Parameterizing the Raindrop Size Distribution

Ziad S. Haddad, Stephen L. Durden, and Eastwood Im

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California

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Abstract

This paper addresses the problem of finding a parametric form for the raindrop size distribution (DSD) that 1) is an appropriate model for tropical rainfall, and 2) involves statistically independent parameters. Such a parameterization is derived in this paper. One of the resulting three "canonical" parameters turns out to vary relatively little, thus making the parameterization particularly useful for remote sensing applications. In fact, a new set of $\Gamma$-drop-size-distribution-based $Z-R$ and $k-R$ relations is obtained. Only slightly more complex than power laws, they are very good approximations to the exact radar relations one would obtain using Mie scattering. The coefficients of the new relations are directly related to the shape parameters of the particular DSD that one starts with. Perhaps most important, since the coefficients are independent of the rain rate itself, the relations are ideally suited for rain retrieval algorithms.

1. Introduction

Since Marshall and Palmer's pioneering 1948 paper, much attention has been focused on obtaining relatively simple analytic expressions involving as small a number of parameters as possible to model measured drop size distributions (DSD's). The three-parameter $\Gamma$ distribution model

$$N(D) = N_0 D^\mu e^{-\Lambda D}$$

proposed by Ulbrich (1983) has been tested using different datasets (see, e.g., Kozu and Nakamura 1991; Goddard and Cherry 1984; Ulbrich 1983), and it has proved to be sufficiently versatile to fit most data satisfactorily, as long as one is willing to allow a relatively wide range of values for the parameter $\Lambda$.

However, Ulbrich (1983) pointed out, and the present work confirms, that the parameters $N_0$, $\mu$, and $\Lambda$ are not mutually independent. In practice, this makes the representation (1) difficult to use in rain retrieval algorithms. To illustrate the problem, suppose one has measurements of a rain-related quantity $Z = Z(a)$ at various altitudes $a$ in the atmosphere. One may then try to determine the distribution $N(D; a)$ at the corresponding altitudes. A priori, all three parameters in (1) may vary with $a$. Yet given one's single observed quantity $Z$, it is unrealistic to expect to successfully determine, at each altitude, the triple $(N_0, \mu, \Lambda)$ that produced the observed value of $Z$. In this case, one way to circumvent this problem is to assume that the typically less-variable parameters are constant, for example, make $N_0$ and $\mu$ constant, and determine $\Lambda(a)$ as a function of the observed $Z(a)$. The specific (constant) values of $N_0$ and $\mu$ need not be known beforehand; one may try to determine them using ancillary observations or archived historical data. The problem with this approach is that it makes little sense to assume $N_0$ constant and let $\Lambda$ change according to the observation when one already knows that $N_0$ and $\Lambda$ are strongly correlated.

It is therefore very useful to derive an expression like (1) but involving statistically independent parameters, ones that are preferably physically meaningful. That is the aim of this paper.

2. Statistical analysis of the Darwin data

The data analyzed were measured by a Joss–Waldvogel disrometer (Joss and Waldvogel 1967; Sheppard and Joe 1994) located at Berrimah near Darwin, Australia. The measurements were taken during the Southern Hemisphere summer seasons of 1988/89 and 1989/90. The disrometer recorded the number of drops in each of 20 drop-diameter bins as in Table 1, reporting a sample distribution every 30 s. To fit a model such as (1) to any such sample distribution, one could proceed in several ways. One way is to express the predicted moments of the DSD as functions of $(N_0, \mu, \Lambda)$, then use three suitable sample moments computed from one’s observations to perform the inversion and deduce the values of the three parameters. This approach is quite unappealing because its estimates would depend on the moments used and, more importantly, because sample moments (and, a fortiori, complicated functions of several of them) are biased estimates of the actual moments, with often large vari-
Table 1. Disdrometer bin values [the 20% discrepancy with the values in Joss and Waldvogel (1967) is due to a miscalibration of the instrument at Darwin].

