On the Computation of Gradients from Observations over Complex Terrain

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ABSTRACT

A mathematical scheme is developed to compute the gradients of observations taken over complex terrain. The method is applied to an artificial example to demonstrate the scheme. An application is made to surface pressure observations between Little Rock, Arkansas, and Amarillo, Texas. Divergence computations are made with the scheme using observed wind data over the Black Hills of South Dakota.

1. Introduction

Bluestein and Crawford (1997) concluded that computing gradients of wind and other quantities is complicated by sloping terrain. They used adjustments suggested by Schaefer (1973) to take the terrain into account. Schaefer (1973) noted that when horizontal divergence is computed over nonhorizontal surfaces, extra terms appear in the equations. The extra terms can be estimated and are probably important enough that they should not be ignored. Computations made from observations taken over flat terrain are straightforward, but most observations of interest are made over land surfaces with a certain amount of irregular topography.

This issue arose in the Upper Missouri River Basin Pilot Project (UMRBPP), which gathered data during the period of 5 April through 5 May 1999 over the Black Hills of South Dakota and Wyoming. Observations around a small watershed were the focus of the project, with the objective to refine the understanding of the water budgets in the watershed. Rawinsondes were launched from three sites, and a wind profiler and radiometer (water vapor) installation made up a fourth site enclosing the watershed area. The measurements were made in mountainous terrain with topography relief of about 1 km. A scheme was developed to compute the gradients of the observations over the watershed.

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2. Analysis

Schaefer (1973) used equations developed by Wigley (1964) to get an estimate of the divergence over terrain that is not level. Wigley's equations were developed from a Taylor series expansion (Taylor 1955); however, a slightly different approach is instructive. Let q represent a general coordinate (x or y); then some quantity A may be represented to first order as

$$A(q + \Delta q, z + \Delta z) = A(q, z) + \Delta q \left.\frac{\partial A}{\partial q}\right|_z + \Delta z \frac{\partial A}{\partial z}. \tag{1}$$

Note that there are two observing points, (q, z) and $(q + \Delta q, z + \Delta z)$, on the terrain E at elevations $z = E(q)$ and $z + \Delta z = E(q + \Delta q)$. Substituting this into Eq. (1), rearranging, and dividing by Δq results in

$$\frac{A[q + \Delta q, E(q + \Delta q)] - A[q, E(q)]}{\Delta q} = \left.\frac{\partial A}{\partial q}\right|_z + \frac{\partial A}{\partial z}\left[\frac{E(q + \Delta q) - E(q)}{\Delta q}\right]. \tag{2}$$

The limit as Δq becomes very small is

$$\left.\frac{\partial A}{\partial q}\right|_E = \left.\frac{\partial A}{\partial q}\right|_z + \frac{\partial A}{\partial z}\frac{\partial E}{\partial q}. \tag{3}$$

Equation (3) is Schaefer's (1973) Eq. (1) and Wigley's (1964) Eq. (11). The left-hand side is the gradient along q, which can be measured on the terrain surface. The first term on the right-hand side is the value that is

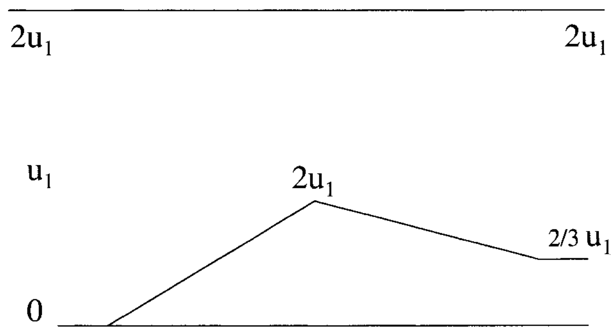

FIG. 1. Schematic diagram of a hill 1 km high and 20 km long, with surface wind speeds and upper winds as indicated. The right side of the hill is a plateau 0.5 km higher than the left side. The mass flux is constant across the diagram.

desired. The last quantity on the right side involves the unknown gradient of A with respect to z and the slope of the terrain. Schaefer (1973) suggests that the gradient of A in the vertical may be estimated from a sounding. However, soundings do not measure A on the surface of the terrain, but rather they measure the variation of A above the surface. Equation (3) has two unknowns—the gradients of A with respect to q and z. It is identical to Eq. (1) for all practical purposes. Equation (1) or (2) can be used with observational data, which are taken at finite intervals, rather than at infinitesimal intervals as suggested by Eq. (3).

