A New Process for the Evaluation of Upper Winds

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(Manuscript received 20 December 1965)

ABSTRACT

The present system of evaluating upper winds by means of radar/radio tracking of pilot and radiosonde balloons is in its most general form to be considered as an overdetermined observation system. The redundancy of information in such systems may be used to improve the overall precision of measurement by applying the well-known theory of adjustment of observations. In this paper a scheme for computation is presented, not only for the case of a flat earth, but also for the case including the effect of the earth's curvature. The scheme is particularly feasible for implementation in practice by means of electronic data processing.

The proposed scheme has the advantage that the numerical process also incorporates the conventional modes of operation. Apart from an adjustment procedure a simple smoothing process is described which depends on the accuracy attainable by the measuring technique. This effectuates a partial separation of the instrumental errors from real wind fluctuations. Some numerical data give an impression of the gain in accuracy and the effect of a certain amount of smoothing.

1. Introduction

At present most aerological stations are equipped with some type of radar or radio direction finding device for wind measuring purposes. These devices have replaced the optical theodolites of earlier days, thus enabling wind observations to be made under all weather conditions. Meanwhile, automatic following, data printing and electronic data processing have become new tools for refinement and speeding up of the observation in order to fulfil international standards.

The wind as such is depicted from the three-dimensional path of a balloon-borne target by means of a finite-difference technique, the principles of which are to be found in any textbook on meteorological observation methods.

The horizontal displacement of the target is a direct measure of the wind averaged in a layer which is traversed by the target in a certain time unit and the position to which the wind applies is determined by the target height. With both passive radar techniques (reflectors) and active systems (transponders), the wind is derived from the radar measurements of elevation angle ϵ, azimuth angle α and slant range r by using some elementary trigonometric relations. In the case of radio direction finding, e.g., the GMD-1 system, the lack of slant range information requires some other element to complete the computation. When tracking a pilot balloon, the additional element is the balloon height based on an assumed mean ascent rate, while the height is determined by integrating the hydrostatic equation when tracking a radiosonde. Depending on the available measuring device, one must make a selection out of three basic methods of wind computation, *viz.*, the height-elevation (h,ϵ), range-elevation (r,ϵ) and height-range (h,r) methods.

However, when a pilot balloon or radiosonde is tracked by a distance measuring radio or radar equipment (GMD-2, Decca WF-2 wind finding radar, Selenia Meteor II C), all three methods may be applied alternatively. The results, of course, would be identical if no deterministic and random errors occurred. The presence of instrumental errors, however, causes a differentiation in the precision of measurement. It is this feature which offers the possibility of selecting, in a certain sense, the best method in the radar operation domain. The decision as to which method should be preferred has to be based on a thorough analysis of the accuracy attainable by each method separately. A solution to this problem was given by de Jong (1958). It was shown there that the working domain of radar operation could be divided into three sub-spaces in which one of the basic methods mentioned above gives the most reliable wind structure. Consequently, in establishing a suitable scheme of computation, one should determine in practice where the target traverses the boundaries of these spaces in order to change over from one method to the other. From a practical point of view this solution is not tractable. It is intuitively clear, however, that there should still exist a better approach when an operation can be organized involving, apart from the azimuth α, *all three elements* h, ϵ and r. In other words, there should exist an (h,ϵ,r) method which guarantees a higher precision than could be obtained from any of the basic methods separately in the whole working area of the radar/radio measurement. Moreover, by proper definition of this (h,ϵ,r) operation, its precision should be a maximum.

One can observe that the system under consideration is an *overdetermined* system inasmuch as the number of variables available is in excess of the number necessary for the wind evaluation. The variables, however, are not independent, since for true or exact values of the elements h, ϵ, r, α, a constraint exists of the form

$$\varphi(h,\epsilon,r,\alpha)=0,$$

a specification of which is given in Eq. (2.2).

In order to construct an optimal scheme of computation, one should apply a mathematical discipline known as the theory of adjustment of observations, which is well-known among astronomers and surveyors. The principles of this theory can be found in any textbook on planning and analysis of experiments and methods of least squares, e.g., Linnik (1961).

In this paper the theory is developed for the case of a flat earth. A description of the theory including the effect of the earth's curvature appears to be superfluous as its solution may be presented in precisely the same form as that found for the flat earth case.

In the normally applied theory of adjustment a scheme of computation is used which involves such elements as constraints and normal equations. This scheme has the disadvantage that it lacks symmetry. In this paper a new scheme of computation (Bouman and de Jong, 1964) is followed, ultimately leading to the same numerical result but avoiding this shortcoming by evading constraints and the normal equations. A summary of this method is outlined in the Appendix.

The adjustment procedure only effectuates a displacement of the balloon position in the vertical plane through the radar site, the azimuth angle remaining unchanged. Thus, the present investigation is confined to the horizontal distance and vertical height only. Thereafter, the overall effect is studied in terms of the actual wind computation.

2. Adjustment procedure in terms of horizontal distance

The position of the balloon target is determined by its polar coordinates α, ϵ and h or, alternatively, by the coordinates α, ϵ and r or α, h and r, as shown in Fig. 1.

The horizontal wind component is derived from the projection of the track in polar coordinates α and d, where d represents the ground distance. In case a radiosonde or pilot balloon is tracked by a distance measuring device and the observation is properly synchronized, then it is apparent that the system of wind measurement is really an overdetermined system. Due to instrumental errors it will make a difference whether the balloon track is computed from the sets

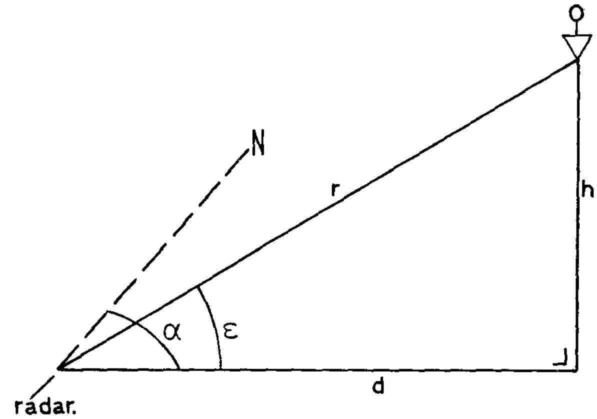

FIG. 1. Arrangement of radar-radiosonde system for wind measuring purposes.

(α,h,ϵ), (α,r,ϵ) or (α,r,h). Consequently, the wind profiles will differ from one another inasmuch as the ground distances are determined from the variables (h,ϵ), (r,ϵ) or (h,r).