<table>
<thead>
<tr>
<th>Bin number</th>
<th>Actual drop diameters (mm)</th>
<th>Reported diameter $D_i$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.36--0.48</td>
<td>0.42</td>
</tr>
<tr>
<td>2</td>
<td>0.48--0.6</td>
<td>0.54</td>
</tr>
<tr>
<td>3</td>
<td>0.66--0.72</td>
<td>0.66</td>
</tr>
<tr>
<td>4</td>
<td>0.72--0.84</td>
<td>0.78</td>
</tr>
<tr>
<td>5</td>
<td>0.84--0.96</td>
<td>0.9</td>
</tr>
<tr>
<td>6</td>
<td>0.96--1.2</td>
<td>1.08</td>
</tr>
<tr>
<td>7</td>
<td>1.2--1.44</td>
<td>1.32</td>
</tr>
<tr>
<td>8</td>
<td>1.44--1.68</td>
<td>1.56</td>
</tr>
<tr>
<td>9</td>
<td>1.68--1.92</td>
<td>1.8</td>
</tr>
<tr>
<td>10</td>
<td>1.92--2.16</td>
<td>2.04</td>
</tr>
<tr>
<td>11</td>
<td>2.16--2.52</td>
<td>2.34</td>
</tr>
<tr>
<td>12</td>
<td>2.52--2.88</td>
<td>2.7</td>
</tr>
<tr>
<td>13</td>
<td>2.88--3.24</td>
<td>3.06</td>
</tr>
<tr>
<td>14</td>
<td>3.24--3.6</td>
<td>3.42</td>
</tr>
<tr>
<td>15</td>
<td>3.6--3.96</td>
<td>3.78</td>
</tr>
<tr>
<td>16</td>
<td>3.96--4.44</td>
<td>4.2</td>
</tr>
<tr>
<td>17</td>
<td>4.44--4.92</td>
<td>4.68</td>
</tr>
<tr>
<td>18</td>
<td>4.92--5.4</td>
<td>5.16</td>
</tr>
<tr>
<td>19</td>
<td>5.4--6.0</td>
<td>5.7</td>
</tr>
<tr>
<td>20</td>
<td>6.0--6.6</td>
<td>6.3</td>
</tr>
</tbody>
</table>

A smaller way is to find those values of $(N_0, \mu, \Lambda)$ that minimize the sum of the squared differences between the observed counts $C_j$ and those predicted by (1). This least squares approach implicitly assumes that the difference between the observation and one’s model is entirely due to white noise evenly spread among the sampling bins. While such an assumption is appealingly simple, it does not allow one to use all the information at hand. A maximum-likelihood (ML) approach does. Indeed, one can view the drop-size distribution $N(D)$ as the product of a drop-size density function $p_{\mu,\Lambda}(D) = \Lambda e^{-\Lambda D}/\Gamma(\mu + 1)$, which depends on $\mu$ and $\Lambda$ only, with the total number of drops $N_0(\mu + 1)\Lambda^{\mu+1}$. Since the latter is directly related to the observed total count, the problem of estimating $\mu$ and $\Lambda$ reduces to finding the values of these two parameters that maximize the likelihood

$$L = \prod_{j=1}^{20} \{ p_{\mu,\Lambda}(D_j) \}^{N_j}$$

of obtaining the number frequencies $N_j$ corresponding to the observed counts $C_j$, with $D_j$ as in Table 1. The relation between the observed counts $C_j$ and the corresponding number of drops per unit volume $N_j$ is given by

$$N_j = \frac{C_j}{(50 \text{ cm}^2)[965(1 - e^{-0.53D}) \text{ cm s}^{-1}](30 \text{ s})} \text{ cm}^{-3}$$

since the sensing surface area of the disdrometer is 50 cm$^2$, the averaging period for each measurement was 30 s, and the fall velocity of a drop of diameter $D$ mm is approximately $965(1 - e^{-0.53D})$ cm s$^{-1}$ [see Gossard et al. (1992)—unlike the exponential velocity formula in Atlas et al. (1973), the form used here has the advantage of being always positive for positive $D$ and is at worst within 4% of the measurements of Gunn and Kinzer (1949)].

The logarithmic partial derivative of (2), with respect to $\Lambda$, turns out to be $-(\sum_j N_j D_j)/(\mu + 1)(\sum_j N_j)/\Lambda$, so the ML value of $\Lambda$ is given by

$$\Lambda = \frac{\mu + 1}{(\sum_j N_j D_j)/(\sum_j N_j)}.$$  

The ML value of $\mu$ can now be determined numerically by substituting this expression for $\Lambda$ in (2) and determining that $\mu$ for which (2) is minimized.

In practice, instead of the abstract parameters $N_0$, $\mu$, and $\Lambda$, we used the more physically meaningful variables

$$D^* \text{ (mass-weighted mean drop diameter)} = \frac{\mu + 4}{\Lambda} \text{ mm},$$

$$s^* \text{ (relative mass-weighted rms deviation of drop diameter)} = (\mu + 4)^{-1/2},$$

$$R \text{ (instantaneous rain rate)} = 6\pi \times 10^{-4} \int 9.65(1 - e^{-0.53D})D^3N_0D^se^{-\Lambda D}dD$$

$$= 0.0182 \frac{\Gamma(\mu + 4)}{\Lambda^{\mu+4}} \times \left[ 1 - \left(1 + \frac{0.53}{\Lambda} \right)^{-\mu+4} \right] N_0 \text{ mm h}^{-1}.$$  

