Marked Point Process Models of Raindrop-Size Distributions

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(Manuscript received 14 June 1991, in final form 28 January 1992)

ABSTRACT

The principal process considered in this paper is the flux of raindrops through a volume of the atmosphere. This process is of fundamental importance for a wide variety of engineering and environmental problems, notably remote sensing of precipitation, infiltration of rainfall, soil erosion, atmospheric deposition of pollutants, and design of microwave communication systems. A marked point process model is developed in which the point process represents the arrival times of drops at the upper surface of a sample volume and the mark associated with a drop is its diameter. In the model, both the rate of occurrence of raindrops and the distribution of drop diameters vary randomly over time. Results that relate the drop-size distribution within the sample volume to the probability law of the drop-arrival process are presented. These results allow straightforward comparisons between temporal characterizations of drop-size distributions and spatial characterizations. Representations for derived processes such as rainfall rate and reflectivity are shown to be quite accurate using raindrop data from North Carolina.

1. Introduction

Rainfall rate and associated geophysical processes can be represented in terms of the number, size, and velocity of raindrops. The principal goal of this paper is to develop a simple parameterization of drop-size distributions that is useful for studying time evolution of rainfall rate and related geophysical processes.

In the 1940s and 1950s, the study of drop-size distributions provided a common avenue for hydrologists and meteorologists to examine precipitation research problems. Hydrologists examined problems of soil erosion and infiltration (see, for example, Laws and Parsons 1943; Chapman 1948; Horton 1948; Wischmeier and Smith 1958). Meteorologists examined problems of remote sensing of the atmosphere by radar (Marshall and Palmer 1948; Spilhaus 1948; Best 1950; Atlas and Plank 1953). A wide range of current research problems (see, for example, NASA 1986; Chahine 1991; Simpson et al. 1988) point to the utility of an integrated approach to hydrometeorological research. An important conclusion of this paper is that the study of drop-size distributions, with its roots in both land-surface processes and atmospheric remote sensing, provides an important element to an integrated program of hydrometeorological research.

Development of analytical tools for relating characteristics of drop-size distributions to geophysical processes is of principal importance to this study. Two fundamental types of processes can be distinguished: those related to the drop-size distribution within a sample volume (for example, radar reflectivity factor, liquid water content, and microwave attenuation) and processes that are tied to the flux of raindrops (for example, rainfall rate, kinetic energy of rainfall, and precipitation scavenging coefficient). In this paper, utility of the marked point process model is illustrated for one flux process, rainfall rate, and one sample volume process, radar reflectivity factor. Similar results can be derived for a wide range of geophysical processes.

Drop-size distributions are represented by a marked point process model in which the point process represents the arrival times of drops at the upper surface of a sample volume and the mark associated with a drop is its diameter [marked point process models are discussed in Karr (1991)]. In the model, both the rate of occurrence of raindrops and the distribution of drop diameters vary randomly over time. Results that relate the drop-size distribution within the sample volume to the probability law of the drop arrival process are obtained. These results allow straightforward comparisons between temporal characterizations of drop-size distributions and spatial characterizations.

The arrival process for raindrops is modeled by a Poisson process with randomly varying rate of occurrence [also known as a Cox process or doubly stochastic Poisson process; see Karr (1991)]. Two parametric families of distribution functions are examined in detail as models of drop-diameter distributions. They are the two-parameter lognormal and gamma distributions. In each case, the parameters of the drop-diameter distribution are treated as time-varying random processes. The lognormal distribution has important antecedents...
as a model for drop-size distributions (see, for example, Feingold and Levin 1986) as does the gamma distribution (see, for example, Ulbrich 1983; Willis 1984; Chandrasekhar and Bringi 1987). The gamma model also provides an important link with standard approaches to modeling drop-size distributions based on the Marshall–Palmer model (see Marshall and Palmer 1948; Waldvogel 1974).

Raindrop data from North Carolina are used to illustrate performance of the models developed in this paper. The observations were collected during the period 1960–62 using the Illinois State Water Survey raindrop camera (Mueller and Sims 1967). Previous analyses of these data have appeared in Cataneo and Stout (1968). The principal justification of the assumptions used for the marked point process model is that they lead to very accurate representations of rainfall rate and reflectivity. This point is illustrated with the North Carolina data.

2. Definitions and notation

In this section, the modeling framework for drop-size distributions is introduced. The sampling environment that underlies model development can be visualized as follows. A sample volume of approximately 1 m³ is located at or near the ground surface. The history of raindrops passing through the sample volume from an arbitrary time origin is the fundamental process to be modeled. This process can be characterized either by an arrival process, which represents the times and diameters of raindrops arriving at the upper surface of the sample volume, or by a sample-volume process, which represents the counts and diameters of raindrops within the sample volume as a function of time. The drop-size data used for empirical analyses are introduced in section 4. A concrete picture of the sampling environment can be obtained from the discussion in section 4 of the data-collection system.

