Optimization of Multiparameter Radar Estimates of Rainfall

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ABSTRACT

The estimates of rainfall rate derived from a multiparameter radar based on reflectivity factor ($R_{ZH}$), differential reflectivity ($Z_{DR}$), and specific differential propagation phase ($K_{DP}$) have widely varying accuracies over the dynamic range of the natural occurrence of rainfall. This paper presents a framework to optimally combine the three estimates, $R_{ZH}$, $Z_{DR}$, and $K_{DP}$, to derive the best estimate of rainfall using coherent multiparameter radars. The optimization procedure is demonstrated for application to multiparameter radar measurements at C band.

1. Introduction

Application of polarimetric techniques to remote measurement of rainfall is an area of continuing research. The problem of rainfall estimation using multiparameter technique has been studied in detail by many researchers (Bringi et al. 1982; Seliga et al. 1986; Aydin et al. 1987; Sachidananda and Zrnić 1987; Chandrasekar et al. 1990). The various researchers have studied techniques based on reflectivity factor $Z$ and differential reflectivity $Z_{DR}$ as well as those based on specific differential propagation phase $K_{DP}$. The inaccuracies in the estimates of rainfall rate using different multiparameter techniques have two aspects, namely, (a) the ability of the estimates to follow rainfall due to physical considerations and, (b) the effect of measurement fluctuations. Reflectivity factor and differential reflectivity estimates are based on backscattered power measurements whereas $K_{DP}$ is obtained from the measurements of differential propagation phase. Chandrasekar and Bringi (1988) studied the error structure of rainfall-rate estimate $R_{DR}$ based on $Z$ and $Z_{DR}$ and concluded that for light rain, $R_{DR}$ does not significantly outperform the conventional $Z$-$R$ relationship ($R_{ZH}$). Sachidananda and Zrnić (1987) and Chandrasekar et al. (1990) have shown that $K_{DP}$ is a noisy measurement at low and moderate rain rates and therefore the estimate of rain rates $R_{DP}$ based on $K_{DP}$ is also noisy at low and moderate rain rates. However at high rain rates, $R_{DP}$ is more accurate than $R_{DR}$ as well as $R_{ZH}$. Thus the different estimators of rainfall rate perform differently over the full dynamic range of the natural occurrence of rainfall. However all the estimators with different accuracies attempt to measure the same rainfall.

In this paper we study techniques to optimally combine the three different estimates of rainfall, namely, $R_{ZH}$, $R_{DR}$, and $R_{DP}$ to derive the best estimate of rainfall from multiparameter radars. The accuracy of all the estimators changes with the mean rainfall rate under consideration and this precludes us from obtaining a single optimal algorithm for all ranges of rainfall. On the contrary it is impractical to use an algorithm that changes continuously with mean rainfall rate. As a compromise we consider three distinct regions, namely, (a) light rain, (b) moderate rain, and (c) heavy rain, and study the optimal way to combine the individual multiparameter radar estimates in each range. Most of the applications of this research work are focused toward rainfall measurement in Europe and as a result we have chosen C-band as our primary frequency of analysis (because all of the European multiparameter radars are C-band systems). This analysis can be easily extended to S-band systems. We note here that we study the C-band measurements in the context of optimization only. The C-band observables could be potentially perturbed by propagation effects due to the medium between the radar and the measurement cell. The analysis of the propagation effects are beyond the scope of this paper and have been discussed extensively by Scarchilli et al. (1991).

Our paper is organized such that section 2 presents the optimization procedure, section 3 models the variabilities in the rainfall estimates, section 4 presents the results of the optimal estimates based on simulations, and section 5 summarizes the key results of this paper.

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2. Optimization procedure

The three estimators we have considered in the optimization procedure are $R_{ZH}, R_{DR}$, and $R_{DP}; R_{ZH}$ is a conventional $Z-R$ relationship of the form

$$R_{ZH} = C_{ZH} Z^g_H.$$

(1)

The expression for $R_{DR}$ has been obtained by Scarchilli et al. (1991) in a power-law relationship of the form

$$R_{DR} = C_{DR} Z^g_H Z^g_{DR}.$$

(2a)

Rainfall-rate estimate based on $K_{DP}$ is also a power-law relationship and the exponent in the power-law expression for $R_{DP}$ happens to be very close to unity at C band; the expression is given by

$$R_{DP} = C_{DP} K_{DP}.$$

(2b)

The constants $\gamma, \alpha, \beta$, $C_{ZH}, C_{DR}$, and $C_{DP}$ at C-band frequencies are discussed in section 4.

