A Note on Businger's Derivation of Nondimensional Wind and Temperature Profiles under Unstable Conditions

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ABSTRACT

Using heuristic arguments, Businger presented a derivation of the so-called Dyer–Businger formulas for nondimensional wind and temperature profiles. In this note it is shown that by slightly changing this "derivation," different expressions are obtained that almost coincide with the original ones for $0 < -z/L < -1$, but that behave differently in the free-convection region. The new formula for temperature appears to have a correct free-convection behavior; that is, it becomes proportional to $(-z/L)^{-1/3}$. This note is primarily "historical" in nature.

1. Introduction

As a lecturer of micrometeorology, I have to explain to my students the essentials of the Monin–Obukhov similarity (MOS) theory leading to the so-called Dyer–Businger expressions for nondimensional wind and temperature profiles in the atmospheric surface layer. Sometimes I use the "heuristic" derivation that can be found in Fleagle and Businger (1980, 275; hereafter denoted as FB), which was written by Joost Businger (1996, personal communication).

The MOS theory is based purely on dimensional analysis; that is, the nondimensional profiles follow from curve fitting the experimental data that is organized into dimensionless groups. In this procedure the choice of the mathematical form of the fitting function is entirely free. From a lecturer's point of view, the attractive aspect of Businger's approach is that it is based on some physics, although one must realize that for turbulent flows it is impossible to derive nondimensional profiles from first physical principles. It appeared that in the unstable case the functions Businger derived fit most experimental data very well in the region $0 \leq z/L < \approx -1$.

In preparing my lectures, I found that by changing one step in Businger's derivation a new mathematical form for ϕ_m and ϕ_h is obtained that also fit the experimental data for $0 \leq z/L < -1$. In addition, the new ϕ_h function appears to obey the scaling rules of free convection; that is, the new ϕ_h becomes proportional to $(-z/L)^{-1/3}$ for $-z/L \to \infty$, whereas the Businger ϕ_h is proportional to $(-z/L)^{-1/2}$.

Considering the influence of J. Businger in micrometeorology, it is very likely that if Businger would have followed this "new" derivation the new mathematical form for ϕ_m and ϕ_h would have been used in most practical applications. The objective of this paper to point out this new form, so this paper is meant to be an "historical" note.

I have changed the derivation of Businger only at one point. I copied the rest, by which I adopted all the assumptions and simplifications made by him. Some of these are very questionable. Considering the historical nature of this note, it is irrelevant whether Businger's or my derivation is correct or incorrect since it is impossible to derive ϕ_m and ϕ_h functions using purely physical arguments.

2. Alternative derivation

In the framework of the MOS theory, the following ϕ functions can be defined:

$$\frac{\partial \overline{u}}{\partial z}\frac{kz}{u_*} = \varphi_m\left(\frac{z}{L}\right), \qquad \frac{\partial \overline{\theta}}{\partial z}\frac{kz}{\theta_*} = \varphi_h\left(\frac{z}{L}\right), \quad \text{and}$$

$$\frac{\sigma_w}{u_*} = \varphi_w\left(\frac{z}{L}\right),$$

where the notation is standard.

Using heuristic arguments, Businger first derives an expression for ϕ_m. Next he uses one of the empirical results of the Kansas field experiment: the gradient