Model development will proceed from definitions based on the arrival process. In this section, the arrival process is defined, the sample-volume process is represented in terms of the arrival process, and rainfall rate and radar reflectivity factor are represented in terms of the arrival process.

Let \((T_i, D_i)\) be a marked point process denoting the arrival times of raindrops and diameters of raindrops at the upper surface of the sample volume, that is, \(T_i\) is the arrival time (s) of the \(i\)th drop and \(D_i\) is the diameter (mm) of the \(i\)th drop. The counting process for drop arrivals is denoted by

\[
\eta(t) = \sum_{i=1}^{\infty} 1(T_i \leq t),
\]

where

\[
1(T_i \leq t) = \begin{cases} 1, & \text{if } T_i \leq t, \\ 0, & \text{if } T_i > t, \end{cases}
\]

is the indicator function for the event that the \(i\)th arrival time is less than or equal to \(t\). The process \(\eta(t)\) specifies the accumulated number of drops that arrive at the top of the sample volume as a function of time. The time origin is arbitrary; it can represent the beginning of a storm, the beginning of a minute, hour, day, etc. For the empirical results presented in sections 4 and 5, the time origin will typically represent the beginning of a storm period.

In general, the distribution of drop diameters for the arrival process will differ from the distribution of drop diameters for the sample-volume process. The principal reason is that the terminal velocity of raindrops depends on drop diameter. Atlas and Ulbrich (1977) have developed the following power-law model for terminal velocity (m s⁻¹) of raindrops in still air at sea level based on the data of Gunn and Kinzer (1949):

\[
v(D_i) = aD_i^b,
\]

where \(D_i\) is in millimeters, \(a\) is in meters per second, and \(b\) is dimensionless. Atlas and Ulbrich obtain a value of 0.67 for \(b\) and 3.78 mm s⁻¹ for \(a\). The representation is quite accurate for drops between 0.5 and 5.0 mm. Note that the velocity of a 0.5-mm drop is 2.4 mm s⁻¹; the drop velocity for a 2-mm drop is 6.0 mm s⁻¹. Thus, a 0.5-mm drop will take 0.42 s to pass through a 1-m sample volume, while a 2-mm drop will take 0.17 s.

Denote the number of drops in the sample volume at time \(t\) by \(M(t)\). The process can be represented in terms of \((T_i, D_i)\) as follows:

\[
M(t) = \sum_{i=1}^{\eta(t)} 1(T_i + v(D_i)^{-1} > t).
\]

A drop of diameter \(D_i\) will take \(v(D_i)^{-1}\) s to pass through a 1-m sample volume; consequently \(T_i + v(D_i)^{-1}\) is the time at which the \(i\)th drop reaches the bottom of the sample volume. The indicator function in (5) is used to count drops that have not reached the bottom of the sample volume by time \(t\). By restricting the summation to drops that have arrived up until time \(t\) [i.e., by summing over drop arrivals from 1 to \(\eta(t)\)], the total number of drops in the sample volume at time \(t\) is obtained. The drop diameters in the sample volume at \(t\) will be denoted \(D_{1t}, \ldots, D_{M(t)}\).

Rainfall accumulation (mm) can be represented in terms of the marked point process through computations of the water volume of raindrops:

\[
A(t) = \frac{\pi}{6} \times 10^{-6} \sum_{i=1}^{\eta(t)} D_i^3.
\]

Rainfall rate (mm h⁻¹) is defined relative to an accumulation time interval \(\Delta t\):

\[
R(t) = 3600 \left[ \frac{A(t + \Delta t) - A(t)}{\Delta t} \right].
\]

For an area of 1 m³ typical values of drop arrival rate range from less than 100 drops per second to more
than 10,000 drops per second. An averaging interval of 1 s will be used in subsequent sections for representing rainfall rate through a 1-m² sample volume.

Radar reflectivity factor is the sum of the sixth power of drop diameters divided by the sample volume (mm⁶ m⁻³). For a 1-m² sample volume and a given time t, it can be represented as follows:

\[ Z(t) = \sum_{i=1}^{n(t)} D_i^6 \left[ T_i + v(D_i)^{-1} > t \right] \]  \tag{8}

or

\[ Z(t) = \sum_{i=1}^{M(t)} D_i^6. \]  \tag{9}

The term “radar reflectivity factor” will be shortened to “reflectivity.”

3. The marked point process model

Having established the notation and terminology that will be used in modeling, the principal problems to be examined can be described as follows:

1) determine the probability law of the marked point process \((T_i, D_i)\);

2) given the probability law for the marked point process, determine the probability law of the counts and drop sizes within the sample volume; and

3) given the probability law for the marked point process, determine the probability law for derived processes such as rainfall rate and radar reflectivity.