Obviously one has (see Fig. 1):

$$\left.\begin{array}{ll}\text{height elevation:} & d_1=d_{(h,\epsilon)}=h\cot\epsilon\\ \text{range elevation:} & d_2=d_{(r,\epsilon)}=r\cos\epsilon\\ \text{range height:} & d_3=d_{(r,h)}=(r^2-h^2)^{\frac{1}{2}}\end{array}\right\}. \qquad (2.1)$$

The error in distance and, consequently, in the wind may be considerable. In the (h,ϵ) profile, for example, when the target moves far from the radar site, the precision is very questionable due to the rapid variation of $\cot\epsilon$ at low elevation angles.

Other wind profiles may be found based on such expressions for the ground distance as $\frac{1}{2}[d_{(h,\epsilon)}+d_{(r,\epsilon)}]$, $\frac{1}{3}[2d_{(r,\epsilon)}+d_{(r,h)}]$, etc. One can therefore state that the number of measured wind profiles is unrestricted since one can always find a new formal expression for the ground distance! The only condition is that these expressions be mutually dependent due to the following constraint in terms of the basic variables h, ϵ, and r:

$$\varphi=h-r\sin\epsilon=0. \qquad (2.2)$$

The question now arises as to whether, within the class of wind profiles, there may be found one which guarantees a minimum error in the whole working domain of the radar operation. This implies an optimization process to be performed according to the procedure outlined in the Appendix. In order to construct a solution we refer to the computational algorithm explained there. Following this algorithm, an expression for the ground distance is sought of the form

$$\mathcal{S}(d)=\lambda_1 d_1+\lambda_2 d_2+\lambda_3 d_3+\cdots\lambda_N d_N,$$

$$\sum_{i=1}^{N}\lambda_i=1,$$

specified in such a way that the variance in terms of

$\mathcal{S}(d)$ is minimized. This leads to a specification of the weight factors λ_i, including the total number N of these factors to be retained. First, one should establish the covariance matrix $\mathbf{S}$ in terms of the elements r, ϵ and h. Although the theory applies for (stochastic) dependent variables, it will be assumed that the measurement of, say, r does not affect the measurement of ϵ or h. Then, if σ_r, σ_ϵ and σ_h denote the standard deviations of r, ϵ and h, the matrix $\mathbf{S}$ takes the (diagonal) form

$$\mathbf{S}=\begin{vmatrix}\sigma_r^2 & 0 & 0\\ 0 & \sigma_\epsilon^2 & 0\\ 0 & 0 & \sigma_h^2\end{vmatrix}. \tag{2.3}$$

Next a column matrix $\mathbf{F}=\begin{vmatrix}d_1\\ \vdots\\ d_N\end{vmatrix}$ is introduced. Since there exists *one* constraint only as shown in Eq. (2.2), $M=1$, and the number N should not exceed $N=M+1=2$. *A combination of two basic expressions will therefore lead to the ultimate goal.* Which elements in $\mathbf{F}$ are chosen is irrelevant as the optimal $\mathcal{S}(d)$ appears to be invariant with respect to a change of column elements of $\mathbf{F}$.

Then, referring to (2.1), we chose $\mathbf{F}$ so that

$$\mathbf{F}=\begin{vmatrix}d_{(h,\epsilon)}\\ d_{(r,\epsilon)}\end{vmatrix}=\begin{vmatrix}h\cot\epsilon\\ r\cos\epsilon\end{vmatrix},$$

where the Jacobian of $\mathbf{F}$ is

$$\mathbf{F}_x=\begin{vmatrix}\frac{\partial}{\partial r}d_{(h,\epsilon)} & \frac{\partial}{\partial \epsilon}d_{(h,\epsilon)} & \frac{\partial}{\partial h}d_{(h,\epsilon)}\\ \frac{\partial}{\partial r}d_{(r,\epsilon)} & \frac{\partial}{\partial \epsilon}d_{(r,\epsilon)} & \frac{\partial}{\partial h}d_{(r,\epsilon)}\end{vmatrix}$$

$$=\begin{vmatrix}0 & \frac{r}{\sin\epsilon} & \cot\epsilon\\ \cos\epsilon & -r\sin\epsilon & 0\end{vmatrix}.$$

Then the covariance matrix $\mathbf{C}$ associated with $\mathbf{F}$ is

$$\mathbf{C}=\mathbf{F}_x\mathbf{S}\mathbf{F}_x^T$$

$$=\begin{vmatrix}0 & \frac{r}{\sin\epsilon} & \cot\epsilon\\ \cos\epsilon & -r\sin\epsilon & 0\end{vmatrix}\begin{vmatrix}\sigma_r^2 & 0 & 0\\ 0 & \sigma_\epsilon^2 & 0\\ 0 & 0 & \sigma_h^2\end{vmatrix}\begin{vmatrix}0 & \cos\epsilon\\ \frac{r}{\sin\epsilon} & -r\sin\epsilon\\ \cot\epsilon & 0\end{vmatrix}.$$

Hence

$$\mathbf{C}=\begin{vmatrix}\frac{r^2}{\sin^2\epsilon}\sigma_\epsilon^2+\cot^2\epsilon\sigma_h^2 & r^2\sigma_\epsilon^2\\ r^2\sigma_\epsilon^2 & \cos^2\epsilon\sigma_r^2+r^2\sin^2\epsilon\sigma_\epsilon^2\end{vmatrix}.$$

The inverse matrix $\mathbf{C}^{-1}$ becomes

$$\mathbf{C}^{-1}=\frac{1}{\cot^2\epsilon(r^2\sigma_r^2\sigma_\epsilon^2+r^2\sin^2\epsilon\sigma_h^2\sigma_\epsilon^2+\cos^2\epsilon\sigma_r^2\sigma_h^2)}\times\begin{vmatrix}\cos^2\epsilon\sigma_r^2+r^2\sin^2\epsilon\sigma_\epsilon^2 & -r^2\sigma_\epsilon^2\\ -r^2\sigma_\epsilon^2 & \frac{r^2}{\sin^2\epsilon}\sigma_\epsilon^2+\cot^2\epsilon\sigma_h^2\end{vmatrix}.$$

Next the sum of row or column elements of $\mathbf{C}^{-1}$, denoted by Σ_1 and Σ_2, is determined, i.e.,

$$\Sigma_1=\frac{\sin^2\epsilon(\sigma_r^2-r^2\sigma_\epsilon^2)}{r^2\sigma_r^2\sigma_\epsilon^2+r^2\sin^2\epsilon\sigma_h^2\sigma_\epsilon^2+\cos^2\epsilon\sigma_r^2\sigma_h^2},$$

$$\Sigma_2=\frac{r^2\sigma_\epsilon^2+\sigma_h^2}{r^2\sigma_r^2\sigma_\epsilon^2+r^2\sin^2\epsilon\sigma_h^2\sigma_\epsilon^2+\cos^2\epsilon\sigma_r^2\sigma_h^2}.$$