The error structures of the three estimators $R_{ZH}, R_{DR}$, and $R_{DP}$ are completely different and the standard errors in $R_{ZH}, R_{DR}$, and $R_{DP}$ also vary with rainfall rates. Analysis of the error structure of $R_{DR}$ and $R_{DP}$ at S band has shown that for light rain it is preferable to use $R_{ZH}$ (Chandrasekar et al. 1990). This inference is also valid at C band as shown in the following section.

Thus in our optimization procedure we use only $R_{ZH}$ in light rain, but at higher rain rates (moderate and high) we use a linear combination of $R_{DR}$ and $R_{DP}$.

The two multiparameter rainfall estimates $R_{DR}$ and $R_{DP}$ have no mean bias if the radar system is well calibrated. Let $R$ be the estimate of rainfall based on a linear combination of $R_{DR}$ and $R_{DP}$ written as,

$$R = \alpha_1 R_{DR} + \alpha_2 R_{DP}.$$

(3)

Based on the condition that the combined estimate $R$ is unbiased we can write

$$\alpha_1 + \alpha_2 = 1.$$

(4)

We need to minimize the variance of $R$ to optimize the rainfall estimate. The variance of $R$ in (3) can be expressed as

$$\text{var}(R) = \alpha_1^2 \text{var}(R_{DR}) + \alpha_2^2 \text{var}(R_{DP}) + 2\alpha_1\alpha_2 \text{cov}(R_{DR}, R_{DP}).$$

(5)

Minimizing the variance of $R$ under the condition given by (4) gives us the optimum unbiased estimate of rainfall rate. The variances and covariances in (5) are composed of two parts, namely, (a) variabilities between $R_{DR}, R_{DP}$, and $R$ due to statistical measurement fluctuations, and (b) variabilities in the physical conditions of rainfall. The correlation structures of these two variabilities are widely different. The statistical measurement fluctuations are controlled by the radar observation parameters such as the number of samples used in integration, Doppler spectrum width, and the cross correlation between the horizontally and vertically polarized radar returns (Sachidananda and Zrnić 1987). The correlation structure of variabilities due to physical conditions involves the response of $Z_H, Z_{DR}$, and $K_{DP}$ due to the physical variabilities of the rain medium, which are discussed in more detail in section 4.

In principle we can include a combination of the three estimators $R_{ZH}, R_{DR}$, and $R_{DP}$ in all three rainfall regimes in an expression similar to (3) so that $\hat{R} = \alpha_1 \hat{R}_{DR} + \alpha_2 \hat{R}_{DP} + \alpha_3 \hat{R}_H$. However, a rigorous optimization analysis procedure similar to the one described in section 4 yields the result that using $R_{DR}$ and $R_{DP}$ at light rain is redundant. If we use $R_{DR}$ and $R_{DP}$ in conjunction with $R_H$ then at low rainfall rates the scaling constants $\alpha_1$ and $\alpha_2$ of $R_{DR}$ and $R_{DP}$ will be very close to zero. In the same way at moderate and high rates, using $R_{ZH}$ in conjunction with $R_{DR}$ and $R_{DP}$ is also redundant since the estimator $R_{DR}$ includes $Z_H$ in a power-law form. Thus our optimization procedure involves using $R_{ZH}$ in light rain and using a linear combination of $R_{DR}$ and $R_{DP}$ in moderate and heavy rain.