A synthetic example will illustrate how to solve the equations for the unknowns. Figure 1 shows a hill 20 km long and 1 km high. The flow is from left to right with the flow speed $u = 0$ on the left at the surface increasing linearly to $2u_1$ at 2 km above the surface. At the top of the hill the flow is $2u_1$ at all levels, decreasing to $\frac{2}{3}u_1$ on the right. The mass flux (assumed density is 1 kg m^{-3}) is constant at $2000u_1$ kg m^{-1} s^{-1} across the hill. Three observations of the wind may be made along the surface to determine the gradients. After choosing the left side of the hill as the basis point, the following two equations can be constructed from Eqs. (1) or (2):

$$2u_1 = 10\ \text{km} \times \frac{\partial u}{\partial x} + 1\ \text{km} \times \frac{\partial u}{\partial z}, \quad \text{and} \tag{4}$$

$$\frac{2}{3}u_1 = 20\ \text{km} \times \frac{\partial u}{\partial x} + 0.5\ \text{km} \times \frac{\partial u}{\partial z}. \tag{5}$$

Equation (5) is subtracted from 2 times Eq. (4) to solve for the vertical gradient of u, and 2 times Eq. (5) is subtracted from (4) to obtain the horizontal gradient. Clearly the gradient of u with respect to z is $20/9u_1$ km^{-1} and the gradient of u with respect to x is $-1/45u_1$ km^{-1}. Schaefer's (1973) approach would produce different results depending on the value used for the vertical shear in u. On the left side of the hill, the shear is u_1 km^{-1}. Schaefer's method gives

$$\frac{\partial u}{\partial x} = \frac{\frac{2}{3}u_1}{20\ \text{km}} - \frac{2u_1}{2\ \text{km}}\frac{0.5\ \text{km}}{20\ \text{km}} = \frac{1}{120}\ \text{km}^{-1} \times u_1.$$

Because the mass flux is constant across the hill, the correct answer should be no divergence. The winds along the slope do not vary linearly in this example. A second-order solution may be found from the Taylor expansion:

$$u(x, z) = u(0, 0) + x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} + \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}z^2\frac{\partial^2 u}{\partial z^2} + \cdots .$$

Two additional observations are needed to solve the set of equations for the unknown gradients. At least one additional slope is needed to get the second derivative with respect to z. Adding an observation upstream from the left foot of the hill ($u = 0$ at the surface) and one downstream on the plateau [$u = (2/3)u_1$] will result in a set of four equations that can be solved for the gradients. The results are:

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = 0, \qquad \frac{\partial u}{\partial z} = \frac{2}{3}u_1\ \text{km}^{-1}, \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = \frac{8}{3}u_1\ \text{km}^{-2}.$$

Now the horizontal divergence is correct. This example is not particularly realistic but shows how to solve for the gradients.

With three observations in a line on terrain with variable slope, the gradient along the line may be obtained. However, observation points in complex terrain are usually located somewhat randomly, because concerns about accessibility, power availability, and other matters determine good sites. Furthermore, a divergence calculation requires gradients in both x and y directions. With this in mind, a three-dimensional Taylor series expansion in x, y, and z may be written (neglecting higher-order terms):

$$A(x + \Delta x, y + \Delta y, z + \Delta z) = A(x, y, z) + \Delta x\frac{\partial A}{\partial x} + \Delta y\frac{\partial A}{\partial y} + \Delta z\frac{\partial A}{\partial z}. \tag{6}$$

This may be rearranged as

$$\Delta A = A(x + \Delta x, y + \Delta y, z + \Delta z) - A(x, y, z) = \Delta x\frac{\partial A}{\partial x} + \Delta y\frac{\partial A}{\partial y} + \Delta z\frac{\partial A}{\partial z}. \tag{7}$$

There are now three unknowns (the gradients of A in x, y, and z), so four measurements of A are needed to solve for the unknowns. As before, assume that measurements $A_0(x_0, y_0, z_0)$, $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, and $A_3(x_3, y_3, z_3)$ have been made. Use A_0 as the starting point to define the delta values

$$\Delta A_1 = A_1 - A_0 \quad \Delta x_1 = x_1 - x_0 \quad \Delta y_1 = y_1 - y_0 \quad \Delta z_1 = z_1 - z_0,$$
$$\Delta A_2 = A_2 - A_0 \quad \Delta x_2 = x_2 - x_0 \quad \Delta y_2 = y_2 - y_0 \quad \Delta z_2 = z_2 - z_0, \quad \text{and}$$
$$\Delta A_3 = A_3 - A_0 \quad \Delta x_3 = x_3 - x_0 \quad \Delta y_3 = y_3 - y_0 \quad \Delta z_3 = z_3 - z_0.$$