The modeling strategy will be to prescribe the probability law required in 1) for the drop-arrival process. Justification for the assumptions made at this step will be that accurate representations of rainfall rate and reflectivity are obtained (as judged from empirical data analysis). The results for 2) and 3) must be obtained analytically from the assumed probability law for the drop-arrival process. The model for the drop-arrival process is introduced below.

a. Raindrop-arrival process

Let \( \lambda(t) \) be stochastic processes with the following interpretation:

\( \lambda(t) = \) mean number of drop arrivals at the top of the sample volume at time \( t \) (drops per square meter per second);

\( \theta(t) = \) parameters of the drop diameter distribution at time \( t \).

The marked point process model can be described as follows. Raindrops arrive at the upper surface of the sample volume according to a Poisson process with time-varying rate of occurrence \( \lambda(t) \). Drop diameters are independent and the distribution of drop diameters is given by a probability density function \( f[y | \theta(t)] \), where \( \theta(t) \) denotes the time-varying parameters of the drop diameters. The model is specified by the following assumptions:

1) for \( k = 0, 1, 2, \ldots \),

\[ P[\eta(t) = k | \lambda(u), \theta(u); u < t] = \frac{1}{k!} \left[ \int_0^t \lambda(u) du \right]^k \exp \left[ -\int_0^t \lambda(u) du \right] \] \tag{10}

2) for \( 0 < t_1 < \ldots < t_k, \eta(t_k) - \eta(t_{k-1}), \ldots, \eta(t_1) - \eta(0) \) are conditionally independent given \( \{\lambda(t), \theta(\cdot)\} \); and

3) \[ P[D_i \leq x | \lambda(u), \theta(u); u \leq T_i] = \int_0^x f[y | \theta(T_i)] dy, \quad x \geq 0 \] \tag{11}

where \( f[y | \theta] \) is a probability density function with parameter(s) \( \theta \).

Conditions 1 and 2 above specify that the arrival process for drops at the top of the sample volume is a Poisson process with time-varying rate of occurrence \( \lambda(t) \). Similar models have been used for rainfall modeling by Hosking and Stow (1987) and Smith and Karr (1983).

The third assumption specifies that the conditional density function of drop sizes arriving at the top of the sample volume is \( f[y | \theta] \), with parameters specified by the random process \( \theta(t) \). Two-parameter lognormal and gamma distributions will be examined in detail at the end of this section as models for drop-diameter distributions.

b. Sample-volume process

Having specified the probability law of the drop-arrival process, the next task is to obtain the probability law of the sample-volume process in terms of the probability model for the drop-arrival process.

The probability law for drop counts \( M(t) \) for the sample volume at time \( t \) is specified as follows:

for \( k = 0, 1, 2, \ldots \),

\[ P[M(t) = k | \lambda(t), \theta(t)] = \frac{\Lambda(t)^k}{k!} \exp[-\Lambda(t)], \]  \tag{12}

where

\[ \Lambda(t) = \lambda(t) \int_0^t v(y)^{-1} f[y | \theta(t)] dy \] \tag{13}

and has units of drops per cubic meter. Using the power-law model for terminal velocity (4), the result becomes

\[ \Lambda(t) = \frac{\lambda(t)}{a} \int_0^y y^{-b} f[y | \theta(t)] dy. \]  \tag{14}

The proof of (12) is given in the Appendix.
The result states that the number of drops in the sample volume has a Poisson distribution with mean \( \Lambda(t) \). Note that if \( b \) were 0, the velocity of drops would take the constant value \( a \) for all drop diameters and the mean number of drops in the sample volume would be \( \lambda(t)/a \).

The probability density function \( g[x|\theta(t)] \) for drop diameters in the sample volume is defined by

\[
P[D_i \leq y|\lambda(t), \theta(t)] = \int_0^y g[x|\theta(t)]dx, \quad y \geq 0.
\]  

(15)

The relationship between the distribution of drop diameters for the arrival process and sample-volume process is given by the following:

\[
g[x|\theta(t)] = \frac{f[x|\theta(t)]}{\int_0^\infty \left[ v(x)/v(y) \right] f[y|\theta(t)]dy}.
\]  

(16)

Using the terminal-velocity model of (4), the result becomes

\[
g[x|\theta(t)] = \frac{x^{-b}f[x|\theta(t)]}{\int_0^\infty y^{-b}f[y|\theta(t)]dy}.
\]  

(17)