3. Rainfall model

Fluctuations of radar rainfall rate estimates are composed of two parts: (a) the statistical measurement fluctuations and (b) the physical variabilities. Let $\hat{R}_{DR}$ and $\hat{R}_{DP}$ be the radar estimates of rainfall rate based on $Z_{DR}$ and $K_{DP}$, respectively; $\hat{R}_{DR}$ and $\hat{R}_{DP}$ can be expressed in terms of the measurement fluctuations as follows:

$$\hat{R}_{DR} = \overline{R}_{DR} + \epsilon_{DR},$$

(6a)

$$\hat{R}_{DP} = \overline{R}_{DP} + \epsilon_{DP},$$

(6b)

where $\overline{R}_{DR}$ and $\overline{R}_{DP}$ are the mean values averaged over only their respective measurement fluctuations, and $\epsilon_{DR}$ and $\epsilon_{DP}$ are the random measurement fluctuations.

Further, $\overline{R}_{DR}$ and $\overline{R}_{DP}$ can be expressed in terms of their differences with actual rainfall rate $R$ as

$$\overline{R}_{DR} = R + \epsilon_{DR},$$

(7a)

$$\overline{R}_{DP} = R + \epsilon_{DP},$$

(7b)

where $\epsilon_{DR}$ and $\epsilon_{DP}$ represent variabilities between the mean value of the estimates and the actual rainfall rate $R$. Combining (6) and (7), $\hat{R}_{DR}$ and $\hat{R}_{DP}$ can be expressed as

$$\hat{R}_{DR} = R + \epsilon_{DR} + \epsilon_{DR}$$

(8a)

$$\hat{R}_{DP} = R + \epsilon_{DP} + \epsilon_{DP}.$$  

(8b)

We can now proceed to evaluate the variances of $\hat{R}_{DR}$ and $\hat{R}_{DP}$ using the representation in (6). We can write the variance of $\hat{R}_{DR}$ in terms of conditional variance with respect to $R_{DR}$ as (Papoulis 1965)

$$\text{var}(\hat{R}_{DR}) = \langle \text{var}(\hat{R}_{DR} | R_{DR}) \rangle + \text{var}[(\hat{R}_{DR} | R_{DR})].$$

(9)
where the angle brackets $\langle \rangle$ represent mean values. We can simplify (9) using (6)–(8) as
\[ \text{var}(\hat{R}_{DR}) = \text{var}(\epsilon_{DR}^4 | \bar{R}_{DR}) + \text{var}(\epsilon_{DR}^6), \] (10)
where var$(\epsilon_{DR}^4)$ is the variance due to measurement fluctuations and var$(\epsilon_{DR}^6)$ is the variance due to the construction of the estimator. We defer the detailed discussion of var$(\epsilon_{DR}^6)$ to the next section.

In the above equation var$(\epsilon_{DR}^4 | \bar{R}_{DR})$ is not practical to use. However, var$(\epsilon_{DR}^4)$ is a smooth function of $R$ in addition to the property that on average $\hat{R}_{DR}$ follows $R$. Therefore we can approximate var$(\epsilon_{DR}^4 | \bar{R}_{DR})$ by var$(\epsilon_{DR}^4 | \bar{R})$.

Thus we can simplify (10) as
\[ \text{var}(\hat{R}_{DR}) = \text{var}(\epsilon_{DR}^4 | \bar{R}) + \text{var}(\epsilon_{DR}^6). \] (11)
Similarly we can write var$(\hat{R}_{DP})$ as
\[ \text{var}(\hat{R}_{DP}) = \text{var}(\epsilon_{DP}^4 | \bar{R}) + \text{var}(\epsilon_{DP}^6). \] (12)

In (11) and (12) the terms var$(\epsilon_{DR}^4 | \bar{R})$ and var$(\epsilon_{DP}^4 | \bar{R})$ are due to measurement fluctuations only, and can be expressed in terms of the standard errors in the measurements of $Z_H$, $Z_{DR}$, and $K_{DP}$. If $R_{DR}$ is expressed as a power law of the form shown in (2a) then var$(\epsilon_{DR}^6)$ can be expressed as (Chandrasekar and Bringi 1988)
\[ \text{var}(\epsilon_{DR}^6) = R^2 \left\{ \alpha \left[ \frac{\text{var}(Z_H)}{Z_H^2} \right] + \beta \left[ \frac{\text{var}(Z_{DR})}{Z_{DR}^2} \right] \right\}. \] (13)

Similarly if $R_{DP}$ is expressed as the power law of the form (2b) then var$(\epsilon_{DP}^6)$ can be expressed as
\[ \text{var}(\epsilon_{DP}^6) = R^2 \gamma^2 \frac{\text{var}(K_{DP})}{K_{DP}^2}. \] (14)

We note here that the parameters $\epsilon_{DR}^4$ and $\epsilon_{DP}^4$ can be used to model all the variabilities that we can possibly have in the estimate of rainfall due to physical reasons.