The sum Σ of all elements of $\mathbf{C}^{-1}$ is

$$\Sigma=\Sigma_1+\Sigma_2=\frac{\sin^2\epsilon\sigma_r^2+r^2\cos^2\epsilon\sigma_\epsilon^2+\sigma_h^2}{r^2\sigma_r^2\sigma_\epsilon^2+r^2\sin^2\epsilon\sigma_h^2\sigma_\epsilon^2+\cos^2\epsilon\sigma_r^2\sigma_h^2}.$$

Then, finally, the adjusted value $\mathcal{S}(d)$ becomes

$$\mathcal{S}(d)=\lambda_1 d_{(h,\epsilon)}+\lambda_2 d_{(r,\epsilon)}=\frac{\Sigma_1}{\Sigma}d_{(h,\epsilon)}+\frac{\Sigma_2}{\Sigma}d_{(r,\epsilon)},$$

or

$$\boxed{\mathcal{S}(d)=\frac{\sin^2\epsilon(\sigma_r^2-r^2\sigma_\epsilon^2)d_{(h,\epsilon)}+(r^2\sigma_\epsilon^2+\sigma_h^2)d_{(r,\epsilon)}}{\sin^2\epsilon\sigma_r^2+r^2\cos^2\epsilon\sigma_\epsilon^2+\sigma_h^2}}. \tag{2.4}$$

The associated minimum variance $\sigma^2\mathcal{S}_{(d)}=1/\Sigma$, is

$$\boxed{\sigma^2\mathcal{S}_{(d)}=\frac{r^2\sigma_\epsilon^2\sigma_r^2+r^2\sin^2\epsilon\sigma_\epsilon^2\sigma_h^2+\cos^2\epsilon\sigma_r^2\sigma_h^2}{\sin^2\epsilon\sigma_r^2+r^2\cos^2\epsilon\sigma_\epsilon^2+\sigma_h^2}}, \tag{2.5}$$

completing the construction of the adjusted ground distance.

It is seen that $\mathcal{S}(d)$ and $\sigma^2\mathcal{S}_{(d)}$ involve, apart from the variables h, ϵ and r, all σ's. The error reduction is obviously realized by a feed-back of the error analysis into the original computation scheme.

It might be expected that in case one of the σ's of the matrix $\mathbf{S}$ tends to infinity, the expressions (2.4) and (2.5) will reduce to those for the three basic methods separately. For $\varphi=0$, one thus obtains

$$\sigma_r\rightarrow\infty:\quad \mathcal{S}(d)=d_{(h,\epsilon)},$$

$$\sigma^2\mathcal{S}_{(d)}=\frac{h^2}{\sin^4\epsilon}\sigma_\epsilon^2+\cot^2\epsilon\sigma_h^2=\sigma^2 d_{(h,\epsilon)}, \tag{2.6}$$

$$\sigma_h\rightarrow\infty:\quad \mathcal{S}(d)=d_{(r,\epsilon)},$$

$$\sigma^2\mathcal{S}_{(d)}=r^2\sin^2\epsilon\sigma_\epsilon^2+\cos^2\epsilon\sigma_r^2=\sigma^2 d_{(r,\epsilon)}, \tag{2.7}$$

$$\sigma_\epsilon \to \infty: \quad \mathcal{S}(d) = -\tan^2\epsilon d_{(h,\epsilon)} + \frac{1}{\cos^2\epsilon} d_{(r,\epsilon)},$$

$$\sigma^2_{\mathcal{S}(d)} = \frac{1}{r^2-h^2}(r^2\sigma_r^2 + h^2\sigma_h^2) = \sigma^2_{d(h,r)}. \qquad (2.8)$$

It is easily verified that the above variances are nothing but those obtained by application of the propagation law to (2.1). It is striking to see, however, that all basic methods are reproduced except for the (h,r) method (2.8). Here, for $\sigma_\epsilon \to \infty$, the ground distance persists in a "pooling" of the (r,ϵ) and (h,ϵ) modes of operation. Although the (r,ϵ) and (h,ϵ) methods are now highly inaccurate, the weight factors appear to have a compensating effect causing large errors in $d_{(h,\epsilon)}$ and $d_{(r,\epsilon)}$ to cancel each other. These remarks are important in regard to the electronic data processing of the optimal (h,ϵ,r) method. By substituting for one of the σ's as $\sigma \to \infty$, the computer program changes automatically into a program for one of the basic modes of operation presently in use.

3. Adjustment procedure in terms of the balloon height

The height of the balloon has to be evaluated in order to determine the level of reference for the wind. Apparently, there again exist different versions for the height formula.

On the one hand we have

$$h_{\text{radar}} = r\sin\epsilon, \qquad (3.1)$$

while on the other, the height may be found by integrating the hydrostatic equation (radiosonde), i.e.,

$$h = -\int_{p_0}^{p} RT_v d\ln p, \qquad (3.2)$$

where R is the gas constant for dry air, T_v the virtual temperature and p pressure, or by use of a mean ascent rate A (pilot balloon),

$$h = At. \qquad (3.3)$$

Both expressions (3.2) and (3.3) will be brought under one heading h_{balloon} with associated standard deviation σ_h.

One could set up a procedure for determining the optimal quantity $\mathcal{S}(h)$ and minimum variance $\sigma^2_{\mathcal{S}(h)}$ similar to that for $\mathcal{S}(d)$, but a closer examination of h_{radar} and h_{balloon} reveals that they have no variables in common. Under these circumstances the solution to the problem is very simple. A construction of $\mathcal{S}(h)$ analogous to $\mathcal{S}(d)$ may be avoided by merely observing that the best linear combination of h_{radar} and h_{balloon} is nothing but the weighted mean value of both, *i.e.*,

$$\mathcal{S}(h) = \frac{\sigma^2_{h\text{radar}} \cdot h_{\text{balloon}} + \sigma^2_{h\text{balloon}} \cdot h_{\text{radar}}}{\sigma^2_{h\text{radar}} + \sigma^2_{h\text{balloon}}},$$

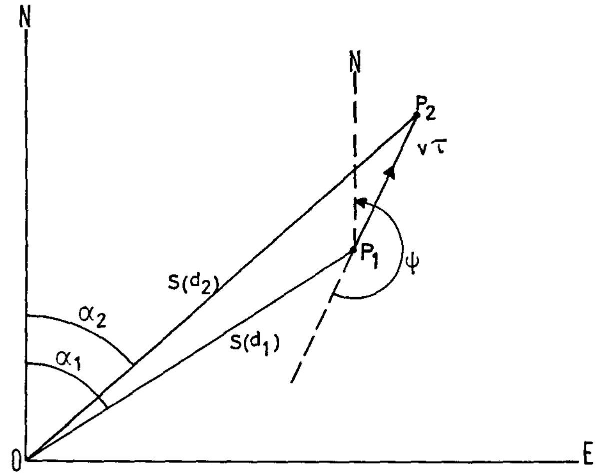

FIG. 2. Horizontal projection of balloon trajectory.