The result shows that the distributional form of drop sizes is modified in a straightforward fashion in the transformation from arrival process to sample-volume process. The density of drop sizes within the sample volume is proportional to the density of drop sizes for the arrival process multiplied by drop diameter raised to \(-b\), where \( b \) is the exponent in the velocity–diameter relationship of (4). This result implies that density functions that contain power-law terms are “invariant” in the sense that the transformation from arrival process to sample-volume process does not change the distributional form. It will be seen below that the lognormal and gamma distributions are examples of such invariant distributions. An important point to note, however, is that although the distributional form remains the same, the parameters of the distribution will change in the transformation from arrival process to sample-volume process. The proof of (16) is given in the Appendix.

c. Rainfall rate and reflectivity

The following representations are obtained for rainfall rate (mm h\(^{-1}\)) and reflectivity (mm\(^2\) m\(^{-3}\)), using (6) and (8), respectively:

\[
R(t) = (6\pi \times 10^{-4})\lambda(t) \int_0^\infty y^3 f[y|\theta(t)]dy \quad (18)
\]

\[
Z(t) = \Lambda(t) \int_0^\infty y^6 g[y|\theta(t)]dy. \quad (19)
\]

The derivations are given in the Appendix. The key feature of these results is that they allow rainfall rate and reflectivity to be analyzed in terms of simple parameterizations of raindrop distributions. It is shown below ((23), (24), (28), and (29)) that for lognormal- and gamma-distributed drop diameters, particularly simple parametric representations arise. Similar representations can be obtained for a wide range of geophysical processes, including kinetic energy, microwave attenuation, and precipitation scavenging coefficient.

To conclude this section, the preceding results are specialized to the cases in which drop diameters have lognormal and gamma distributions. The lognormal distribution has been used by Feingold and Levin (1986) for modeling drop sizes. The lognormal distribution also has a long history of application to geophysical processes, in particular, those associated with rainfall rate (see, for example, discussions in Waymire and Gupta 1990; Kedem and Chiu 1987; Lovejoy and Schertzer 1985).

Let \( \theta(t) = [\mu(t), \sigma(t)] \), where \( \mu(t) \) is the mean log-diameter of drops and \( \sigma(t) \) is the standard deviation of log-diameter of drops. For the lognormal model, the parameter process has two component processes, one representing time variation in the mean of the natural logarithms of drop diameters, the other representing time variation in the standard deviation in the natural logarithms of drop diameters.

The conditional density function of (11) is given by

\[
f[y|\theta(t)] = \frac{1}{y \left[ 2\sigma(t)^2 \right]^{3/2}} \exp \left[ - \frac{\left[ \log(y) - \mu(t) \right]^2}{2\sigma(t)^2} \right].
\]  

(20)

The following results can be derived from (14) and (17):

\[
\Lambda(t) = \frac{\lambda(t)}{a} \exp \left[ -b\mu(t) + \left( \frac{b^2}{2} \right) \sigma(t)^2 \right] \quad (21)
\]

\[
g[x|\mu(t), \sigma(t)] = \frac{1}{y \left[ 2\sigma(t)^2 \right]^3/2} \times \exp \left[ - \frac{\left[ \log(y) - \left[ \mu(t) - b\sigma(t)^2 \right] \right]^2}{2\sigma(t)^2} \right].
\]  

(22)

Equation (22) shows that the lognormal model of drop diameters is invariant under the transformation from arrival process to sample-volume process. Drop diameters in the sample volume, like drop diameters for the arrival process, have a lognormal distribution. The log-mean of the sample-volume distribution, however, is decreased by \( b\sigma(t)^2 \).

For the lognormal model, rainfall rate and reflectivity can be represented as follows, using (18) and (19):

\[
R(t) = (6\pi \times 10^{-4})\lambda(t) \exp \left[ 3\mu(t) + \frac{9}{2} \sigma(t)^2 \right] \quad (23)
\]

\[
Z(t) = \frac{\lambda(t)}{a} \exp \left[ (6 - b)\mu(t) + \frac{1}{2} \left( 36 - 12b + b^2 \right) \sigma(t)^2 \right].
\]  

(24)
Another important model is obtained by allowing the drop diameters to have a gamma distribution (see Ulbrich 1983; Willis 1984). Let \( \lambda(t) = [r(t), m(t)] \) and

\[
f[y|\theta(t)] = \frac{1}{\Gamma[r(t)]} y^{r(t)-1} m(t)^{-r(t)} \exp\left[-\frac{y}{m(t)}\right].
\]

(25)

From (14) and (17), it follows that

\[
\Lambda(t) = \frac{\lambda(t) \Gamma[r(t) - b]}{a \Gamma[r(t)]} m(t)^{-b}
\]

(26)

and

\[
g[x|\theta(t)] = \frac{1}{\Gamma[r(t) - b]} x^{r(t)-b-1} m(t)^{-r(t)-b} \\
\times \exp\left[-\frac{x}{m(t)}\right].
\]

