Comments on "An Investigation of the Slope–Shape Relation for Gamma Raindrop Size Distribution"

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Here I comment on the "theoretical relation" between the gamma distribution slope and shape parameters reportedly discovered by Chu and Su (2008, hereinafter CS08). These authors have presented another analysis of raindrop size observations that contributes to the ongoing debate (e.g., Zhang et al. 2001, 2003; Seifert 2005; Moisseev and Chandrasekar 2007) about the existence of an empirical relationship between the parameters Λ and μ of the gamma drop size distribution function

$$N(D) = N_0 D^{\mu} \exp(-\Lambda D). \tag{1}$$

In the paper they also claim to have derived a "theoretical relation" between those two parameters, but what they have actually done is just to introduce a quantity that could aid in estimating values of the aforementioned parameter pair from empirical data. Indeed, since the gamma distribution has been under study by mathematicians and statisticians for decades it would be remarkable if such a relationship were only now discovered.

To illustrate the problem with the derivation in their section 3b, note that the nth moment M_n of the distribution in Eq. (1) is

$$M_n = \frac{N_0 \Gamma(\mu + n + 1)}{\Lambda^{\mu+n+1}}, \tag{2}$$

where $\Gamma(x)$ is the gamma function. From this, the value of any moment M_n can be expressed in terms of M_0 as

$$M_n = \frac{M_0}{\Lambda^n} \frac{\Gamma(\mu + n + 1)}{\Gamma(\mu + 1)}. \tag{3}$$

Substituting this into CS08's Eq. (20), one obtains

$$\ln M_0 - \ln[\Gamma(\mu + 1)] + \ln\Lambda = \ln[N(D_m)] - \mu \ln(\mu + 1) + (\mu + 1), \tag{4}$$

where D_m is the mean drop diameter. [It is unfortunate that CS08 chose to use D_m to represent the *mean* drop diameter, because that symbol has frequently been used to represent the *mass-weighted mean* diameter (e.g., Weber 1976; Smith 1982; Ulbrich 1983; Bringi and Chandrasekar 2001; Testud et al. 2001). For clarity, the CS08 notation is retained here.] Taking exponentials of both sides and rearranging yields the exact expression

$$\Lambda = \frac{N(D_m)}{M_0} \frac{\exp(\mu + 1)\Gamma(\mu + 1)}{(\mu + 1)^{\mu}}. \tag{5}$$

Comparing Eq. (5) with Eq. (25) of CS08 shows that their essential finding is the approximation (exact value on the right-hand side)

$$\sqrt{2\pi(\mu + 1)} \approx \frac{\exp(\mu + 1)\Gamma(\mu + 1)}{(\mu + 1)^{\mu}}. \tag{6}$$

The approximation improves as μ increases, being in error by just 4.2% when $\mu = 1$ and 1.4% when $\mu = 5$. However, its utility is unclear because calculating values for the right-hand side of Eq. (6) is not difficult.

What of the ratio $N(D_m)/M_0$, which they claim to be "the decisive factor governing the property of the developed μ–Λ relation"? This ratio is

$$\frac{N(D_m)}{M_0} = \frac{\Lambda^{\mu+1} D_m^{\mu} \exp(-\Lambda D_m)}{\Gamma(\mu + 1)} \tag{7}$$

and in fact is just the value of the gamma probability density function (PDF) at $D = D_m$ [i.e., $p(D_m; \Lambda, \mu)$].

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Substituting the value of D_m, $(\mu + 1)/\Lambda$ [CS08's Eq. (9)] into Eq. (7) yields

$$\frac{N(D_m)}{M_0} = \frac{\Lambda(\mu + 1)^{\mu} \exp[-(\mu + 1)]}{\Gamma(\mu + 1)}. \tag{8}$$

The value of the ratio on the left (call it ρ) is readily obtained from the gamma PDF and is fixed by the independent parameters Λ and μ. Thus this ratio provides only a numerical (and not a theoretical) relation between μ and Λ. Figure 10 in CS08 serves only to demonstrate the accuracy of the approximation in their Eq. (25). The expression for the variance of the gamma distribution, CS08's Eq. (29), is well known, and their Fig. 14 also serves only to demonstrate the accuracy of this approximation.

In the present context, one could in principle use Eq. (5) above, or the approximation in CS08's Eq. (25), to substitute a function of ρ for one of the parameters in the gamma drop size distribution function. However, doing so does not yield any useful form of the function. Empirical values of ρ could also be useful in fitting gamma distribution parameters to observed drop size spectra, through Eq. (5) or the CS08 approximation, if the sample sizes are large enough. This would require a second independent relationship between some other empirically based quantity and the two gamma parameters Λ and μ. That is the approach used to develop Fig. 12 in CS08; not surprising, in view of the known bias and error in moment estimators for such parameters (Robertson and Fryer 1970; Smith et al. 2009), is that they find that the use of low-order moments gives better results.

In essence, the CS08 contention that they have developed a ''theoretical relation'' between parameters μ and Λ is equivalent to taking the well-known relationship between the median volume diameter D_v (here again we depart from the conventional notation because CS08 used D_0 to denote the mode of the gamma distribution) and those parameters to be such a theoretical relation:

$$D_v = \frac{\mu + 3.672}{\Lambda}. \tag{9}$$

In principle, one could use this relationship and Eq. (5) above to fit the parameters Λ and μ to empirical drop size observations. However, the solution is extremely sensitive to the numerical values of ρ and D_v and is even double valued over part of the range.

Manipulating expressions involving the ratio $N(D_m)/M_0$ or the value of D_v are comparable exercises in the analysis of raindrop size data, and do not demonstrate any inherent relationship between the independent gamma parameters Λ and μ. Nevertheless, the CS08 data analysis does add to the body of evidence suggesting an empirical relationship between those two parameters in observed raindrop size distributions.

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