The correlation structure of $\hat{R}_{DR}$ and $\hat{R}_{DP}$ is also important to complete the rainfall model. The correlation structure of $\hat{R}_{DR}$ and $\hat{R}_{DP}$ can be studied in terms of the covariance structure of $\epsilon_{DR}^4$, $\epsilon_{DP}^4$, $\epsilon_{DR}^6$, and $\epsilon_{DP}^6$. The estimator $R_{DR}$ is based on power measurements whereas the estimator $R_{DP}$ uses only phase measurements. Therefore we can conclude $\epsilon_{DR}^4$ and $\epsilon_{DP}^4$ are uncorrelated. However both $R_{DR}$ and $R_{DP}$ use the same physical principle that raindrops are oblate as they become larger in size. Thus intuitively we can expect that $\epsilon_{DR}^6$ and $\epsilon_{DP}^6$ are correlated. We study the detailed structure of the variances in $\hat{R}_{DR}$ and $\hat{R}_{DP}$ in the next section using simulations.

4. Simulation study

We use simulations in this section to analyze the correlation structure of the various rainfall estimates and optimization procedure. Simulations have been used by Chandrasekar et al. (1990) for comparing the estimators $R_{DR}$ and $R_{DP}$. We follow similar procedures for the study of optimization. Our models simulate random measurement errors in $Z_H$, $Z_{DR}$, and $K_{DP}$. Physical variability in rainfall is introduced by varying the parameters of a general gamma raindrop size distribution (Ulbrich 1983). The principal parameters of our simulation are as follows: (a) wavelength $\lambda = 5.5$ cm, (b) sampling time $T_s = 1$ ms, (c) number of sample pairs $N = 64$, (d) Doppler velocity spectrum width, $\sigma_v = 2$ m s$^{-1}$, and (e) zero-lag cross correlation between $H$ and $V$ signals, $\rho_{HV}(0) = 0.99$, and the pathlength over which $K_{DP}$ is estimated 1 km. Scarchilli et al. (1991) have obtained the expressions for $R_{DR}$ and $R_{DP}$ at C-band frequencies and they are
\[ R_{DR} = 3.61 \times 10^{-3} Z_H^{0.95} Z_{DR}^{-0.28}, \] (15)
\[ R_{DP} = 19.8 K_{DP}. \] (16)

We utilize (15) and (16) to estimate the rainfall rates $\hat{R}_{DR}$, $\hat{R}_{DP}$, and $\hat{R}_{ZH}$ from simulated measurements of $Z_H$, $Z_{DR}$, and $K_{DP}$. These estimates of rainfall are used in the subsequent optimization analysis.

Figure 1a shows the fractional standard deviation (FSD) in the estimation of $R_{DR}$ versus the actual rain rate $R$, where the fractional standard deviation is defined as the standard deviation normalized with respect to its mean. We can see from Fig. 1a that FSD$(R_{DR})$ is nearly constant with rain rate (shows a slight increase at high rain rates). Figure 1b shows the FSD of $R_{DP}$ as a function of $R$. We can observe from the results of Fig. 1b that FSD$(R_{DP})$ decreases monotonically with increase in rainfall rate, assuming very high values at low rain rates. The two curves FSD$(R_{DR})$ and FSD$(R_{DP})$ cross over around the rain rates of approximately 90–100 mm h$^{-1}$. Figure 1c shows the FSD of $R_{ZH}$ obtained from the Marshall–Palmer relationship as
\[ R_{ZH} = 0.0365 Z_H^{0.625}. \] (17)