(27)

The following representations for rainfall rate and reflectivity can be obtained for the gamma model:

\[
R(t) = (6.0\pi \times 10^{-4}) \lambda(t) \frac{\Gamma[r(t) + 3]}{\Gamma[r(t)]} m(t)^3
\]

(28)

and

\[
Z(t) = \frac{\lambda(t) \Gamma[r(t) - b + 6]}{a \Gamma[r(t)]} m(t)^{6-b}.
\]

(29)

Several interesting corollaries follow from (27). If the arrival process has a gamma distribution, the sample-volume process will also have a gamma distribution but with transformed parameters. Thus, the gamma distribution, like the lognormal distribution, is invariant under the transformation from arrival process to sample-volume process. A special case of the gamma distribution is the exponential distribution. The exponential distribution is the cornerstone of the Marshall–Palmer theory (Marshall and Palmer 1948). Equation (27) implies that if the drop-arrival process has an exponential distribution, then the sample-volume process will have a (nonexponential) gamma distribution. If the sample-volume process has an exponential distribution, then the arrival process will have a (nonexponential) gamma distribution.

4. Empirical results for the lognormal model

Observations of drop-size distributions at the ground surface were made in North Carolina from 1960 to 1962 using the Illinois State Water Survey raindrop camera (see Mueller and Sims 1967; Catanoe and Stout 1968). These observations will be utilized in this section to assess performance of the model proposed in the previous section. In this section, the lognormal distribution will be used for drop-diameter distributions. In the following section, results are presented using the gamma distribution for drop diameters.

The raindrop camera provides data on drop counts and sizes within a sample volume of approximately 1 m³ at a 1-min time interval. The observation process can be viewed as follows. At the beginning of each minute, the camera becomes active and takes a sequence of seven pictures, each representing one-seventh of a cubic meter. The camera is then inactive for the remainder of the minute. The observations are reported as drop counts in 65 equally spaced 0.1-mm size classes ranging from 0.5 to 7 mm.

Using the velocity–drop-diameter relationship of (3), sample-volume observations can be converted to drop-arrival observations. For example, if 100 drops are observed for the 1.0-mm drop class, multiplying by the velocity of 3.78 m s⁻¹ yields a drop rate of 378 drops per second. The sample-volume representation incorporates observations over the several seconds required to obtain the seven photographs. The drop-arrival representation is effectively an average arrival rate over the several seconds of observations. Drop-arrival data will be represented in drops per second with the interpretation that the characteristic time of the data is on the order of 1 s. As noted above, the time interval between these observations is 1 min.

The raindrop observations will be represented as follows: the number of drops in a 1-s time interval for minute \( i \) of observation for a storm will be denoted \( \eta_i \). Diameters will be denoted \( D_j \). Sample statistics that are used to describe the raindrop data include the sample mean, standard deviation, and coefficient of variation; these three statistics can be represented as follows:

\[
\hat{u}_i = \frac{1}{\eta_i} \sum_{j=1}^{\eta_i} D_j
\]

(30)

\[
\hat{s}_i = \sqrt{\frac{1}{\eta_i} \sum_{j=1}^{\eta_i} (D_j - \hat{u}_i)^2}
\]

(31)

\[
\hat{\beta}_i = \frac{\hat{s}_i}{\hat{u}_i}
\]

(32)

Figure 1 illustrates time sequences of sample statistics derived from the data for a 75-min storm that occurred on 22 May 1961. The figure shows the drop-arrival rate (in drops per second), the sample mean of the drop diameters [from (30)], and the coefficient of variation of the drop diameters [from (32)]. Also shown in Fig. 1 is rainfall rate, computed from (6) and (7) (using a time interval of 1 s). The time axis represents time of day; thus, the observations extend from approximately 1025 LST until 1140 LST 22 May.

Each of the four processes shown in Fig. 1 exhibits pronounced temporal variability. The drop-arrival rate ranges from less than 50 m⁻² s⁻¹ to more than 5000 m⁻² s⁻¹ over a period of minutes. Over the same time period, the mean diameter ranges from less than 1.0 mm to more than 2.0 mm, the coefficient of variation of drop diameter ranges from less than 0.2 to
more than 0.4, and the rainfall rate ranges from near 0 to more than 25 mm h\(^{-1}\). A notable feature of Fig. 1 is that the temporal pattern of rainfall rate most closely reflects the temporal pattern of drop-arrival rate. Estimates of the randomly varying parameters of the lognormal distribution can be expressed in terms of the sample estimates of (30) and (32) as follows:

\[
\hat{\mu}_i = \ln(\hat{\mu}_i) - 0.5\hat{\sigma}_i^2
\]

\[
\hat{\sigma}_i = \ln(1 + \hat{\theta}_i).
\]
The model drop-arrival rate can be estimated by the sample drop-arrival rate, that is:

$$\hat{\lambda}_i = \eta_i.$$  \hspace{1cm} (35)

The estimates of drop-arrival rate, mean log-diameter, and standard deviation of log-diameter can be viewed as reconstructions of the random processes \([\lambda(t), \mu(t), \sigma(t)]\). Figure 2 shows the time sequence of estimators for the 22 May storm.