Thus, the ML estimates of $\Lambda$ and $\mu$ directly give estimates of $s^*$ and $D^*$. The rain rate can be obtained directly using the equation

$$R = \frac{\pi}{250} \sum_j D_j^3C_j \text{ mm h}^{-1}.$$  

Examples of the ML estimates are shown in the pairwise scatter diagrams of Figs. 1a–c. Since the estimates of $D^*$ and $s^*$ obtained during very light rain are unreliable because of the small sample size, we imposed a lower-bound condition on $R$. The particular value of 0.7 mm h$^{-1}$ was chosen because it corresponds to the projected Tropical Rainfall Measuring Mission (TRMM) radar’s sensitivity (Kawanishi et al. 1993). This lower bound was exceeded by 1794 samples during the 1988/89 season and by 2731 samples during the 1989/90 season. The sample mean of $D^*$ was 1.82 mm over the two seasons (1.69 mm for 1988/89, and
1.9 mm for 1989/90) with a standard deviation of 0.56 mm over the two seasons (0.54 mm during 1988/89, and 0.55 mm during 1989/90). The sample mean of \( s^* \) was 40.25% over the two seasons (41.44% for 1988/89, and 39.48% for 1989/90) with a standard deviation of 6.43% over the two seasons (5.68% during 1988/89, and 6.76% during 1989/90). The values of the various conditional correlation coefficients (conditioned on \( R > 0.7 \) mm h\(^{-1}\)) are given in Table 2. As one might have expected, there is little correlation between the mean drop diameters \( D^* \) and the relative mean variance \( s^* \) of the diameters. However, the data show that both these quantities are significantly correlated with the rain rate. As to Ulbrich's original variables, the most striking correlation is that of \( \log(N_0) \) with \( \log(\Lambda) \): their correlation coefficient is 0.89 for the 1988/89 season, 0.92 for 1989/90, and 0.9 over the 2-yr period, confirming that it is not consistent to assume that \( N_0 \) is constant for a particular set of measurements, if \( \Lambda \) is allowed to vary significantly over the same measurements.

A popular way to make mathematically explicit the interdependences that underlie the observed correlations is to use power-law regressions and express one variable in terms of another, for example, try to find intervals for \( a, b, c, \) and \( d \) such that \( N_0 = aR^b \) or \( \Lambda = cR^d \) (see, e.g., Ulbrich 1983, 1992). The problems with such an approach are that one then artificially introduces new coef-

**TABLE 2. Correlation coefficients for the 1988/89 and 1989/90 seasons, conditioned on \( R > 0.7 \) mm h\(^{-1}\).**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \log(R) ) and ( \log(D^*) )</td>
<td>0.66</td>
<td>0.66</td>
<td>0.66</td>
</tr>
<tr>
<td>( \log(R) ) and ( \log(s^*) )</td>
<td>-0.63</td>
<td>-0.54</td>
<td>-0.66</td>
</tr>
<tr>
<td>( \log(D^<em>) ) and ( \log(s^</em>) )</td>
<td>-0.2</td>
<td>-0.13</td>
<td>-0.22</td>
</tr>
<tr>
<td>( \log(R) ) and ( \log(N_0) )</td>
<td>0.13</td>
<td>0.01</td>
<td>0.23</td>
</tr>
<tr>
<td>( \log(\Lambda) ) and ( \log(N_0) )</td>
<td>0.90</td>
<td>0.89</td>
<td>0.92</td>
</tr>
</tbody>
</table>

![Fig. 1](image-url)  
(a) Simultaneous \( (D^*, s^*) \) occurrences during the 1988/89 season, \( D^* \) in millimeters.  
(b) Simultaneous \( (D^*, s^*) \) occurrences during the 1989/90 season, \( D^* \) in millimeters.  
(c) Simultaneous \( (R, s^*) \) occurrences during the 1989/90 season, \( R \) in millimeters per hour.
coefficients (namely $a$, $b$, $c$, and $d$) that are not related to any of the original variables in a unique way and whose mutual covariances are therefore impossible to determine. Since, in addition, such power laws produce far more unknowns than the one started with, a more efficient and consistent approach such as a simple (judicious) change of variables should prove more useful.

The simplest way to change variables so as to end up with an independent set is to find the (orthogonal) eigenvectors of the covariance matrix. However, the variables produced using such an approach will not be physically meaningful. In addition, it is very desirable to specifically retain the rain rate $R$ as one of the three variables since it is one of the quantities of most interest. So rather than diagonalize the covariance matrix, let us decide to keep $R$ as the first variable and successively modify $D^*$ then $s^*$ in order to end up with

Table 3. Marginal statistics of $R$, $D'$, and $s'$, conditioned on $R > 0.7$ mm h$^{-1}$.