and

$$\sigma^2_{\mathcal{S}(h)} = \frac{\sigma^2_{h\text{radar}} \cdot \sigma^2_{h\text{balloon}}}{\sigma^2_{h\text{radar}} + \sigma^2_{h\text{balloon}}}.$$

Referring to (3.1), we have, in view of the law of propagation of errors,

$$\sigma^2_{h\text{radar}} = \sin^2\epsilon\sigma_r^2 + r^2\cos^2\epsilon\sigma_\epsilon^2.$$

It follows that

$$\boxed{\mathcal{S}(h) = \frac{(\sin^2\epsilon\sigma_r^2 + r^2\cos^2\epsilon\sigma_\epsilon^2)h_{\text{balloon}} + \sigma_h^2 h_{\text{radar}}}{\sin^2\epsilon\sigma_r^2 + r^2\cos^2\epsilon\sigma_\epsilon^2 + \sigma_h^2}} \qquad (3.4)$$

and

$$\boxed{\sigma^2_{\mathcal{S}(h)} = \frac{\sin^2\epsilon\sigma_r^2\sigma_h^2 + r^2\cos^2\epsilon\sigma_h^2\sigma_\epsilon^2}{\sin^2\epsilon\sigma_r^2 + r^2\cos^2\epsilon\sigma_\epsilon^2 + \sigma_h^2}} \qquad (3.5)$$

completing the adjustment procedure for the height. Incidentally, it is observed that for one of the σ's tending to infinity, $\mathcal{S}(h)$ reduces to h_{radar} and h_{balloon} with associated standard errors $\sigma_{h\text{radar}}$ and $\sigma_{h\text{balloon}}$.

4. Wind

The wind structure is effected indirectly by the procedure of adjustment inasmuch the projection of the balloon track (α,d) is displaced to the position $[\alpha,\mathcal{S}(d)]$ and the level of the wind h is replaced by $\mathcal{S}(h)$.

Referring to Fig. 2, the wind speed is given by

$$v = \frac{\{\mathcal{S}^2(d_1) + \mathcal{S}^2(d_2) - 2\mathcal{S}(d_1)\mathcal{S}(d_2)\cos(\alpha_1-\alpha_2)\}^{\frac{1}{2}}}{\tau}, \qquad (4.1)$$

where τ denotes the time interval between consecutive positions. If the azimuth is measured clockwise with respect to north, the wind direction becomes

$$\psi=\pi+\alpha_2-\arcsin\left\{\frac{\mathcal{S}(d_1)}{v\tau}\sin(\alpha_1-\alpha_2)\right\}. \qquad (4.2)$$

Since wind is a vector quantity one generally introduces as a measure of precision the standard vector deviation σ, defined by

$$\sigma^2=\overline{v^2}-v_R^2,$$

where $\overline{v^2}$ is the mean square wind speed and v_R^2 is the square of the mean wind vector.

The computation of σ proceeds along well-known lines but is somewhat lengthy. The result, however, is quite intelligible. For a point (d,α) with associated standard errors (σ_d,σ_α), one has

$$\sigma=\frac{\sqrt{2}}{\tau}(d^2\sigma_\alpha^2+\sigma_d^2)^{\frac{1}{2}},$$

where, strictly speaking, the point (d,α) refers to an intermediate point on the segment P_1P_2 (Fig. 2), but reference to P_1 or P_2 is a good approximation.

For the three basic modes of operation and the optimal (h,ϵ,r) method, we thus have:

$$\left.\begin{aligned}
(h,\epsilon):&\quad \sigma=\frac{\sqrt{2}}{\tau}(d^2_{(h,\epsilon)}\sigma_\alpha^2+\sigma^2_{d(h,\epsilon)})^{\frac{1}{2}}\\
(r,\epsilon):&\quad \sigma=\frac{\sqrt{2}}{\tau}(d^2_{(r,\epsilon)}\sigma_\alpha^2+\sigma^2_{d(r,\epsilon)})^{\frac{1}{2}}\\
(h,r):&\quad \sigma=\frac{\sqrt{2}}{\tau}(d^2_{(h,r)}\sigma_\alpha^2+\sigma^2_{d(h,r)})^{\frac{1}{2}}\\
(h,\epsilon,r):&\quad \sigma=\frac{\sqrt{2}}{\tau}(\mathcal{S}^2(d)\sigma_\alpha^2+\sigma^2_{\mathcal{S}(d)})^{\frac{1}{2}}
\end{aligned}\right\}. \qquad (4.3)$$

Note that σ is inversely proportional to τ. At a particular point of the track the terms within parentheses remain unchanged, so that

$$\sigma\tau=\text{constant}. \qquad (4.4)$$

To obtain a detailed structure of the wind one could decrease τ indefinitely but at the cost of the accuracy of the wind vector. With an increase of the time unit, σ will decrease proportionally. This is another statement of the fact that the wind averaged in a layer of thickness $A\tau$ will be more accurate in thicker layers. The discussion of the quantitative results is postponed until later, when the effect of the earth's curvature will be incorporated in the computation scheme.

5. Summary

Summarizing the results in the previous sections, we will, in particular, stress the symmetry of the scheme. This not only facilitates the programming for electronic data processing but has the additional advantage that the scheme will also apply to the problem of including the effect of the earth's curvature after a proper choice of some parameters has been made (see Section 6).

For reasons of completeness the scheme also involves expressions for $\mathcal{S}(d)$ which would have resulted after starting the algorithm by pooling of, say, $d_{(h,\epsilon)}$ and $d_{(h,r)}$. In addition, σ_ϵ is replaced by the error in arc distance $r\sigma_\epsilon$.