For the 22 May 1961 storm, rainfall rate and reflec-

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**Fig. 2.** Time-varying parameter estimates of the lognormal model for the 22 May storm.
tivity are computed from drop-size data using (7) and (8) and from the model representation using (23) and (24). In applying the model formulation, sample estimates of drop-arrival rate, mean log-diameter, and standard deviation of log-diameter (as illustrated in Fig. 2) are used. Figure 3 compares the resulting rainfall-rate and reflectivity values. The rainfall-rate estimates from the model formulation are virtually indistinguishable from the observed rainfall rates. The reflectivity values show only slightly larger differences. The results of Fig. 3 are typical. Figure 4 shows similar results for a storm that occurred on 21 June 1961. For the entire sample of 3100 h of data, model-derived estimates of rainfall rate are within 5% of observed value for more than 98% of the samples. For reflectivity, model-derived estimates are within 5% of observed value for more than 95% of the samples.

These results show that if drop-arrival rate, mean diameter, and standard deviation of drop diameter are known, then rainfall rate and reflectivity can be estimated quite accurately. Another way of interpreting these results is that for estimating rainfall rate and reflectivity, the information lost in condensing the 65 values of the original drop-size data (i.e., counts in 65 size classes) to the 3 values of drop-arrival rate, mean diameter, and standard deviation of diameter is negligible.

5. Results for the gamma model

In this section, empirical results are presented for the marked point process model with gamma-distributed drop diameters. Analytical results are developed to facilitate comparisons with the Marshall–Palmer model (Marshall and Palmer 1948) and a generalization of the Marshall–Palmer model described in Waldvogel (1974). The Marshall–Palmer model is typically developed in terms of the drop spectrum for the sample-volume process. This section begins with the background material needed to relate the different model formulations.

The drop spectrum at time $t$, $N(t, x)$, is the expected number of drops per unit volume per unit drop diameter, that is,

$$N(t, x) = \frac{d}{dx} E[\sum_{i=1}^{M(t)} 1(\tilde{D}_i \leq x)].$$

(36)

Recall that $M(t)$ is the number of drops in the sample volume at time $t$. The spectrum $N(t, x)$ has units of drops per cubic meter per millimeter of drop diameter, and $N(t, x)dx$ can be interpreted as the expected number of drops with diameter in the range $(x, x + dx)$ in a cubic meter sample volume.

It is straightforward to show from (36) that for the gamma model, the drop spectrum is given by...
model that results for \( r(t) = 1 \). The drop spectrum for the Waldvogel model is

\[
N(t, x) = N_0(t) \exp \left[ -\frac{x}{m(t)} \right].
\]  

(40)

Waldvogel (1974) estimates model parameters \( N_0(t) \) and \( m(t) \) from drop-size observations but in reverse fashion to the approach of section 4. First, \( R(t) \) and \( Z(t) \) are calculated from drop-size observations, then model parameters are inferred from (28) and (29) with \( r(t) = 1 \) (Waldvogel actually uses liquid water content instead of rainfall rate). We want to examine, as in section 4, how well rainfall rate and reflectivity can be represented using estimates of the physical attributes of drop-size distributions, such as drop rate and mean diameter.

The estimates of rainfall rate and reflectivity obtained from the gamma model [(28) and (29)] are very similar to the lognormal model results of section 4. Figure 5 shows model-derived estimates of rainfall rate and reflectivity for the 22 May 1961 storm (used in Figs. 1–3). For the sample of 3100 storm hours, the gamma-model estimates of rainfall rate are within 5% of the observed values for 98% of samples; model estimates of reflectivity are within 5% of observed values for 95% of samples.

Figure 6 shows estimates of rainfall rate and reflectivity using the model with \( r(t) \) constrained to equal

\[
N(t, x) = N_0 \exp \left[ -\frac{x}{m(t)} \right].
\]  

(37)

where \( r(t) \) and \( m(t) \) are parameters of the gamma distribution for drop diameters in (25). The ratio of mean drop density to mean diameter is typically denoted \( N_0(t) \), that is,

\[
N_0(t) = \frac{\Lambda(t)}{m(t)}.
\]  

(38)

The Marshall–Palmer model is obtained by taking \( r(t) = 1 \) and \( N_0(t) = N_0 \). Under these assumptions, the drop spectrum of (37) simplifies to the Marshall–Palmer spectrum:

\[
N(t, x) = N_0 \exp \left[ -\frac{x}{m(t)} \right].
\]  

(39)

Marshall and Palmer (1948) take the constant \( N_0 \) to be 8000 drops per cubic meter per millimeter of drop diameter.