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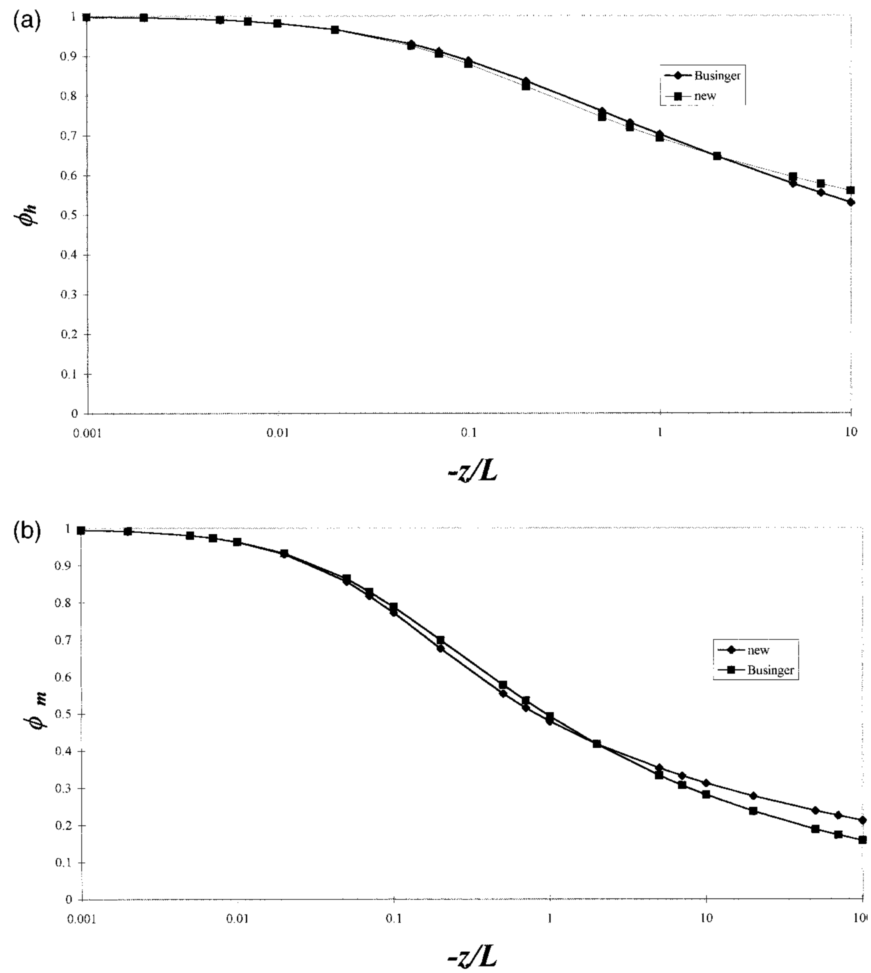

FIG. 1. (a) Comparison between the ϕ_h function (1), denoted as "Businger" and the new expression (5). (b) As in (a) except for ϕ_m.

Richardson number is approximately equal to z/L for unstable conditions. This leads to

$$\varphi_h = \varphi_m^2 = \left(1 - \alpha\frac{z}{L}\right)^{-1/2}, \tag{1}$$

in which α is a constant. Note that by adopting (1), the findings of Dyer (1982), which state that ϕ_m and ϕ_h are 1 at $z/L = 0$, are used. I do not follow Businger et al. (1971), which implies that I use a von Kármán constant of 0.4 instead of 0.35.

The alternative derivation of ϕ_m presented here is based on Businger's expression [FB, Eq. (6.39), 276], which we write here in a slightly different form:

$$\overline{w^2} = \alpha_1\overline{\ell_m^2}\left(\frac{\partial\overline{u}}{\partial z}\right)^2 - \alpha_2\frac{g}{\rho}\overline{\ell_m\rho'}. \tag{2}$$

Here, α_1 and α_2 are constants and ℓ_m is a mixing length (assumed to be equal to kz). Businger multiplies (2) with ℓ_m^2 and then applies $K_m = \overline{w\ell_m}$ [FB, Eq. (6.26), 270].

In this note, the last step is altered as follows. First, (2) is divided by u_*^2 and with $\rho'/\overline{\rho} = -\theta_v'/\overline{\theta_v}$, $\theta_v' = -\ell_m\partial\overline{\theta_v}/\partial z$ and $\varphi_h = \varphi_m^2$. Using $L = -(\overline{\theta}/kg)(u_*^3/\overline{w'\theta_v'})$ the following expression is obtained:

$$\varphi_w = \sqrt{\alpha_1}\varphi_m\left(1 - \frac{\alpha_2}{\alpha_1}\frac{\varphi_h}{\varphi_m^2}\frac{z}{L}\right)^{1/2}$$
$$= \sqrt{\alpha_1}\varphi_m\left(1 - \frac{\alpha_2}{\alpha_1}\frac{z}{L}\right)^{1/2}. \tag{3}$$

So far I only did some algebraic reshuffling. The next step is new. I propose to apply the expression for ϕ_w, which was already known in the 1960s (see Monin and Yaglom 1971) in the form mentioned in recent textbooks (e.g., Panofsky and Dutton 1984):

$$\varphi_w = 1.25\left(1 - 3\frac{z}{L}\right)^{1/3}. \tag{4}$$

This is the essential step of the alternative derivation presented here. Equation (4) is generally accepted, and it obeys the free-convection scaling rules (Monin and

Yaglom 1971) because ϕ_m and ϕ_h are 1 at $z/L = 0$, $\alpha_1 = 1.25^2$.