Then the following matrix equation relates the observations with the gradients:

$$\begin{bmatrix}\Delta A_1\\ \Delta A_2\\ \Delta A_3\end{bmatrix} = \begin{bmatrix}\Delta x_1 & \Delta y_1 & \Delta z_1\\ \Delta x_2 & \Delta y_2 & \Delta z_2\\ \Delta x_3 & \Delta y_3 & \Delta z_3\end{bmatrix}\begin{bmatrix}\dfrac{\partial A}{\partial x}\\ \dfrac{\partial A}{\partial y}\\ \dfrac{\partial A}{\partial z}\end{bmatrix}. \tag{8}$$

Multiplying both sides of Eq. (8) by the inverse of the 3×3 matrix will determine the unknown gradients:

$$\begin{bmatrix}\dfrac{\partial A}{\partial x}\\ \dfrac{\partial A}{\partial y}\\ \dfrac{\partial A}{\partial z}\end{bmatrix} = \begin{bmatrix}\Delta x_1 & \Delta y_1 & \Delta z_1\\ \Delta x_2 & \Delta y_2 & \Delta z_2\\ \Delta x_3 & \Delta y_3 & \Delta z_3\end{bmatrix}^{-1}\begin{bmatrix}\Delta A_1\\ \Delta A_2\\ \Delta A_3\end{bmatrix}. \tag{9}$$

The 3×3 matrix will have an inverse if the four observations are not coplanar. Equation (9) can be applied to any observations, including ones from an elevated (but not planar) surface. The observation points should form a quadrangular space that is approximately square to optimize the results. If the points are nearly in a line (for example, along the x direction), the gradients in the y direction will be poorly resolved. Substantial departures in the vertical from a planar surface similarly facilitate resolution of the gradients in z.

To compute divergence, observations of the winds are needed. The u and v components of the wind can then be resolved, and the gradients of the wind in three dimensions can be computed using Eq. (9). The horizontal divergence D is then calculated directly from the gradients:

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{10}$$

There is no need to adjust the results as in Eq. (3). The gradients are applicable to the area enclosed by the observation sites. Of course, the results depend on the quality of the wind measurements.

Using Fig. 1, another synthetic example may be constructed. Assume observations at points (0, 0, 0), (10, 5, 1), (20, 0, 0.5), and (10, −5, 1). These points are on the left side of the hill, two on top of the hill, and on the right side of the hill. Now, from Eq. (9), the solution matrix can be set up:

$$\begin{bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial u}{\partial y}\\ \dfrac{\partial u}{\partial z}\end{bmatrix} = \begin{bmatrix}10 & 5 & 1\\ 20 & 0 & 0.5\\ 10 & -5 & 1\end{bmatrix}^{-1}\begin{bmatrix}2u_1\\ (2/3)u_1\\ 2u_1\end{bmatrix} = \frac{1}{-150}\begin{bmatrix}2.5 & -10 & 2.5\\ -15 & 0 & 15\\ -100 & 100 & -100\end{bmatrix}\begin{bmatrix}2u_1\\ (2/3)u_1\\ 2u_1\end{bmatrix} = \begin{bmatrix}-\dfrac{1}{45}u_1\\ 0\\ \dfrac{20}{9}u_1\end{bmatrix}. \tag{11}$$

3. Error analysis

Equation (9) has two sources of error: siting and measurement. Errors in the site locations will produce errors in the matrix. In keeping with the recommendation that the observation sites should form a quadrilateral space, consider the rectangular layout of sites shown in Fig. 2. The sites are numbered 0 through 3, with site 0 at the origin (0, 0, 0), and site 3 at (x, y, z). The sides of the rectangle are x and y units long. Without loss of generality, the sites 1 and 2 may be assumed to be at the same elevation as site 0. Recall that three nonlinear points determine a plane. Thus the four observation sites are in two distinct planes, with site 3 on high ground. If the site positions are known exactly, then Eq. (9) gives

$$\begin{bmatrix}\dfrac{\partial A}{\partial x}\\ \dfrac{\partial A}{\partial y}\\ \dfrac{\partial A}{\partial z}\end{bmatrix} = \begin{bmatrix}x & 0 & 0\\ 0 & y & 0\\ x & y & z\end{bmatrix}^{-1}\begin{bmatrix}\Delta A_1\\ \Delta A_2\\ \Delta A_3\end{bmatrix} = \frac{1}{xyz}\begin{bmatrix}yz & 0 & 0\\ 0 & xz & 0\\ -xy & -xy & xy\end{bmatrix}\begin{bmatrix}\Delta A_1\\ \Delta A_2\\ \Delta A_3\end{bmatrix}.$$