<table>
<thead>
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<tbody>
<tr>
<td>$R$</td>
<td>13.82</td>
<td>11.10</td>
<td>15.57</td>
<td>24.00</td>
<td>21.8</td>
<td>25.18</td>
</tr>
<tr>
<td>$D'$</td>
<td>1.38</td>
<td>1.32</td>
<td>1.43</td>
<td>0.33</td>
<td>0.31</td>
<td>0.33</td>
</tr>
<tr>
<td>$s'$</td>
<td>0.41</td>
<td>0.43</td>
<td>0.41</td>
<td>0.045</td>
<td>0.039</td>
<td>0.046</td>
</tr>
<tr>
<td>log($R$)</td>
<td>1.62</td>
<td>1.44</td>
<td>1.74</td>
<td>1.35</td>
<td>1.26</td>
<td>1.38</td>
</tr>
<tr>
<td>log($D'$)</td>
<td>0.30</td>
<td>0.25</td>
<td>0.33</td>
<td>0.24</td>
<td>0.26</td>
<td>0.22</td>
</tr>
<tr>
<td>log($s'$)</td>
<td>0.90</td>
<td>-0.86</td>
<td>-0.90</td>
<td>0.12</td>
<td>0.10</td>
<td>0.12</td>
</tr>
</tbody>
</table>

Table 4. Correlation coefficients of $R$, $D'$, and $s'$, for the 1988/89 and 1989/90 seasons, conditioned on $R > 0.7$ mm h$^{-1}$.

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>log($R$) and log($D'$)</td>
<td>1.03 $\times$ 10$^{-4}$</td>
<td>0.11</td>
<td>-0.11</td>
</tr>
<tr>
<td>log($R$) and log($s'$)</td>
<td>$-6.9 \times 10^{-3}$</td>
<td>0.075</td>
<td>-0.0052</td>
</tr>
<tr>
<td>log($D'$) and log($s'$)</td>
<td>5.22 $\times 10^{-5}$</td>
<td>-0.084</td>
<td>0.12</td>
</tr>
</tbody>
</table>
Using the 1988–90 Darwin data to estimate the second-order moments in (11), one finds that

$$\alpha = 0.155.$$  

(12)

As to $s^*$, the scatter diagrams 1e and 1f suggest that the $s^*-R$ correlation is due mostly to data corresponding to high rain rates. In fact, when conditioned on $R < 12$ mm h$^{-1}$, the $s^*-R$ conditional correlation coefficient drops from the original $-0.63$ to $0.1$ for the 1988–90 Darwin data. Therefore, rather than a linear change of variables, an exponential form

$$\log(s^*) = \log(s) + \beta R^{0.74}$$  

(13)

seems more suitable. The exponent of $R$ was found using a least squares fit to the $R-s^*$ scatter diagram. The value of $\beta$ that will make the correlation between $R$ and the new variable $s''$ zero can be derived as before. In this case, one finds

$$\beta = -0.017.$$  

(14)

Finally, one needs to replace $s^*$ by a variable that is uncorrelated with $R$ or $D'$. Calling the new variable $s'$, if it is defined by a linear change of variables

$$\log(s') = \log(s'') + \gamma \log(D'),$$  

(15)

it will be automatically uncorrelated with $R$ (because $s''$ and $D'$ are), so it suffices to choose $\gamma$ in such a way that $s'$ and $D'$ are uncorrelated. Proceeding as before, one finds

$$\gamma = 0.2.$$  

(16)

Thus, our three uncorrelated variables are

$$R = \text{instantaneous rain rate},$$

(17)

$$D' = D R^{-0.155},$$

(18)
and

\[ s' = s \cdot D^{ \cdot 0.2} R^{0.031} e^{0.017 R^{0.74}}. \] (19)

The scatter diagrams in Figs. 2a–c show values of these new variables for the Darwin DSDs. The marginal statistics for each individual variable are summarized in Table 3. It is quite encouraging to note that the standard deviation of \( s' \) seems very small. Finally, Table 4 confirms that the correlation coefficients are all negligibly small.

