5599	5895
19	6252	6164	6138	6170	5960	6300	6154
20	6540	6454	6453	6426	6237	6573	6439
21	6819	6768	6796	6785	6694	6843	6790
22	7030	7146	7119	7112	7033	7155	7115
23	7402	7404	7424	7393	6892	7674	7403
24	7787	7756	7784	7825	7182	8078	7805

Whereas in the first few minutes h_2, h_3 and h_4 are in fairly good agreement with $S(h)$ the values of h_1 are in considerable excess. This can be explained by the fact that shortly after release the balloon moves in the vicinity of the base near point P_2. There the measurement of h_1 is inaccurate as can be seen in Fig. 3b. The inaccuracy of h_2, h_3 and h_4 is still restricted there. In Fig. 7 the data are plotted in a (h,t) diagram. The ascent curve based on the h_1 values suggests an ascent

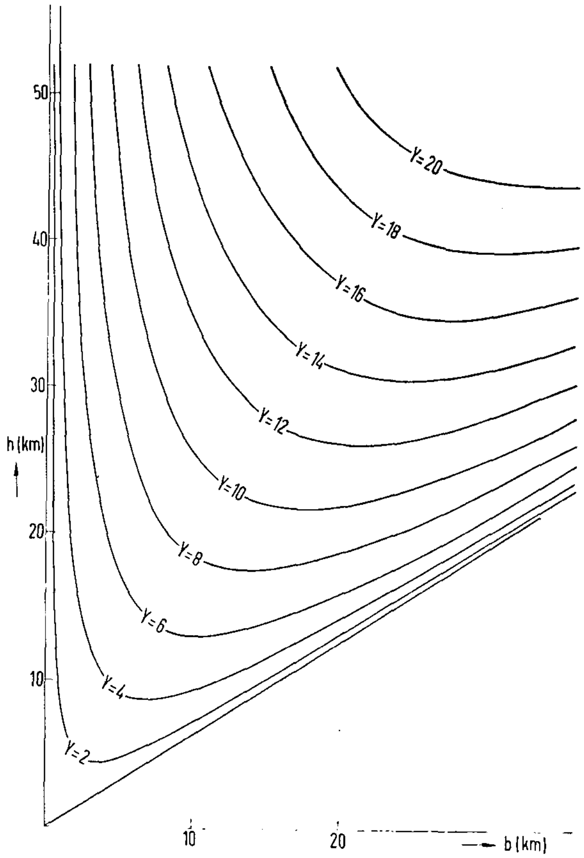

Fig. 6. Isopleths of the distance y of the minimum in the error field σ_S in terms of the base length b and the height h.

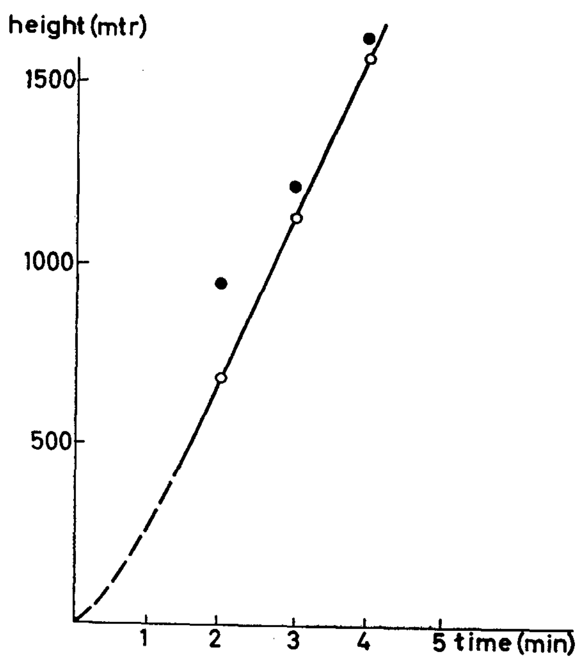

Fig. 7. Height-time diagram.

rate which on physical grounds is very doubtful. The curve through the circles based on $S(h)$ however indicates an acceleration shortly after release in agreement with what should be expected. The points for h_2, h_3 and h_4 are located within the small circles. The table reveals furthermore that at higher levels the inaccuracy of h_3 and h_4 increases rapidly. This is to be expected, since the balloon then moves within areas of high inaccuracy, in the error fields of h_3 and h_4 (cf., Fig. 4b). The values of h_1 and h_2 are there in agreement with those of $S(h)$.