In reality however, there is some uncertainty in the positions:

$$\begin{bmatrix} \dfrac{\partial A}{\partial x} \\ \dfrac{\partial A}{\partial y} \\ \dfrac{\partial A}{\partial z} \end{bmatrix} = \begin{bmatrix} x_1 & \delta_{y1} & \delta_{z1} \\ \delta_{x2} & y_2 & \delta_{z2} \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} \Delta A_1 \\ \Delta A_2 \\ \Delta A_3 \end{bmatrix}.$$

The lowercase deltas indicate the uncertainties in the differences as a result of uncertainties in the exact positions for the sites. Note that x_3 and y_3 are x_1 and y_2 with some uncertainty built in. So the determinant is

$$\begin{vmatrix} x_1 & \delta_{y1} & \delta_{z1} \\ \delta_{x2} & y_2 & \delta_{z2} \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1 y_2 z_3 + \delta_{x2} y_3 \delta_{z1} + x_3 \delta_{y1} \delta_{z2} - (x_3 y_2 \delta_{z1} + \delta_{x2} \delta_{y1} z_3 + x_1 y_3 \delta_{z2}).$$

Clearly the double-delta terms are second order and may be neglected. Also recall that the x_3 and y_3 include error terms that are multiplied by the respective z error terms. Thus, to first order, the determinant simplifies to

$$\begin{vmatrix} x_1 & \delta_{y1} & \delta_{z1} \\ \delta_{x2} & y_2 & \delta_{z2} \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1 y_2 z_3 \left(1 - \frac{\delta_{z1}}{z_3} - \frac{\delta_{z2}}{z_3}\right).$$

The inverse is then

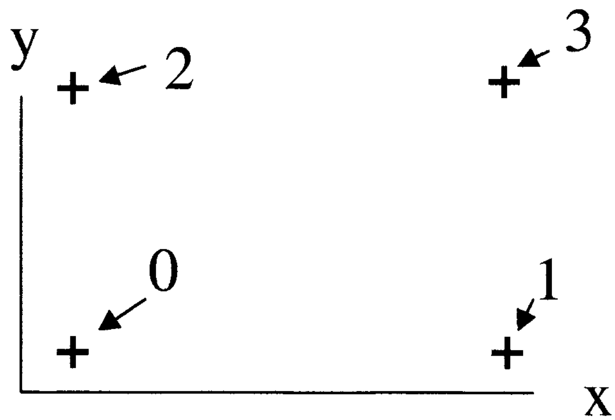

FIG. 2. Schematic layout of sites for the error analysis.

$$\frac{1 + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3}}{x_1 y_2 z_3} \begin{bmatrix} y_2 z_3 \left(1 - \dfrac{\delta_{z2}}{z_3}\right) & -y_2 z_3 \left(\dfrac{\delta_{y1}}{y_2} - \dfrac{\delta_{z1}}{z_3}\right) & -y_2 z_3 \dfrac{\delta_{z1}}{z_3} \\ -x_1 z_3 \left(\dfrac{\delta_{x2}}{x_1} - \dfrac{\delta_{z2}}{z_3}\right) & x_1 z_3 \left(1 - \dfrac{\delta_{z1}}{z_3}\right) & -x_1 z_3 \dfrac{\delta_{z2}}{z_3} \\ -x_1 y_2 \left(1 - \dfrac{\delta_{x2}}{x_1} + \dfrac{\delta_{x3}}{x_1}\right) & -x_1 y_2 \left(1 - \dfrac{\delta_{y1}}{y_2} + \dfrac{\delta_{y3}}{y_2}\right) & x_1 y_2 \end{bmatrix},$$

where second-order terms are neglected. Multiplying by the inverse of the determinant and neglecting second-order terms gives

$$\begin{bmatrix} \dfrac{1}{x_1}\left(1 + \dfrac{\delta_{z1}}{z_3}\right) & -\dfrac{1}{x_1}\left(\dfrac{\delta_{y1}}{y_2} - \dfrac{\delta_{z1}}{z_3}\right) & -\dfrac{1}{x_1}\dfrac{\delta_{z1}}{z_3} \\ -\dfrac{1}{y_2}\left(\dfrac{\delta_{x2}}{x_1} - \dfrac{\delta_{z2}}{z_3}\right) & \dfrac{1}{y_2}\left(1 + \dfrac{\delta_{z2}}{z_3}\right) & -\dfrac{1}{y_2}\dfrac{\delta_{z2}}{z_3} \\ -\dfrac{1}{z_3}\left(1 - \dfrac{\delta_{x2}}{x_1} + \dfrac{\delta_{x3}}{x_1} + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3}\right) & -\dfrac{1}{z_3}\left(1 - \dfrac{\delta_{y1}}{y_2} + \dfrac{\delta_{y3}}{y_2} + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3}\right) & \dfrac{1}{z_3}\left(1 + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3}\right) \end{bmatrix}.$$