Waldvogel (1974) introduced a generalization of the Marshall–Palmer model by allowing for time variability in \( N_0 \). Waldvogel's model is a special case of the gamma

![Fig. 5. Comparison of observed rainfall rate and reflectivity (solid lines) with model-derived estimates (dashed lines) for the 22 May storm (gamma model).](image)

![Fig. 6. Comparison of observed rainfall rate and reflectivity (solid lines) with model-derived estimates (dashed lines) for the 22 May storm [gamma model with \( r(t) = 1 \).](image)
1. This corresponds to the situation for which drop diameters have an exponential distribution and $N_0(t)$ varies with time. The estimates of both reflectivity and rainfall rate are severely biased.

One distinction between the results of Figs. 5 and 6 is that three time-varying parameters are used in the results of Fig. 5 and only two are used for the results of Figure 6. The dramatic contrasts in performance, however, are not primarily due to the change in information content but to model error. Figure 7 shows the

![Graphs](image)

**Fig. 7.** Time-varying parameter estimates of the gamma model for the 22 May storm.
temporal variation of the gamma-model parameters with time. The estimates of $r(t)$ range from a minimum of 13 to a maximum of more than 80. Recall that for an exponential distribution $r(t)$ equals 1. Severe bias in the results of Fig. 6 is due primarily to the fact that the drop-diameter data are poorly represented by an exponential distribution.

One possible explanation for the poor performance of the exponential distribution is that small drops are not reported. Catancea and Stout (1968) note, however, that drops smaller than 0.5 mm were occasionally observed at the North Carolina site, but that they were generally not significant in comparison with the total drop count. This assessment is consistent with the histogram of drop-diameter distributions for the entire dataset (Fig. 8).

6. Summary and conclusions

A model for the arrival rate and diameters of raindrops has been developed and assessed with drop-size data from North Carolina. The model represents the arrival times and diameters of raindrops at the upper surface of a sample volume. The arrival times are represented by a Poisson process with randomly varying rate of occurrence $\lambda(t)$. Drop diameters are represented as independent random variables with time-varying probability density function $f[x | \theta(t)]$, where $\theta(t)$ are the time-varying parameters that characterize the distribution of drop diameters. The processes $[\lambda(t), \theta(t)]$ constitute the fundamental parameterization of the model.

Given the specification of the drop arrival and diameter process, results are derived for the sample volume process as a function of time. Specifically, it is shown that 1) the drop counts have a Poisson distribution with simple dependence on the drop arrival and diameter process, and 2) the probability density function of drop sizes within the sample volume is a power-law multiple of the density function of drop diameters at the upper surface of the sample volume. Simple analytical representations are developed for rainfall rate and reflectivity in terms of the model parameterization.

Two distributional families are particularly attractive for representing drop diameters by virtue of the power-law transformation from arrival process to sample-volume process. It is shown that the lognormal and gamma families are invariant under the transformation from arrival process to sample volume, that is, both the sample volume and arrival process have lognormal (gamma) distributions if the other has a lognormal (gamma) distribution. The parameters are transformed in a straightforward fashion from arrival process to sample volume.

Using the lognormal model and drop-size data from North Carolina, it is shown that the model representations of rainfall rate and reflectivity are extremely accurate. For rainfall rate, 98% of model estimates are within 5% of the true value. For reflectivity, 95% of model estimates are within 5% of the true value.

The Marshall–Palmer model can be incorporated into the modeling framework of section 3 using the gamma model for drop diameters. It is shown that the exponential assumption underlying the Marshall–Palmer models leads to significant errors in model estimates of rainfall rate and reflectivity.

A number of model enhancements can be recommended. The model does not incorporate drop interactions. This feature is justified in the present study by restricting consideration to a small sample volume at the ground surface. For detailed modeling of large sample volumes in the atmosphere, it would be useful to extend the modeling framework to include drop interactions (as in Srivastava 1971). It would also be useful to examine the spatial distribution of raindrops within the sample volume (as in Lovejoy et al. 1991).

The model serves principally as a tool for analyzing geophysical processes. An example includes the assessment of time-varying error of rainfall rate estimates based on rainfall rate–reflectivity relationships. The model is also being used to study rainfall–runoff processes, especially infiltration and soil erosion. Of particular interest is comparison of the role of rainfall rate in runoff processes between humid and semiarid climates.