Thus, one can conclude that the Darwin data strongly suggests that \( \{ R, D', s' \} \) are indeed uncorrelated variables parameterizing the drop size distribution, with the mean of \( \log(D') \) approximately 0.3 and its standard deviation approximately 0.24, while the mean of \( \log(s') \) is approximately –0.9 and its standard deviation approximately 0.12. The original DSD parameters can be calculated from \( \{ R, D', s' \} \) using the relations

\[ \mu = \frac{e^{0.034 R^{0.74}}}{s^{2} D^{0.64}} - 4, \] (20)

\[ \Lambda = \frac{e^{0.034 R^{0.74}}}{s^{2} D^{1.4} R^{0.155}}, \] (21)

\[ N_0 = 55 \frac{\Gamma(\mu + 4)[1 - (1 + 0.53/\Lambda)^{-\mu-4}]}{\Gamma(\mu + 4)} R. \] (22)

Additional systematic DSD measurements from other tropical locations should prove particularly useful in verifying these observations. Rainfall retrieval schemes that use a parameterized expression for the DSD, such as Kozu and Nakamura (1991), Kumagai et al. (1993), and Haddad et al. (1996), can benefit from (20)

through (22). Specifically, these formulas would allow one to avoid the inconsistency of assuming \( \mu \) and/or \( N_0 \) constant and letting \( \Lambda \) vary when in fact the three variables are significantly correlated. Indeed, it is entirely consistent to make the corresponding assumptions about the uncorrelated variables \( s', D', \) and \( R \). Moreover, since the variance of \( s' \) is so relatively small, one could further simplify the retrieval problem and fix the value of \( s' \) at its mean 0.41, thus reducing to two the number of unknown quantities.

3. Application

In Ulbrich (1983), formulas relating the radar reflectivity \( Z \) and the radar attenuation coefficient \( k \) to the rain rate given the drop size distribution are given. Specifically, to the DSD (1) is associated the \( Z-R \) relation

\[ Z = \frac{10^6 \Gamma(\mu + 7)N_0^{\beta-8}}{[33.31 \Gamma(\mu + 4.67)]^\beta} R^\beta, \] (23)

with \( \beta = (\mu + 7)(\mu + 4.67)^{-1} \) and with \( N_0 \) per cubic meter per centimeter, \( D \) in centimeters. In Haddad et al. (1995), it is shown that this is not the only \( Z-R \) relation that one can derive from the DSD given by (1). In fact, by “eliminating” \( N_0 \) rather than \( \Lambda \), the relation

\[ Z = 140 \frac{\Gamma(\mu + 7)}{\Gamma(\mu + 4.67)} \Lambda^{3.33} R \] (24)

can be derived (with \( \Lambda \) in units of per millimeter). Jameson (1993, 1994) adopts a different approach: starting with several realizations of the DSD (1) at varying rain rates, a polynomial relation is fit to the
results of T-matrix calculations of the expected radar attenuation coefficient $k$. For example, the $k$–$R$ relation thus obtained at 13.8 GHz is

$$k = (0.012 + 0.0058 D^* + 0.0083 D^*^2 - 0.0015 D^*^3) R,$$  \hspace{1cm} (25)

with the mass-weighted mean drop diameter $D^*$ in millimeters.

Yet neither one of these three relations is satisfactory because the coefficients in all cases are not independent of $R$. Until one makes this “dependence” explicit, or, more exactly, until this correlation is explicitly accounted for, one cannot consider the relation to be complete. This is difficult to do directly in the case of formulas (23) and (24) because the coefficients of these relations depend on the DSD parameters in a rather complicated way. Jameson’s approach looks more promising, because while $D^*$ and $R$ are correlated, substituting a correlating formula for $D^*$ into (25) would be easy. Such a correlating formula, however, has not until now been available. Another problem is that in the realizations of (1) used in Jameson’s calculations, the Sekhon–Srivastava relation $N_0 = 7000 (3.8/\Lambda)^{2.643}$ m$^{-3}$ mm$^{-1}$ was imposed. Equation (1), however, already imposes a relation between $N_0$, $\Lambda$, $\mu$, and $R$, namely (22). As a consequence, the calculations leading to (25) turn out to presuppose a coincidental relation between $\Lambda$, $\mu$, and $R$, namely

$$R = 4336.1[1 - (1 + 0.53/\Lambda)^{-\nu-\mu}] \Gamma(\mu + 4)/\Lambda^\nu 6.643,$$

which may or (most likely) may not be verified in reality.

In Olson et al. (1978), the parameter $\mu$ is in effect set to zero, once and for all, and for each of three different constant values of $N_0$ (each of which then forcing a particular $\Lambda$–$R$ relation), a different $k$–$R$ relation is calculated. For these formulas to be useful in practice, one would need to know that $N_0$ is constant, that its value is one of the three considered in Olson et al. (1978), and that one is sure that $\mu = 0$.