Recall that x_1, y_2, and z_3 have errors that have not been accounted for as yet. The following includes these errors to first order:

$$\begin{bmatrix} \dfrac{1}{x_1}\left(1 + \dfrac{\delta_{z1}}{z_3} - \dfrac{\delta_{x1}}{x_1}\right) & -\dfrac{1}{x_1}\left(\dfrac{\delta_{y1}}{y_2} - \dfrac{\delta_{z1}}{z_3}\right) & -\dfrac{1}{x_1}\dfrac{\delta_{z1}}{z_3} \\ -\dfrac{1}{y_2}\left(\dfrac{\delta_{x2}}{x_1} - \dfrac{\delta_{z2}}{z_3}\right) & \dfrac{1}{y_2}\left(1 + \dfrac{\delta_{z2}}{z_3} - \dfrac{\delta_{y2}}{y_{y2}}\right) & -\dfrac{1}{y_2}\dfrac{\delta_{z2}}{z_3} \\ -\dfrac{1}{z_3}\left(1 - \dfrac{\delta_{x2}}{x_1} + \dfrac{\delta_{x3}}{x_1} + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3} - \dfrac{\delta_{z3}}{z_3}\right) & -\dfrac{1}{z_3}\left(1 - \dfrac{\delta_{y1}}{y_2} + \dfrac{\delta_{y3}}{y_2} + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3} - \dfrac{\delta_{z3}}{z_3}\right) & \dfrac{1}{z_3}\left(1 + \dfrac{\delta_{z1}}{z_3} + \dfrac{\delta_{z2}}{z_3} - \dfrac{\delta_{z3}}{z_3}\right) \end{bmatrix}.$$

TABLE 1. Sounding sites.

	Amarillo, TX	Norman, OK	Little Rock, AR
Elev (MSL)	1099	357	165
Lat	35.23	35.21	34.73
Long	−101.70	−97.45	−92.23
Parameters:			
P0	1033.8	1031.3	1027.7
P1	−0.124 80	−0.123 07	−0.121 38
P2	6.0963	5.5797	5.0494
Mean sea level pressure	1030	1031	1028

TABLE 2. Soundings—00 UTC 18 Apr 2001.

Amarillo, TX		Norman, OK		Little Rock, AR	
Pressure	Height	Pressure	Height	Pressure	Height
904.0	1099.0	988.0	357.0	1007.0	165.0
850.0	1599.0	958.2	610.0	1000.0	233.0
838.0	1713.2	925.0	902.0	991.4	305.0
829.0	1799.8	923.6	914.0	955.7	610.0
815.0	1936.3	889.6	1219.0	925.0	881.0
813.0	1956.0	850.0	1589.0	921.3	914.0
807.0	2015.6	829.0	1789.5	887.3	1219.0
800.0	2086.1	824.9	1829.0	850.0	1567.0
737.4	2443.0	820.0	1876.8	822.5	1829.0
715.0	2992.3	794.0	2134.0	808.0	1970.3
713.0	3014.9	779.0	2286.6	799.0	2059.1
710.1	3048.0	764.4	2438.0	791.5	2134.0
700.0	3164.0	751.0	2578.5	761.8	2438.0
		735.7	2743.0	756.0	2498.3
		735.0	2750.5	732.8	2743.0
		729.0	2816.0	700.0	3102.0
		721.0	2904.2		
		708.0	3049.4		
		700.0	3140.0		

The errors in the computation of the inverse matrix depend on the accuracy in determining the locations of the observing sites. There are horizontal measurement errors and elevation measurement errors. They are different and must be considered separately. Elevations for meteorological observation sites are usually rounded off to the nearest meter; thus the accuracy is about 0.5 m. Measurements obviously can be made to a greater accuracy, but for generally available observational data the site elevations are rounded to the nearest meter. Thus the delta zs are on the order of a meter for general data. The implication is that z, the height of the high ground, should be on the order of 100 m to reduce the errors to a few percent in the inverse matrix. Latitude and longitude are usually used as the location coordinates for meteorological sites. These are typically given to the nearest 0.01°. This implies a precision of about 0.5 km. The distance between two sites is then known to about 1 km, which implies that the horizontal distances should be on the order of 100 km or so to reduce errors to a few percent.