In the draft version of this note, I assumed, without any justification, that α_2 is equivalent with the constant α used by Businger, that is, I took $\alpha_2 = 16$. One of the anonymous reviewers proposed to obtain the value of α_2 by using the fact that the small z asymptote of Eq. (1) (for temperature) is $(1 + 8z/L)$, assuming $\alpha = 16$. This leads to $\alpha_2 = 15.6$, and finally we get

$$\varphi_h = \varphi_m^2 = \left(1 - 3\frac{z}{L}\right)^{2/3}\left(1 - 10\frac{z}{L}\right)^{-1}. \qquad (5)$$

In Figs. 1a, b the expressions (2) and (5) are compared. The differences are within the experimental scatter. So, in practice, there is no difference between the old and the new ϕ_m and ϕ_h functions for $0 \le -z/L < -1$.

From a theoretical point of view the expressions differ because for large values of $-z/L$, ϕ_h becomes proportional to $(-z/L)^{-1/3}$, so the new expression obeys the free-convection limit, which was known already in the 1930s (Monin and Yaglom 1971). According to (6), $\phi_m \propto (-z/L)^{-1/6}$ in the free-convection limit. This result will be discussed later. Note that (3) can also be written as

$$\varphi_m = \frac{\varphi_w}{\sqrt{\alpha_1}\left(1 - \dfrac{\alpha_2}{\alpha_1}\mathrm{Ri}\right)^{1/2}}, \qquad (6)$$

which is not based on the assumption $\mathrm{Ri} = z/L$.

3. Discussion and concluding remarks

This note is meant to show that by a slight change in Businger's derivation alternative expressions for ϕ_m and ϕ_h can be obtained for unstable conditions, which, in the range of $0 < -z/L < -1$, are virtually the same as the original ones but that behave differently for very unstable situations. The new expression for ϕ_h has the free-convection behavior, as found by many authors [e.g., Högström (1996) and Kader and Yaglom (1990)]. For an historical overview, see Monin and Yaglom (1971).

The new ϕ_m expression has a "−1/6-power behavior" in the free-convection range. In literature there is still an ongoing debate on the free-convection behavior of ϕ_m. The expression derived by Kader and Yaglom (1990) has a "+1/3-power behavior," which entirely contradicts with the "−1/4-power behavior" of (1); see, for example, Högström (1996). Expression (5) is leaning toward the −1/4 asymptote. There is still a need for reliable data for very unstable conditions in order to be able to conclude which expression is the best.

In principle, one can use formula (6) and an alternative relationship between Ri and z/L, for example, $\mathrm{Ri} = 1.5z/L$ instead of $\mathrm{Ri} = z/L$ as proposed by Högström (1996) to obtain a different expression for ϕ_h. Also, (6) allows expressions for the nondimensional ϕ functions in terms of Ri. It is outside the scope of this note to explore this issue further.

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REFERENCES

Businger, J. A., J. C., Wyngaard, Y. Izumi, and E. F. Bradley, 1971: Flux profile relationships in the atmospheric surface layer. *J. Atmos. Sci.*, **28,** 181–189.

Fleagle, R. G., and J. A. Businger, 1980: *An Introduction to Atmospheric Physics.* 2d ed. Academic Press, 432 pp.

Högström, U., 1996: Review of some basic characteristics of the atmospheric surface layer. *Bound.-Layer Meteor.*, **78,** 215–246.

Kader, B. A., and A. M. Yaglom, 1990: Mean fields and fluctuation moments in unstable stratified turbulent boundary layers. *J. Fluid Mech.*, **212,** 637–661.

Monin, A. S., and A. M. Yaglom, 1971: *Statistical Fluid Mechanics: Mechanics of Turbulence.* Vol. 1, MIT Press, 766 pp.

Panofsky, H. A., and J. A. Dutton, 1984: *Atmospheric Turbulence, Models and Methods for Engineering Applications.* John Wiley, 399 pp.