Thus, the problem of starting with a parameterized $\Gamma$ drop size distribution and obtaining simple accurate $Z$–$R$ and $k$–$R$ relations whose coefficients are independent of $R$ has still not been solved. This section proposes a solution by starting with the parameterization (20)–(22) of the DSD whose parameters $s'$, $D'$, and $R$ have already been empirically verified to be uncorrelated. Specifically, the goal of this section is to establish relations

$$Z = (a + a_0 e^{-b_0 R^{0.74}})(1 + a_1 R^{-b_1} e^{-b_2 R^{0.74}}) R^b$$

$$k = (\alpha + \alpha_0 e^{-\beta_0 R^{0.74}})(1 + \alpha_1 R^{-\beta_1} e^{-\beta_2 R^{0.74}}) R^\beta$$

approximating the Mie scattering results at the 13.8-GHz frequency of interest in the case of TRMM. Once formulas for computing the coefficients $a$, $b$, $a_0$, $a_1$, $b_0$, $b_1$, $b_2$, $\alpha$, $\beta$, $\alpha_0$, $\alpha_1$, $\beta_0$, $\beta_1$, and $\beta_2$ in terms of $s'$ and $D'$ only are derived, this would effectively give the sought after $Z$–$R$ and $k$–$R$ relations at that frequency.

(a) Scattering approximations

The first step is to use suitable power laws to approximate the Mie scattering cross sections at the frequency of interest, 13.8 GHz in the case at hand. Thus, given any upper-bound drop diameter $D_u$, one must find coefficients $\lambda_\alpha(D_u)$, $\lambda_\beta(D_u)$ and exponents $\varepsilon_\alpha(D_u)$, $\varepsilon_\beta(D_u)$ such that the backscattering cross section $\sigma_\alpha(D)$ and the total extinction cross section $\sigma_\beta(D)$ are well approximated over all values of $D$ less than $D_u$. Specifically, the quantities

$$A_k, B_k$$

FIG. 7. Graphs of the coefficients $A_k$ and $B_k$ in Eq. (33) to the first ratio in the $k$–$R$ relation Eq. (37) versus the upper-bound drop diameter range $D_w$. 

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were minimized for values of $D_u$ up to 8 mm. For each fit, calculations were performed at three temperatures, 275, 282.5, and 290 K. Thus, one ends up with “best relative fit” approximations

$$\sigma_b(D) \approx \lambda_2(D_u) D^{6+\varepsilon_z}(D_u)$$  \hspace{2cm} (26)

$$\sigma_l(D) \approx \lambda_k(D_u) 10^{-4} D^{3+\varepsilon_k}(D_u)$$  \hspace{2cm} (27)

valid for diameters $D$ between 0 and $D_u$. Figure 3 shows the graphs of $\lambda_2$ and $\lambda_k$ versus $D_u$. Figures 4a,b show the graphs of $\varepsilon_z$ and $\varepsilon_k$ versus $D_u$.

b. Mathematical simplifications

The next step is to use the approximations (26) and (27) to perform the integrations $\int \sigma(D)N(D) dD$ that give $Z_e$ (when $\sigma = \sigma_b$) and $k$ (when $\sigma = \sigma_l$). Since $N(D)$ is given by (1), the integrations are straightforward and give

$$Z_e = \lambda_2(D_u) \frac{\Gamma[\mu + 7 + \varepsilon_z(D_u)]}{\Lambda^{\mu+7+\varepsilon_z(D_u)}} N_0$$ \hspace{2cm} (28)

$$k = \lambda_k(D_u) \frac{\Gamma[\mu + 4 + \varepsilon_k(D_u)]}{\Lambda^{\mu+4+\varepsilon_k(D_u)}} N_0$$ \hspace{2cm} (30)

$$\approx \frac{0.0055 \lambda_k(D_u) \Gamma[\mu + 4 + \varepsilon_k(D_u)]}{\Gamma(\mu + 4)(\mu + 4)^{\varepsilon_k(D_u)}}$$ \hspace{2cm} (31)

$$\times \frac{1}{1 - [1 + (0.53/\Lambda)]^{-\mu-4}}$$ \hspace{2cm} (32)

$$\times D^{1.05k(D_u)} R^{1.155k(D_u)}$$ \hspace{2cm} (33)

with $\mu = (e^{0.034 R^{6.74} S^{0.2} D^{0.04}} - 4)$ and $D* = (\mu + 4)/D'$. Here, $R^{-0.155}$ is as before. There remains to simplify the two ratios that occur in each of (29) and (31), and at the same time to exhibit explicitly the dependence on the variable $R$. It turns out that, in (29) and (31), the ratios of the form

$$\frac{\text{const} \times \Gamma(\mu + 4 + P)}{\Gamma(\mu + 4)(\mu + 4)^P}$$ \hspace{2cm} (34)