In the case of the data presented here, taken from UMRBPP, the latitude and longitude are given to an accuracy of 0.001° and the vertical coordinate to 1 m. The observation sites are about 30 km apart, and the vertical distance is 500 m. Thus, position errors are very small and sum to a few percent. The errors in calculating the divergence should not exceed a few percent. The precision reported in the wind speeds is to the nearest 0.1 m s^{-1}. The divergence errors due to wind speed error are on the order of 5×10^{-6} s^{-1}.

4. Example of application to surface pressure calculations

Using observational pressure data from three sites in nearly a straight line, Schaefer's (1973) scheme and the proposed scheme can be compared and evaluated. Table 1 lists the sites and positions; all three are at about latitude 35°N. The correct answer is also known, because the National Weather Service (NWS) computes the mean sea level pressure and the horizontal gradient can be computed from those values. Because these three sites are upper-air stations, a sounding is available from each site. The soundings are given in Table 2; the parameters P0, P1, and P2 in Table 1 are least squares curve fits for the pressure versus height:

$$p = \mathrm{P0} + \mathrm{P1} \times z + \mathrm{P2} \times 10^{-6} \times z^2.$$

From this equation the vertical pressure gradient at the surface can be computed. The observed meteorological conditions for 18 April 2001 were a high pressure center located over Texas, Oklahoma, and Kansas with weak winds over this region. Thus the pressure gradient was weak over this region.

Using the data in Tables 1 and 2 and Schaefer's (1973) equation, we can compute the horizontal pressure gradient between Little Rock and Amarillo. Because Norman is located about one-half of the way between, the vertical pressure gradient from there will be used. The gradient according to Eq. (3) is

$$\begin{aligned}\frac{\partial p}{\partial x} &= \left.\frac{\partial p}{\partial x}\right|_E - \frac{\partial p}{\partial z}\frac{\partial E}{\partial x} \\ &= \frac{1007 - 904}{864.52\ \mathrm{km}} - \left(-0.119\ \frac{\mathrm{hPa}}{\mathrm{m}}\right)\frac{(165 - 1099)\ \mathrm{m}}{864.52\ \mathrm{km}} \\ &= -0.009\,42\ \mathrm{hPa\ km^{-1}}.\end{aligned}$$

The mean sea level gradient is

$$\frac{\partial p}{\partial x} = \frac{1028 - 1030}{864.52} = -0.0023\ \mathrm{hPa\ km^{-1}},$$

about one-quarter of the value from Schaefer's equation.

Utilizing Eq. (1) and the data from Tables 1 and 2, two equations in two unknowns can be set up with Amarillo as the base site:

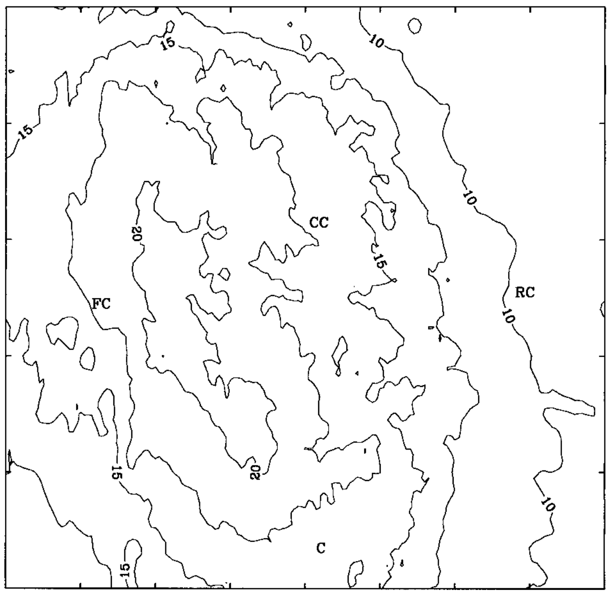

FIG. 3. Topography of the Black Hills region of South Dakota and Wyoming, contoured with 250-m intervals. The four UMRBPP observation sites are Rapid City, SD (RC); Custer Crossing, SD (CC); Four Corners, WY (FC); and Custer, SD (C). The first character is centered over the site. The lower-left corner is at 43°40′N, 104°20′W, and the upper-right corner is 44°30′N, 103°0′W. The highest elevation is greater than 2000 m MSL.

$$(1007 - 904)\ \text{hPa} = 864.52\ \text{km}\frac{\partial p}{\partial x} + (165 - 1099)\ \text{m}\frac{\partial p}{\partial z}, \quad \text{and}$$

$$(988 - 904)\ \text{hPa} = 387.98\ \text{km}\frac{\partial p}{\partial x} + (357 - 1099)\ \text{m}\frac{\partial p}{\partial z}.$$