Introducing the parameters

$$\mu_1=\sin\epsilon$$
$$\mu_2=\cos\epsilon$$
$$\mu_3=(\mu_1^2+\mu_2^2)^{\frac{1}{2}}$$

we obtain the following summary:

a. Ground distance

(h,ϵ) mode:

$$d_{(h,\epsilon)}=\mu_2 r,$$

$$\sigma^2_{d(h,\epsilon)}=\frac{1}{\mu_3^2\mu_1^2}(\mu_3^2\sigma_\epsilon^2+\mu_2^2\sigma_h^2).$$

(r,ϵ) mode:

$$d_{(r,\epsilon)}=\frac{\mu_2}{\mu_1}h,$$

$$\sigma^2_{d(r,\epsilon)}=\frac{1}{\mu_3^2\mu_3^2}(\mu_2^2\sigma_r^2+\mu_1^2\sigma_\epsilon^2).$$

(h,r) mode:

$$d_{(h,r)}=(r^2-h^2)^{\frac{1}{2}},$$

$$\sigma^2_{d(h,r)}=\frac{1}{\mu_3^2\mu_2^2}(\mu_1^2\sigma_h^2+\mu_3^2\sigma_r^2).$$

(h,ϵ,r) mode:

$$\mathcal{S}(d)=\frac{\mu_1^2(\sigma_r^2-\sigma_\epsilon^2)d_{(h,\epsilon)}+\mu_3^2(\sigma_\epsilon^2+\sigma_h^2)d_{(r,\epsilon)}}{\mu_1^2\sigma_r^2+\mu_2^2\sigma_\epsilon^2+\mu_3^2\sigma_h^2},$$

or

$$\mathcal{S}(d)=\frac{\mu_2^2(\sigma_\epsilon^2+\sigma_h^2)d_{(h,r)}+\mu_1^2(\sigma_r^2+\sigma_h^2)d_{(h,\epsilon)}}{\mu_1^2\sigma_r^2+\mu_2^2\sigma_\epsilon^2+\mu_3^2\sigma_h^2},$$

or

$$\mathcal{S}(d)=\frac{\mu_3^2(\sigma_r^2+\sigma_h^2)d_{(r,\epsilon)}+\mu_2^2(\sigma_\epsilon^2-\sigma_r^2)d_{(h,r)}}{\mu_1^2\sigma_r^2+\mu_2^2\sigma_\epsilon^2+\mu_3^2\sigma_h^2},$$

and

$$\sigma^2_{\mathcal{S}(d)}=\frac{\mu_1^2\sigma_h^2\sigma_\epsilon^2+\mu_2^2\sigma_r^2\sigma_h^2+\mu_3^2\sigma_r^2\sigma_\epsilon^2}{\mu_1^2\sigma_r^2+\mu_2^2\sigma_\epsilon^2+\mu_3^2\sigma_h^2}.$$

b. Height

(r,ϵ) mode:

$$h_{\text{radar}} = \mu_1 r,$$

$$\sigma^2_{h\text{radar}} = \frac{1}{\mu_3^2}(\mu_1^2\sigma_r^2 + \mu_2^2\sigma_\epsilon^2).$$

(h,ϵ); (h,r) modes:

$$h_{\text{balloon}} = -\int_{p_0}^{p} RT_v d\ln p,$$

or

$$h_{\text{balloon}} = At,$$

and

$$\sigma^2_{h\text{balloon}} = \frac{1}{\mu_3^2}\mu_3^2\sigma_h^2.$$

(h,ϵ,r) mode:

$$\mathcal{S}(h) = \frac{\mu_3^2\sigma_h^2 h_{\text{radar}} + (\mu_1^2\sigma_r^2 + \mu_2^2\sigma_\epsilon^2)h_{\text{balloon}}}{\mu_1^2\sigma_r^2 + \mu_2^2\sigma_\epsilon^2 + \mu_3^2\sigma_h^2},$$

$$\sigma^2_{\mathcal{S}(h)} = \frac{\mu_1^2\sigma_r^2\sigma_h^2 + \mu_2^2\sigma_\epsilon^2\sigma_h^2}{\mu_1^2\sigma_r^2 + \mu_2^2\sigma_\epsilon^2 + \mu_3^2\sigma_h^2}.$$

c. Wind

(h,ϵ) mode: $$\sigma^2 = \frac{2}{\tau^2}(d^2_{(h,\epsilon)}\sigma_\alpha^2 + \sigma^2_{d(h,\epsilon)}).$$

(r,ϵ) mode: $$\sigma^2 = \frac{2}{\tau^2}(d^2_{(r,\epsilon)}\sigma_\alpha^2 + \sigma^2_{d(r,\epsilon)}).$$

(h,r) mode: $$\sigma^2 = \frac{2}{\tau^2}(d^2_{(h,r)}\sigma_\alpha^2 + \sigma^2_{d(h,r)}).$$

(h,ϵ,r) mode: $$\sigma^2 = \frac{2}{\tau^2}(\mathcal{S}(d)^2\sigma_\alpha^2 + \sigma^2_{\mathcal{S}(d)}).$$

6. The effect of the earth's curvature

In strong winds the balloon-borne target drifts far from the radar site. Under these conditions the influence of the earth's curvature must be taken into account. This necessitates a review of the whole scheme of wind computation where the effect is strongest in the balloon height. Obviously, we have to repeat the calculations for coordinates which are now partly curvilinear.

Consider Fig. 3. The balloon target B is tracked by a precision radar or radio-goniometer situated at point O. M is the midpoint of the earth. Using the notation in the figure and applying the sine rule in triangle OMB,

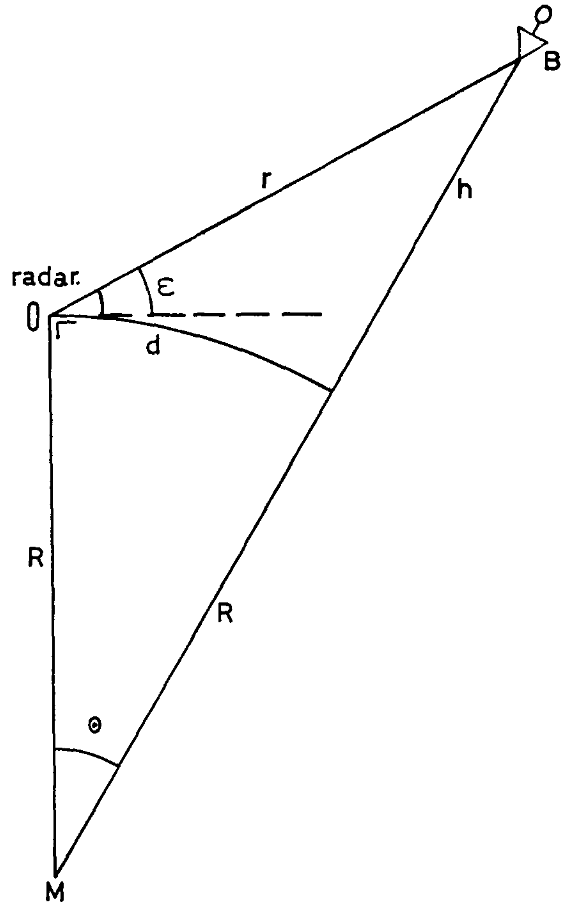

FIG. 3. Arrangement of radar-radiosonde system including the effect of the earth's curvature.

we have

$$\frac{\sin\theta}{\cos\epsilon} = \frac{r}{R+h}, \tag{6.1}$$

and

$$r^2 = R^2 + (R+h)^2 - 2R(R+h)\cos\theta, \tag{6.2}$$

when applying the cosine rule.