can be very well approximated by the simpler expression

$$A + \frac{B}{(\mu + 4)^m}$$ \hspace{2cm} (35)

if one determines the coefficients $A$, $B$, and $m$ by minimizing the relative mean-square error between (33) and (32), both considered as functions of $\mu$. Since both expressions tend to an (constant) asymptote when $\mu$ is large, in practice it is only necessary to consider a small interval of values of $\mu$, say $-1 < \mu < 4$. It also turns out that the ratio

$$1 - \left(\frac{1 + 0.53 D^*}{\mu + 4}\right)^{-\mu-4}$$ \hspace{2cm} (36)

is very well approximated by the simpler expression

$$1 + \frac{0.82}{(\mu + 4)^{0.11}(0.53 D^*)^{1.42}}$$ \hspace{2cm} (37)

One then transforms (29) and (31) into the relations

$$Z_e \approx \left[A_2(D_u) + \frac{B_2(D_u)}{(\mu + 4)^{m_2(D_u)}}\right]$$ \hspace{2cm} (38)

$$\times \left[1 + \frac{2 D^{-1.42} R^{-0.22}}{(\mu + 4)^{0.11}}\right]$$ \hspace{2cm} (39)

$$k \approx \left[A_k(D_u) + \frac{B_k(D_u)}{(\mu + 4)^{m_k(D_u)}}\right]$$ \hspace{2cm} (40)

$$\times \left[1 + \frac{2 D^{-1.42} R^{-0.22}}{(\mu + 4)^{0.11}}\right]$$ \hspace{2cm} (41)

$$\times D^{1.05k(D_u)} R^{1.155k(D_u)}$$ \hspace{2cm} (42)

Figures 5a,b show the graphs of $A_2$ and $B_2$ versus $D_u$. Figure 6 shows the graphs of $m_2$ and $m_k$ versus $D_u$. Figure 7 shows the graphs of $A_k$ and $B_k$ versus $D_u$. One can now replace all the terms involving $\mu$ in (36) and (37) by their expression (20) in terms of $S'$, $D'$, and $R$. 
c. Z–R and k–R relations

There only remains to identify the appropriate values of \( D_0 \) one should choose for the initial approximations (26) and (27). These will naturally depend on the DSD shape parameters \( s' \) and \( D' \). In the case at hand, the appropriate values of \( D_0(s', D') \) were determined by choosing that value whose approximation, (36) for \( Z_e \) or (37) for \( k \), was closest in the mean-square sense to the results of the exact Mie calculation, for rain rates between 1 and 130 mm h\(^{-1}\). It turns out that the appropriate values can be fitted by the quadratics

\[
D_0(s', D') = 0.1 - 1.4s' + 2.5D' + 2.52s'D'
+ 3.34s'^2 - 0.246D'^2 \quad \text{for } Z_e, \quad (38)
\]

\[
= 4.6 - 5.1s' - 3D' + 2.52s'D'
+ 4.18s'^2 + 1.7D'^2 \quad \text{for } k, \quad (39)
\]

Thus, one finally obtains the following closed-form Z–R and k–R relations (whose coefficients are indeed independent of \( R \)):

\[
Z_e = (a + \alpha_0 e^{-\beta_0 R^{0.74}})(1 + CR^{-0.22}e^{0.0037R^{0.74}})R^b
\]

\[
k = (\alpha + \alpha_0 e^{-\beta_0 R^{0.74}})(1 + CR^{-0.22}e^{0.0037R^{0.74}})R^b
\]

with

\[
a = A_Z(s', D')D'^{3+\varepsilon_Z(s', D')}
\]

\[
b = 1.465 + 0.155\varepsilon_Z(s', D')
\]

\[
a_0 = B_Z(s', D')D'^{3+\varepsilon_Z(s', D')}(s'^2D'^{0.4})m_0(s', D')
\]

\[
b_0 = 0.034m_0(s', D')
\]

and

\[
\alpha = A_k(s', D')D'^{\varepsilon_k(s', D')}
\]

\[
\beta = 1 + 0.155\varepsilon_k(s', D')
\]

\[
\alpha_0 = B_k(s', D')D'^{\varepsilon_k(s', D')}(s'^2D'^{0.4})m_0(s', D')
\]

\[
\beta_0 = 0.034m_0(s', D')
\]

and

\[
C = 2s^{0.22}D'^{-1.379}.
\]