These two independent equations can be solved for the unknowns with the results,

$$\frac{\partial p}{\partial x} = -0.007\,27\ \text{hPa km}^{-1}, \quad \text{and}$$

$$\frac{\partial p}{\partial z} = -0.117\ \text{hPa m}^{-1}.$$

Both computations give an absolute gradient that is greater than the observed. (The mean sea level pressure computation is assumed to give accurate results.) Given the horizontal and vertical gradients, the pressure can be computed for points p between the observation points:

$$p = p_{\text{Amarillo}} + (x - x_{\text{Amarillo}})\frac{\partial p}{\partial x} + (z - z_{\text{Amarillo}})\frac{\partial p}{\partial z}.$$

This ability is perhaps the greatest usefulness of these results. If the gradients found with Schaefer's (1973) method are used, the pressure at Little Rock is correctly computed as expected (because it was the value used in the computation), but the pressure at Norman is off by 0.64 hPa. Using the derived gradients from the proposed scheme gives values for both Norman and Little Rock to 0.01 hPa. The vertical pressure gradient found by the proposed scheme is approximately the mean of the vertical gradients at the three sites.

TABLE 3. Site locations.

Site	Lat (north)	Long (west)	Elev (m)
Rapid City	44.1000	103.2000	1029
Custer Crossing	44.2052	103.6493	1652
Four Corners	44.0772	104.1385	1768
Custer	43.7322	103.6138	1708

5. Example of application to divergence calculations from upper-air observations

The same approach can be applied to compute gradients and divergence values from upper-air observations, provided the observations are selected to represent noncoplanar "layers." Figure 3 shows a contour plot of the Black Hills with the four UMRBPP observation sites depicted. Details about the site locations are given in Table 3. Note the high ground directly north and west of Custer (the southernmost site), which was the site with the wind profiler and microwave radiometer. The profiler made measurements continuously and recorded an observation 2 times per hour. The Custer Crossing and Four Corners sites had the National Center for Atmospheric Research cross-chain long-range navigation atmospheric sounding system (CLASS) rawinsonde system; Four Corners, the highest site, is west of the highest terrain in the Black Hills. The watershed of interest is west of Rapid City (the easternmost site), the location of the NWS rawinsonde data.

Figures 4 and 5 show the observed wind components

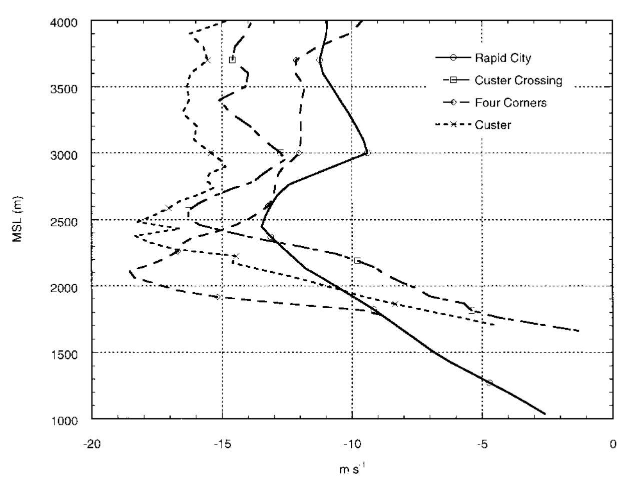

FIG. 4. East–west wind components for 1800 UTC 22 Apr 1999.

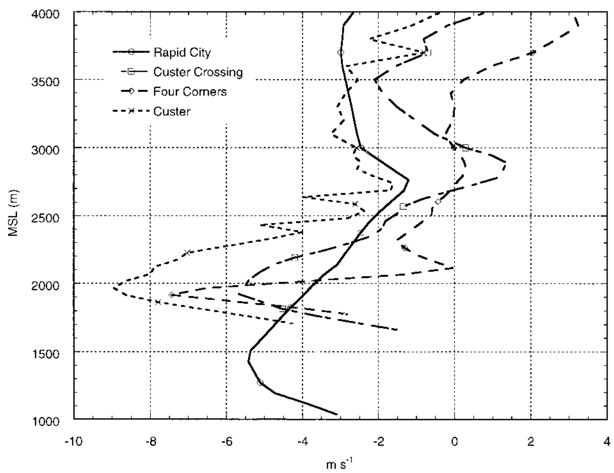

FIG. 5. North–south wind components for 1800 UTC 22 Apr 1999.