As the system is overdetermined, a constraint exists between slant range r, balloon height h and elevation angle ϵ. The constraint is easily derived by application of another cosine rule in triangle OMB, i.e.,

$$\varphi = R^2 + r^2 - (R+h)^2 + 2Rr\sin\epsilon = 0. \tag{6.3}$$

In the curvilinear coordinate system the location of B is determined further by the arc distance $d = R\theta$.

Starting from the arc distance, expressions can now be found which only involve two of the three elements h, ϵ and r. Thus in the (h,ϵ) mode, for example, we have

$$d_{(h,\epsilon)} = R\arcsin\left(\frac{-R\sin\epsilon \pm \{(R+h)^2 - R^2\cos^2\epsilon\}^{\frac{1}{2}}}{R+h}\cos\epsilon\right) \tag{6.4}$$

after eliminating r from (6.1) and (6.3). By eliminating

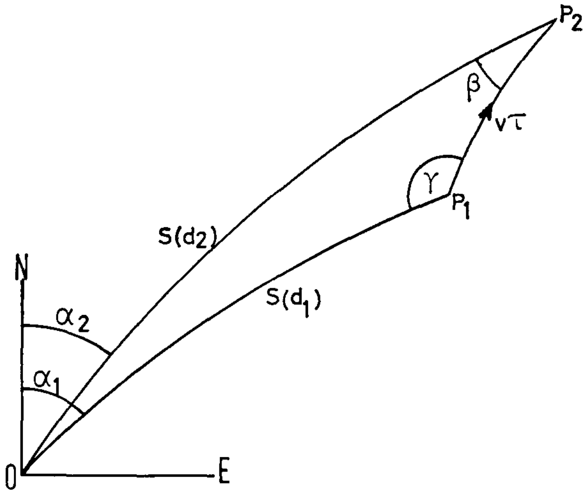

FIG. 4. Projection of balloon trajectory on the earth's surface.

h in (6.1) and (6.3),

$$d_{(r,\epsilon)}=R\text{ arc sin}\left(\frac{r\cos\epsilon}{(R^2+r^2+2rR\sin\epsilon)^{\frac{1}{2}}}\right), \quad (6.5)$$

$$d_{(h,r)}=R\text{ arc cos}\left(\frac{R^2+(R+h)^2-r^2}{2R(R+h)}\right). \quad (6.6)$$

From (6.3) we obtain

$$h_{\text{radar}}=(r^2+R^2+2rR\sin\epsilon)^{\frac{1}{2}}-R. \quad (6.7)$$

The formulae for h_{balloon} remain unchanged.

The adjustment procedure may now be performed in a manner analogous to that for the flat earth's problem using the algorithm explained in the Appendix. Since the calculation may be repeated almost verbally, its explicit presentation is therefore omitted. If one introduces the parameters

$$\mu_1=\sin\epsilon+\frac{r}{R},$$

$$\mu_2=\cos\epsilon,$$

$$\mu_3=(\mu_1{}^2+\mu_2{}^2)^{\frac{1}{2}},$$

it turns out that the final result is exactly the same as that presented in the summary in Section 5, with the understanding, of course, that the formulae for the ground distance and height are to be taken from (6.4), (6.5), (6.6) and (6.7) in this section.

One notes that for a flat earth solution ($R\rightarrow\infty$), $r/R\rightarrow 0$, and the parameters μ_1, μ_2, μ_3 reduce to those in Section 5.

One should be aware of the fact that, due to the earth's curvature, the correspondence of wind and balloon displacement must be based on an analysis in spherical triangles. Let OP_1P_2 be such a triangle as projected from the midpoint M on the earth's surface (Fig. 4). Then the cosine rule gives

$$\cos P_1P_2=\cos S(d_1)\cos S(d_2)+\sin S(d_1)\sin S(d_2)\cos(\alpha_2-\alpha_1).$$

It follows that the wind speed at the level h will be

$$v=\left(1+\frac{h}{R}\right)\frac{\text{arc}P_1P_2}{\tau}.$$

To find the angle β opposite arc OP_1, one has

$$\cos S(d_1)=\cos P_1P_2\cos S(d_2)+\sin P_1P_2\sin S(d_2)\cos\beta.$$

A similar expression holds for the angle γ opposite arc OP_2.

The wind direction then follows from an appropriate reference with respect to the local north-south direction in P_1 and P_2, respectively.

It is to be expected that the expressions for the standard vector error σ will be very complicated. However, in view of the small working domain relative to the earth's radius, the expressions for σ in Section 5 may be maintained without introducing gross errors.

7. Quantitative analysis

It is of practical interest to have an impression of the efficiency of the new approach and, in particular, of the overall error reduction in terms of the conventional methods. An appropriate quantitative analysis has been performed for the system in use in the Netherlands consisting of the British Mark II B sonde tracked by a Selenia Meteor 200 RMT/1C radar. The investigation involves an analysis of the accuracy of the balloon height and wind based on the scheme in Section 5 including the effect of the earth's curvature.

a. Height. First of all we ought to have a good estimate of the covariance matrix **S**. In the performance characteristics the manufacturer mentions a range accuracy for the Selenia radar of $\sigma_r=25$ m and azimuth and elevation accuracies of $\sigma_\alpha=\sigma_\epsilon=0.1°=1.745$ milliradians. σ_h relates to an ensemble of radiosonde ascents where experimental values are seldom available. The main sources of error are to be attributed here by the instability and drift of the pressure and temperature sensors. According to figures published by Harrison (1962), a relative error of 2 per cent is a rather good approximation, although at heights above 20 km the pressure error causes somewhat greater deficiencies. For correct temperature measurements the adoption of the 2 per cent relative error corresponds to a standard error in the geopotential at 700 mb of 2.5 gpm, at 500 mb of 8.5 gpm and at 300 mb of 28 gpm.

A good estimate for **S** is therefore

$$\mathbf{S}=\begin{vmatrix}25^2 & 0 & 0\\ 0 & 0.00174^2 & 0\\ 0 & 0 & (0.02h)^2\end{vmatrix}.$$

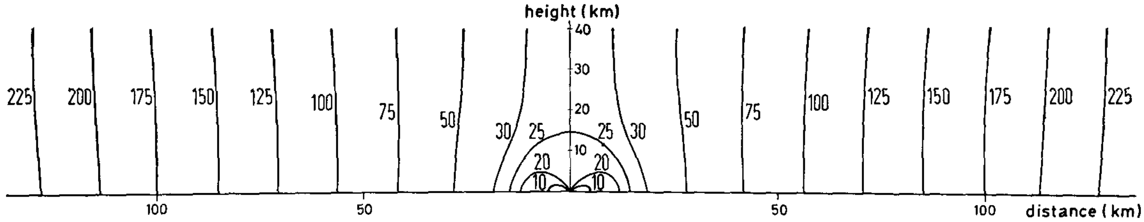

FIG. 5. Isopleths in meters for the standard height error $\sigma_{(r,\epsilon)}$ in the vertical plane for a range-elevation operation. $\sigma_r=25$ m, $\sigma_\epsilon=0.1°$.