These are the Z–R and k–R relations that best approximate the Mie scattering results at 13.8 GHz for a drop size distribution with shape parameters \( s', D' \). To check that (40) and (41) are not drastically different from the traditional power laws, note that for all values of \( s' \) and \( D' \) within two standard deviations of their respective means, the values of \( A_Z, m_0, A_k, \) and \( B_k \) remain close to 30, 1.2, 0.0045, 0.001, and 0.95, respectively, and the values of \( \varepsilon_Z, \varepsilon_k, \) and \( B_Z \) are never far from 0.1, 1.2, and 145, respectively. If these values are used in (42)–(49), and if one also uses the average values \( s' = 0.41 \) and \( D' = 1.4 \), formulas (40) and (41) become

\[
Z_e \approx (85 + 57e^{-0.041R^{0.74}})
\]

\[
\times (1 + R^{-0.22}e^{-0.0037R^{0.74}})R^{1.48}
\]

\[
k \approx (0.0067 + 0.0003e^{-0.032R^{0.74}})
\]

\[
\times (1 + R^{-0.22}e^{-0.0037R^{0.74}})R^{1.19}
\]

Fig. 9. (a) Exact Mie k–R curves when the DSD parameters are \( s' = 0.51 \) and \( D' = 2.2 \) (at 282.5, 275, and 290 K), along with the graph of Eq. (41). (b) Exact Mie k–R curves when the DSD parameters are \( s' = 0.32 \) and \( D' = 0.83 \) (at 282.5, 275, and 290 K), along with the graph of Eq. (41).
or, after replacing the exponential terms by their second-order polynomial approximation (namely)

\[ e^{-0.041 R^{0.74}} \approx 1 - 0.057 R^{0.61} + 0.00088 R^{1.22}, \]

and

\[ e^{-0.032 R^{0.74}} \approx 1 - 0.042 R^{0.64} + 0.00088 R^{1.28}, \]

\[ Z_e \approx 284 R^{1.48} (1 - 0.023 R^{0.61} + 0.00035 R^{1.21}) \times \left( \frac{1 + R^{-0.22} - 0.0037 R^{0.52}}{2} \right) \]

\[ k \approx 0.014 R^{1.19} (1 - 0.0018 R^{0.64} + 0.000021 R^{1.28}) \times \left( \frac{1 + R^{-0.22} - 0.0037 R^{0.52}}{2} \right). \]

Manifestly, the factors in parentheses vary slowly with \( R \) (and less significantly for \( k \) than for \( Z_e \), as expected). Furthermore, these factors differ significantly from 1 only for larger values of \( R \). Thus, formulas (40) and (41) are not drastically different from the traditional power laws, at least for typical values of \( s' \) and \( D' \).

d. Numerical results

Figure 8 shows the Mie and approximate \( Z-R \) curves when \( s' = 0.51 \) and \( D' = 2.2 \). While the Mie curves were computed assuming temperatures of 275, 282.5, and 290 K in this case, there was little variability evident in the reflectivity at 13.8 GHz. The attenuation, however, did exhibit some variability, especially for drop distributions with relatively smaller drops. Figure 9a shows Mie and approximate \( k-R \) curves when \( s' = 0.51 \) and \( D' = 2.2 \), corresponding to relatively large drops—the Mie curves were computed assuming temperatures of 275, 282.5, and 290 K. Figure 9b shows the same curves when \( s' = 0.32 \) and \( D' = 0.83 \), corresponding to small drops. Figure 10 confirms that the \( Z-R \) approximation (40) remains very accurate even for small drops. A more systematic analysis of the difference between the approximate formulas (40) and (41) and the exact Mie calculations at 275, 282.5, and 290 K was conducted for values of \( s' \) and \( D' \) within two standard deviations of their respective means, namely \( 0.32 \leq s' \leq 0.51 \) and \( 0.83 \leq D' \leq 2.2 \). It turns out that (40) is never farther than 4.5% from the Mie-calculated dBZ value (the maximum error is 4.3% in the case of Fig. 8, 1.5% in the case of Fig. 10). The difference between (41) and the Mie-calculated attenuation coefficient was tracked at 10 and 100 mm h\(^{-1}\). The relative difference at 10 mm h\(^{-1}\) never exceeded 18% (representing an absolute error of about 0.13 dB km\(^{-1}\)), and the one at 100 mm h\(^{-1}\) never exceeded 15% (representing an absolute error of about 0.75 dB km\(^{-1}\)). In any case, this difference is due to the significant dependence of the 13.8-GHz attenuation on the temperature.

4. Conclusions

Based on two years’ worth of data from Darwin, one can parameterize drop size distributions using the variables \( R, D' \), and \( s' \) defined above and assume that these parameters are uncorrelated. Moreover, the variances of \( D' \) and \( s' \) are relatively small. These properties should make this parameterization particularly useful in retrieval problems. In fact, the coupled \( Z-R \) and \( k-R \) relations derived from the parameterization are simple, accurate, and have coefficients that are independent of \( R \).

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REFERENCES