4. Method of adjustment with respect to the trajectory's projection

To determine accurate upper winds the projection of the balloon's trajectory should be known with highest precision. With respect to this projection the system is also overdetermined, so that the method of adjustment again comes to the fore. The target's projection is determined by both the azimuth, angles α_1 and α_2. Therefore the adjustment procedure should be performed in terms of the base variables proper. Then referring to the basic formula (7) one merely needs to substitute for f:

$$f = x_i$$

to obtain:

$$S(x_i) = x_i - S\phi_x{}^T\Gamma^{-1}\phi(x) \tag{22}$$

respectively:

$$\sigma_S{}^2 = \sigma_i{}^2 - S\phi_x{}^T\Gamma^{-1}S.$$

When applied to our problem all four base variables play a part. For reasons of completeness all four angles will be adjusted.

Referring to (15) and (17) the results can immediately be written down.

$\phi(x)$ and $\Gamma = \phi_x S\phi_x{}^T$ are scalar, so in the second term of (22) we have the factor P:

$$P = \frac{\phi(x)}{\phi_x S\phi_x{}^T} = b(\sin\alpha_1 \tan\epsilon_2 - \sin\alpha_2 \tan\epsilon_1) - a\sin(\alpha_1+\alpha_2)$$

$$\times[\{b\cos\alpha_1 \tan\epsilon_2 - a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_1}{}^2$$

$$+\{b\cos\alpha_2 \tan\epsilon_1 - a\cos(\alpha_1+\alpha_2)\}^2\sigma_{\alpha_2}{}^2$$

$$+\sin^2\alpha_2 \sec^4\epsilon_1\sigma_{\epsilon_1}{}^2 + \sin^2\alpha_1 \sec^4\epsilon_2\sigma_{\epsilon_2}{}^2]^{-1}.$$

Then after writing out the matrix equation one has successively:

$$S(\alpha_1) = \alpha_1 - \{b\cos\alpha_1 \tan\epsilon_2 - a\cos(\alpha_1+\alpha_2)\}P\sigma_{\alpha_1}{}^2$$

$$S(\alpha_2) = \alpha_2 + \{b\cos\alpha_2 \tan\epsilon_1 + a\cos(\alpha_1+\alpha_2)\}P\sigma_{\alpha_2}{}^2$$

$$S(\epsilon_1) = \epsilon_1 + b\sin\alpha_2 \sec^2\epsilon_1 P\sigma_{\epsilon_1}{}^2$$

$$S(\epsilon_2) = \epsilon_2 - b\sin\alpha_1 \sec^2\epsilon_2 P\sigma_{\epsilon_2}{}^2.$$

$S(\alpha_1)$ and $S(\alpha_2)$ guarantee a uniquely defined projection point of the target on the horizontal plane, since they fulfill the conditional equation. After the projection of the trajectory is known, the winds may be derived in the usual way. The adjusted values of all four angles uniquely define not only the projection but also the three-dimensional position of the balloon. This again suggests another method to determine the target height, namely by substituting for the adjusted angles in one of the original height formulae, e.g.,

$$S'(h)=h_1\{S(\alpha_1),S(\alpha_2),S(\epsilon_1)\},$$

but it can be proved that this new estimate, when developed in a Taylor series, corresponds with that of the previously introduced estimation in so far terms of the second order and higher may be neglected. So:

$$S'(h)=h_1\{S(\alpha_1),S(\alpha_2),S(\epsilon_1)\}=\cdots$$
$$=h_4\{S(\epsilon_1),S(\epsilon_2),S(\alpha_2)\}=S(h).$$

However, if we had followed the new scheme the derivation of the expressions like (18) and (19) would have been very cumbersome.

5. Discussion

Although the currently used formulae for the location of an airborne target by detection of two theodolites in general give fairly good results, care should be taken when the target happens to move in certain sub-domains where, due to observation errors, the formulae may destroy the accuracy. Then the more sophisticated methods of Thyer and the one in the present study should be strongly recommended. When quick results are required programming of the expressions for electronic computers is unavoidable.

Throughout the investigation it has been assumed that the base distance is not too great and the target is not moving too far off the theodolite sites. Otherwise the earth's curvature should be incorporated in the computations. Then from the onset all formulae should be revised. It seems that up to now an attempt to include the earth's curvature has not been undertaken.

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