The graphs in Figs. 5, 6 and 7 show isopleths of the standard error σ_h for the (r,ϵ) mode, for a radiosonde ascent and for the (h,ϵ,r) mode in a vertical cross section with the origin at the radar station. For an impression of the spacial distribution the figures have to be rotated about the h-axis. Due to the earth's radius the horizontal axis is slightly curved. A close examination of the graphs indicates that the optimal mode of operation is a compromise between the (r,ϵ) method at high levels and the radiosonde ascent proper at low levels. The gain in accuracy is considerable.

b. Wind. Before proceeding with an investigation of the precision of wind measurements, it is important to note that the standard error σ_h in terms of the balloon height has hitherto referred to an ensemble of radiosonde or pilot balloon ascents. The wind vector, however, is affected by the height error in an individual ascent. In the ensemble of ascents, each ascent involves, in general, a systematic deviation due to drift of the sensor elements, i.e., a deviation of real ascent rate in terms of the theoretical value.

In the ensemble these systematic deviations are obscured. In an individual ascent the interdependency of consecutive locations considerably reduces the overall effect of the systematic deviations on the wind. The insertion of interdependency is mathematically difficult but it can be avoided by reasoning as follows: Suppose one knows the systematic deviation of h in an individual ascent. Let this deviation be denoted by Δ (positive or negative). Then $\mathcal{S}(h\pm\Delta)$ and $\mathcal{S}(d)$ can be calculated by replacing h by $h\pm\Delta$ and σ_h by the random error σ_h^* which is superimposed on Δ and which is inherent in the random fluctuations of the sensors. Then $\mathcal{S}(h\pm\Delta)$ and $\mathcal{S}(d)$ will incorporate a smaller residual error than before. The same holds for the wind. Since in reality, however, Δ is unknown and Δ/h, in general, is not excessively high, it is plausible to calculate the standard vector error with good approximation by merely inserting for σ_h the value σ_h^* and replacing $h\pm\Delta$ again by h in the scheme of Section 5. From this it is permissible to investigate the error distribution of the wind vector by substituting for $\mathbf{S}$ the matrix

$$\mathbf{S}=\begin{vmatrix} 25^2 & 0 & 0 \\ 0 & 0.00174^2 & 0 \\ 0 & 0 & 20^2 \end{vmatrix}$$

in the proposed scheme where 20 m has been taken for σ_h^*.

Figs. 8, 9, 10 and 11 show the results for the (h,ϵ), (r,ϵ), (h,r) and optimal (h,ϵ,r) modes, respectively. In addition, Fig. 12 indicates the error distribution and subdivision of the domain of radar operation when the best of the three elementary modes is selected. Here σ is defined by

$$\sigma=\min[\sigma_{(h,\epsilon)},\sigma_{(r,\epsilon)},\sigma_{(h,r)}].$$

This subdivision reviews the state of affairs reached in an earlier investigation (de Jong, 1958).

The efficiency of the adjustment procedure can be studied from a comparison of Fig. 11 with Figs. 8, 9 and 10. The reduction in wind vector error is considerable with respect to the (h,ϵ) mode at lower levels and is also considerable with respect to the (h,r) method in a region about the zenith. With respect to the (r,ϵ) method, however, the gain in accuracy is not impressive.

The conclusion seems justified that the new approach is definitely worthy of application in practice, especially in terms of the height to which the wind refers.

8. Error dependent smoothing

In routine practice, where by international agreement, the wind evaluation, coding and distribution should be performed in due time, the elementary methods referred

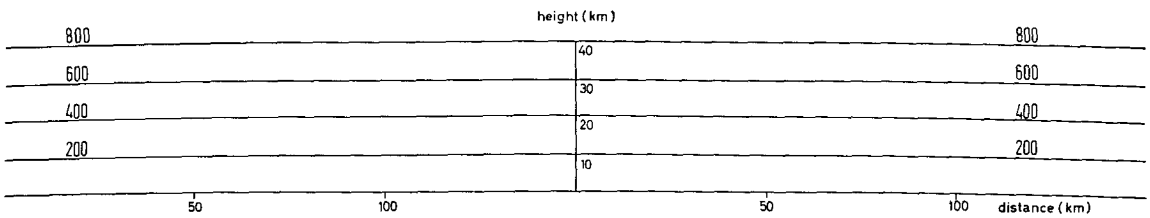

FIG. 6. Isopleths in meters for the standard height error $\sigma_h=0.02h$ in the vertical plane for a radiosonde.

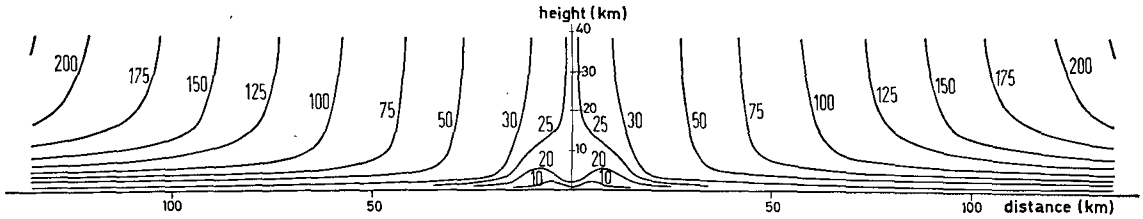

FIG. 7. Isopleths in meters for the standard height error $\sigma_{(h,\epsilon,r)}$ in the vertical plane for the (optimal) height-elevation-range operation.

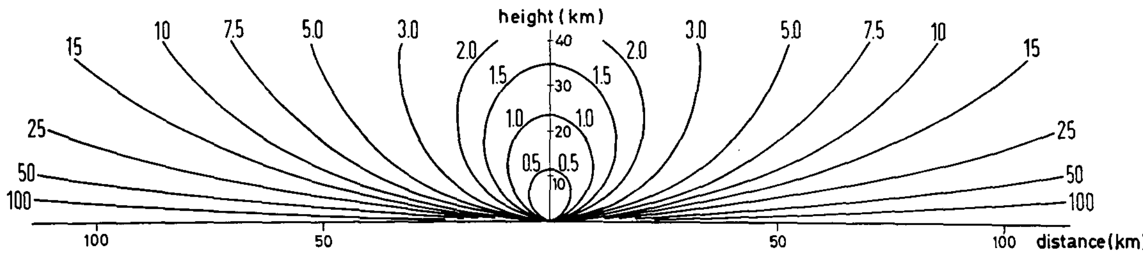

FIG. 8. Isopleths in m sec^{-1} for the standard vector error σ for the wind in the vertical plane using the height-elevation mode of operation. $\sigma_h{}^*=20$ m, $\sigma_\alpha=\sigma_\epsilon=0.1°$.