Acknowledgments. This research was supported in part by NOAA (Climate and Global Change Research Program, Grant NA16RC044-01) and by NASA (“The Utilization of EOS Data in Quantifying the Processes Controlling the Hydrologic Cycle in Arid/Semi-Arid Regions”). This support is gratefully acknowledged. The drop-size data were provided by the Illinois State Water Survey.

APPENDIX

Proofs of Computational Results

Proofs of the principal computational results presented in the paper are sketched in this appendix. Results are derived for the case in which $\eta(t)$ is a Poisson process with a constant rate of occurrence $\lambda$. Applicability of the results to the situation in which the rate
of occurrence is a time-varying random process is based on the assumption that temporal variability in \( \lambda(t) \) is small over the time it takes a raindrop to pass through the sample volume (approximately 0.25 s).

a. Proof of (12) and (13)

The counting process \( M(t) \) has the same distribution as

\[
\tilde{M}(t) = \sum_{i=1}^{\eta(t)} 1[U_i + v(D_i)^{-1} > t], \tag{A1}
\]

where \( U_1, \ldots, U_{\eta(t)} \) are iid. with a uniform distribution on the interval \([0, t]\). This is based on the "uniform property" of Poisson processes, that is given the number of arrivals in a stationary Poisson process, the locations are iid. with a uniform distribution (see Cinlar 1975). It follows that \( M(t) \) has a Poisson distribution with mean

\[
E[M(t)] = (\lambda t)P[U_i + v(D_i) > t]. \tag{A2}
\]

The second term in (A2) can be evaluated using the observation that for \( t > v(D_i)^{-1} \),

\[
P[U_i > t - v(x)^{-1}] = 1 - P[U_i \leq t - v(x)^{-1}] = 1 - \frac{t - v(x)^{-1}}{t} = \frac{v(x)^{-1}}{t}. \tag{A3}
\]

It follows that

\[
P[U_i + v(D_i)^{-1} > t] = E\{P[U_i + v(D_i)^{-1} > t | D_i]\} = E\left\{ \frac{t - [t - v(D_i)^{-1}]}{t} \right\} = t^{-1}E[v(D_i)^{-1}] = t^{-1} \int_0^\infty v(x)^{-1} f(x) dx. \tag{A4}
\]

The first result follows.

b. Proof of (16)

The key observation in deriving (14) is that the distance traveled by the \( i \)th drop at time \( t \) is given by \( t - T_i \) \( v(D_i) \). It follows that the distribution of drops in the sample volume is given by the following conditional probability:

\[
P(\tilde{D}_i \in B | (t - T_i) v(D_i) \leq 1, \eta(t)) = \frac{P(\tilde{D}_i \in B, (t - T_i) v(D_i) \leq 1 | \eta(t))}{P((t - T_i) v(D_i) \leq 1 | \eta(t))}. \tag{A5}
\]

The right-hand side can be computed as follows:

\[
P(\tilde{D}_i \in B | (t - T_i) v(D_i) \leq 1, \eta(t)) = \frac{P((t - T_i) v(D_i) \leq 1 | D_i \in B, \eta(t)) P(D_i \in B)}{P((t - T_i) v(D_i) \leq 1 | \eta(t))}. \tag{A6}
\]

The result follows by noting that

\[
P((t - T_i) v(D_i) \leq 1 | D_i = d) = \frac{v(d)^{-1}}{t} \tag{A7}
\]

and

\[
P((t - T_i) v(D_i) \leq 1 | D_i = d) = P(T > t - v(d)^{-1}) \tag{A8}
\]

and that given the number of events up until \( t \), \( T_i \) is uniformly distributed in \([0, t]\). Now the computational results of (A2)--(A4) can be used to evaluate the denominator of (A6) and complete the proof.

c. Proof of (18) and (19)

Equations (18) and (19) are obtained from large-sample arguments. The expression for rainfall accumulation is

\[
A(t) = K \sum_{i=1}^{\eta(t)} D_i^3. \tag{A9}
\]

It follows from the law of large numbers for random sums (see Serfling 1980) that

\[
A(t) \approx E[A(t) | \eta(t)]. \tag{A10}
\]

The drop-arrival rate typically ranges between 100 and 10 000 drops per second. Consequently, the large-sample approximation is quite accurate.

The right-hand side can be computed as follows:

\[
E[A(t) | \eta(t)] = K \eta(t) E[D_i^3]. \tag{A11}
\]

It follows that

\[
R(t) = K \frac{\eta(t + \Delta t) - \eta(t)}{\Delta t} E[D_i^3]. \tag{A12}
\]

If \( \Delta t \) and \( \lambda(t) \) are "large," then

\[
\lambda(t) \approx \frac{\eta(t + \Delta t) - \eta(t)}{\Delta t}. \tag{A13}
\]

The result for rainfall rate follows immediately. Similar computations yield the result for reflectivity.

REFERENCES