We can see from the results of Fig. 1c that we have substantially higher FSD beyond 10 mm h$^{-1}$, compared to $R_{DR}$. To investigate the light rainfall region in detail, we show the FSD of $R_{DR}$ and $R_{ZH}$ between 0–20 mm h$^{-1}$ in Figs. 2a and 2b using an expanded scale. We can infer from the results of Figs. 2a and 2b that at very small rain rates $R_{ZH}$ performs better than $R_{DR}$. Similar results were reported for S-band frequencies by Chandrasekar and Bringi (1988). However, extensive space–time averaging could improve the estimate $R_{DR}$, at low rainfall rates, as shown by Aydin et al. (1987). Based on the above discussions we can conclude that at C-band frequencies $R_{ZH}$ is the best estimator below 10 mm h$^{-1}$ (light rain), $R_{DR}$ is the best estimator for rain rates between 10–90 mm h$^{-1}$ and $R_{DP}$ is the best estimator for rain rates higher than 90–100 mm h$^{-1}$ (the improvement over $R_{DR}$ becomes significant at rain rates larger than 150 mm h$^{-1}$). Chandrasekar et al. (1990) obtained similar demarcation at
S-band frequencies, and they found that the crossover point where $R_{DP}$ performs better than $R_{DR}$ is around 70 mm h$^{-1}$. Further, $R_{DR}$ and $R_{DP}$ have very high FSD at low rain rates. Therefore it is preferable to use $R_{ZH}$ in light rain. However at moderate or high rain rates the FSD of $R_{DR}$ and $R_{DP}$ are comparable, and we can use a combination of $R_{DR}$ and $R_{DP}$ as explained in section 2 to optimize the estimate of rainfall rate. We need to know the correlation structure of $R_{DR}$ and $R_{DP}$ to analyze the optimum way to combine $R_{DR}$ and $R_{DP}$.

The fluctuation in $R_{DR}$ and $R_{DP}$ due to measurement errors, namely, $\epsilon_{DR}$ and $\epsilon_{DP}$ are uncorrelated. We also noted in section 3 that $\epsilon_{DR}$ and $\epsilon_{DP}$ are correlated, because both the measurements of $Z_{DR}$ and $K_{DP}$ are based on the physical principle that raindrops become more oblate with increased size. In this section we utilize simulation of $R_{DR}$ and $R_{DP}$ to study the correlation between $\epsilon_{DR}$ and $\epsilon_{DP}$. The correlation between $R_{DR}$ and $R_{DP}$ with the measurement fluctuation suppressed is the same as the correlation between $\epsilon_{DR}$ and $\epsilon_{DP}$. Our computations show that this correlation is very high, taking values higher than 0.95. This is not a surprising result since both $R_{DR}$ and $R_{DP}$ are constructed on the same physical principle. This high correlation between $\epsilon_{DR}$ and $\epsilon_{DP}$ also indicates that any optimization procedure using $R_{DR}$ and $R_{DP}$ will suppress mainly the fluctuations $\epsilon_{DR}$ and $\epsilon_{DP}$.

Based on the above discussion we construct the op-
timal rainfall estimator as follows: (a) when the rain rate is less than 10 mm h$^{-1}$ we use $R_{ZH}$; (b) when the rain rate is higher than 10 mm h$^{-1}$ we use a combination of $R_{DR}$ and $R_{DP}$ as given in (4). The optimization coefficients $\alpha_1$ and $\alpha_2$ used in (4) for minimum variance can be obtained as (Papoulis 1965)

$$\alpha_1 = \frac{\sigma^2(R_{DP})}{\sigma^2(R_{DR}) + \sigma^2(R_{DP})},$$

$$\alpha_2 = \frac{\sigma^2(R_{DR})}{\sigma^2(R_{DR}) + \sigma^2(R_{DP})},$$

where $\sigma^2(R_{DP})$ and $\sigma^2(R_{DR})$ are the variances of $R_{DR}$ and $R_{DP}$, respectively. The above equation is not very practical since we have demonstrated through Figs. 1a and 1b that $\sigma^2(R_{DR})$ and $\sigma^2(R_{DP})$ vary with $R$. Therefore we divide the region where $R > 10$ mm h$^{-1}$ into two parts: (a) moderate to intense and (b) heavy rain, and then use two different combinations of $R_{DR}$ and $R_{DP}$ as follows:

$$\hat{R} = \alpha_1 R_{DR} + \alpha_2 R_{DP}, \quad 10 < R < R_T; \quad (19a)$$

$$\hat{R} = \alpha'_1 R_{DR} + \alpha'_2 R_{DP}, \quad R \geq R_T; \quad (19b)$$

where $R_T$ is the threshold separating moderate and heavy rainfall rates. The main purpose of this demarcation is that we know a priori that $R_{DP}$ has lower standard error compared to $R_{DR}$ in heavy rain based on the discussions presented in the earlier sections. The next problem to be resolved is the choice of the threshold value $R_T$. We choose the threshold value $R_T$ such that the average standard error is minimized. Based on this criteria the result will depend on the natural

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**Fig. 2.** (a) Histogram of the fractional standard deviation in the estimate of $R_{DP}$ calculated for 1 mm h$^{-1}$ intervals of rainfall rate, shown for $0 < R < 20$ mm h$^{-1}$. (b) Same as (a) except FSD of $R_{ZH}$ is shown.

**Fig. 3.** The distribution of rainfall rates used in the simulation.
0.68, 0.29, 0.32, and 0.71, respectively. Thus the optimal estimate of rain rate under the conditions of our analysis is

\[ \hat{R} = 0.0365Z^{0.625}, \quad R \leq 10 \text{ mm h}^{-1} \]

\[ \hat{R} = \alpha_1\hat{R}_{DR} + \alpha_2\hat{R}_{DP}, \quad 10 < R < 90 \text{ mm h}^{-1} \]

\[ \hat{R} = \alpha_1'\hat{R}_{DR} + \alpha_2'\hat{R}_{DP}, \quad R \geq 90 \text{ mm h}^{-1}. \]

We have obtained this optimization under specific conditions. We recognize that many of our assumptions such as assumed accuracies for \( Z_H, Z_{DR}, \) and \( K_{DP} \) may change depending upon changes in the radar system or observation parameters. Our experiences suggest that whenever the fluctuation in one measurement increases, this is typically accompanied by an increase in the fluctuation of other parameters proportionally. We therefore expect the optimal equations to remain nearly the same. Nevertheless the optimization procedure can be easily modified by substituting any new set of parameters in the analysis.

5. Conclusions

We have compared the error structure of the three rainfall estimates: (a) based on the Marshall–Palmer relationship \( R_{ZH} \), (b) based on differential reflectivity \( R_{DR} \), and (c) based on differential propagation phase \( R_{DP} \) at C-band frequencies, in the absence of propagation effects. The standard error of the three estimators \( R_{ZH}, R_{DR}, \) and \( R_{DP} \) is shown to vary with the rainfall rate. We have obtained an optimum estimator of rainfall rate using a combination of the three estimates \( R_{ZH}, R_{DR}, \) and \( R_{DP}, \) minimizing the resultant standard error. The optimization procedure has shown that \( R_{ZH} \) is the best estimator to use in light rain (less than 10 mm h\(^{-1}\)). However extensive space-time averaging could improve the estimator \( \hat{R}_{DR} \) at low rainfall rates, as shown by Aydin et al. (1987). For moderate and heavy rainfalls the optimum estimate involves only a combination of \( R_{DR} \) and \( R_{DP} \), whose scaling coefficients vary continuously with rainfall rate. This procedure is not practical to use. As a compromise we have divided the rainfall region into two parts, moderate, and heavy, and have obtained distinct equations for each region using a combination of \( R_{DR} \) and \( R_{DP} \). The coefficients of \( R_{DR} \) and \( R_{DP} \) in the optimal estimate are optimized such that the average errors over all rainfall regions are minimized. We recognize that we have obtained this optimization under specific conditions, and many of our assumptions may change depending upon changes in the radar system as well as variations in the observation parameters. Nevertheless this paper provides a framework for using different combinations of the three estimators, \( R_{ZH}, R_{DR}, \) and \( R_{DP} \), in the optimum way so that the resultant error in the estimates of rainfall can be minimized.

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