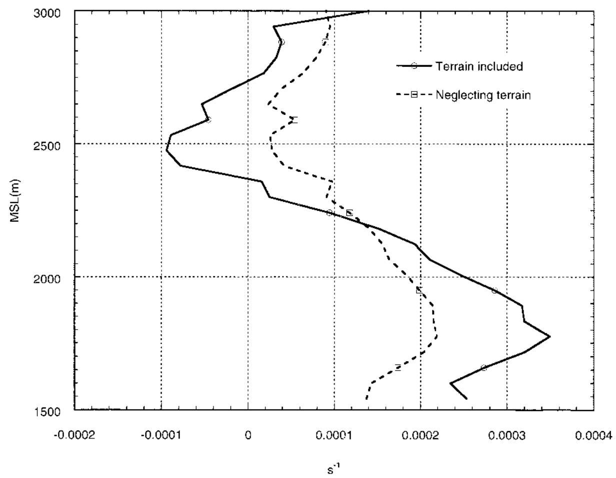

FIG. 6. Horizontal divergence profile for 1800 UTC 22 Apr 1999. See text for details on the computations.

(u and v) from the soundings for 1800 UTC 22 April 1999. Each site had similar wind structure above the ground. The u component increases with height to about 2.5 km above mean sea level (MSL) in Fig. 4 then becomes variable but somewhat weaker with increasing height. The v component is weaker but has a similar structure (Fig. 5). Note that the winds are from the east-northeast, so there were upslope winds on the eastern slopes of the Black Hills. Easterly winds were observed for the previous 12 h and the following 12 h of this observation. Precipitation in the form of light snow was observed over the watershed during the period from 1200 UTC April 22 through 0000 UTC April 23, so one might expect to find convergence at some level.

The measured winds were interpolated into 25 levels beginning at the surface and extending to 3000 m MSL at each site. The resulting (nonplanar) layers are thickest at Rapid City and thinnest at Four Corners. The interpolated data were used to compute the divergence values shown in Fig. 6; two computations are shown. The Rapid City and Four Corners sites are nearly on an east–west line (Fig. 3) and so can be used to compute the gradient of u according to Eq. (2). The Custer and Custer Crossing sites are nearly on a north–south line and can be used to approximate the gradient of v. This has been done to get the ''terrain-neglected'' approximation. The ''terrain-included'' curve was computed using Eq. (9), with Rapid City as the base station. Note that this computation used data from all four sites to get the gradients of u and v below 3000 m MSL.

In principle, there should be more valid information in the terrain-included curve. There is divergence from the surface to 2400 m MSL in both computations. Above 2400 m MSL, there is a layer of convergence about 400 m thick in the terrain-included curve, whereas the terrain-neglected approximation remains divergent. In this particular case, the terrain-included analysis shows enhanced divergence at the surface and convergence above 2400 m MSL that does not appear in the straightforward terrain-neglected analysis. The uncertainties in the results should increase near the upper boundary as the layers become more nearly planar; some indications of this effect appear in Fig. 6.

6. Conclusions

As seen in the synthetic example (section 2), topography complicates the computation of horizontal gradients. Using the maximum vertical shear, Schaefer's (1973) scheme results in some divergence whereas the scheme proposed herein does not. Based on the constant mass flux in the example, the correct answer is thought to be no divergence. Schaefer's scheme gives reasonable results when applied to the surface pressure in the example of section 4. In this case, the vertical gradient of the pressure varies smoothly and a sounding in proximity to the surface data gives an approximate result. However, the proposed scheme gives a more accurate result for interpolation purposes and a horizontal pressure gradient that is closer to the horizontal pressure gradient calculated from the mean sea level pressures. Results from Schaefer's (1973) scheme depend greatly on the accuracy of the vertical gradient that is used.

The example of divergence in complex terrain in section 5 makes use of Schaefer's (1973) scheme difficult. The winds (Figs. 4 and 5) have different shears at different sites. The sites are at markedly different altitudes, so choosing a value of shear is complicated and it would have to be computed numerically. The proposed scheme, by using more data points, is able to resolve a value for both the horizontal and vertical gradients that is accurate to first order. In addition, resolving the solution into meridional and latitudinal gradients is part of the scheme. Schaefer's (1973) scheme, using two points, will give a gradient in the direction defined by the two points. At least one additional point is needed to resolve the orthogonal direction, and then additional computations may be needed to resolve the results into meridional and latitudinal components.

The proposed scheme, based on a Taylor series expansion, may be enhanced to get higher-order derivatives if more data sites are available. There are six second-order terms in the Taylor expansion, requiring 10 observation sites to solve the equations for all of the terms, which greatly complicates the problem. The sites should be either on a 3 × 3 grid with one extra site somewhere or, perhaps, a triangle with four sites per side and one in the center.

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