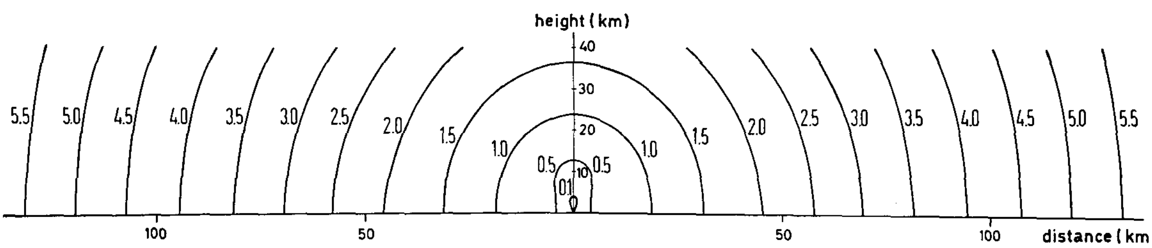

FIG. 9. Isopleths in m sec^{-1} for the standard vector error σ for the wind in the vertical plane using the range-elevation mode of operation. $\sigma_r=25$ m, $\sigma_\alpha=\sigma_\epsilon=0.1°$.

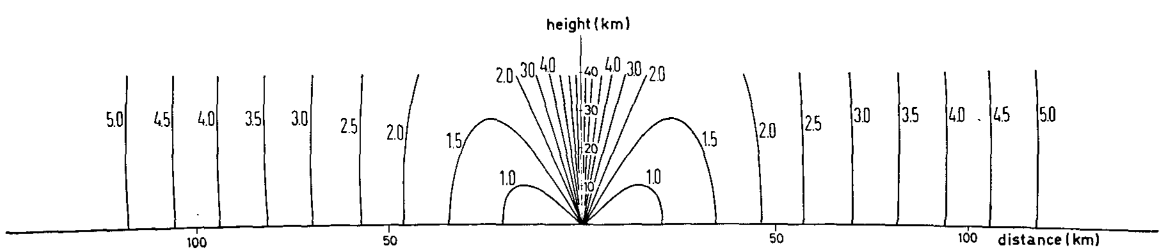

FIG. 10. Isopleths in m sec^{-1} for the standard vector error σ for the wind in the vertical plane using the range-height mode of operation. $\sigma_r=25$ m, $\sigma_h{}^*=20$ m, $\sigma_\alpha=0.1°$.

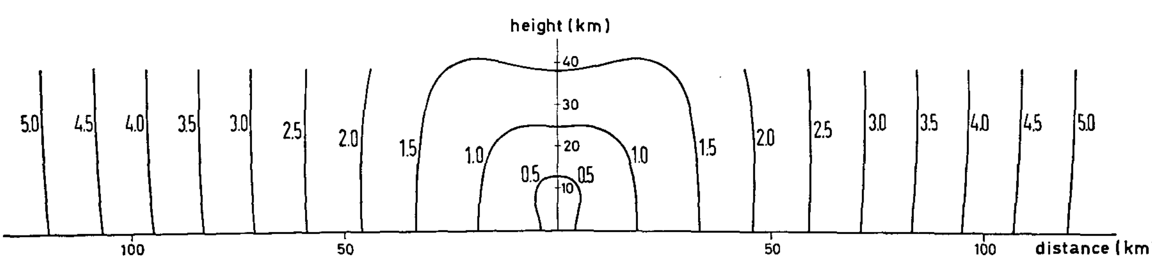

FIG. 11. Isopleths in m sec^{-1} for the standard vector error σ for the wind in the vertical plane using the optimal height-elevation-range method. $\sigma_h{}^*=2.0$ m, $\sigma_\alpha=\sigma_\epsilon=0.1°$, $\sigma_r=25$ m.

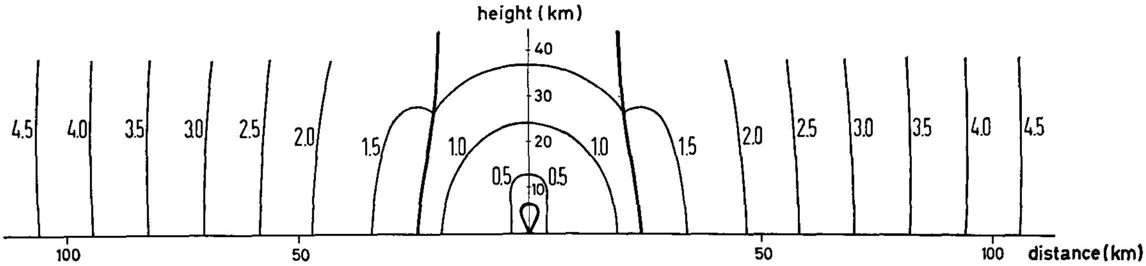

FIG. 12. Isopleths in m sec^{-1} for the standard vector error σ for the wind in case of the best selection of the (h,ϵ), (r,ϵ) and (r,h) modes of operation. Thick lines delineate the subspaces where one of these modes gives the most accurate wind profile.

to above as (h,ϵ), (r,ϵ) and (h,r) lend themselves fairly well to quick graphical and numerical techniques where the track of the balloon is most frequently plotted on a polar diagram. Wind speed and direction are derived from measuring the distance and orientation of segments between consecutive positions, usually one minute apart. In more sophisticated systems with data printing (GMD 1 and 2), the positions may be available each 6 sec. There is thus a certain threshold, determined by such factors as the amount of data, speed of processing and cost, above which electronic computers do a better job. The complexity of the computation scheme is another factor which may ultimately lead to the decision to find the wind structure by electronic data processing. This is particularly true for the scheme of optimum adjustment outlined in the previous sections. However, as soon as machine calculation comes to the fore another problem remains to be solved which is closely related to the precision of measurement. This concerns the problem of smoothing. Even after improving the balloon positions according to the principles of the adjustment theory, there remains some scatter of the points along the track. This is caused not only by instrumental errors but also by fluctuations in the wind which actually exist in the atmosphere, as was shown by Barbé (1958). Usually the operator is inclined to rearrange the scattered points by trial and error until they are regularly distributed in a proper chronological order and the overall trajectory has a smooth appearance. Being unaware of the position error due to instrumental errors, there is, however, a chance that the operator, apart from suppressing the influence of the instrumental errors, will eliminate some real wind fluctuations.

Applying electronic calculation, however, there is a possibility, at least in part, to separate the signal from

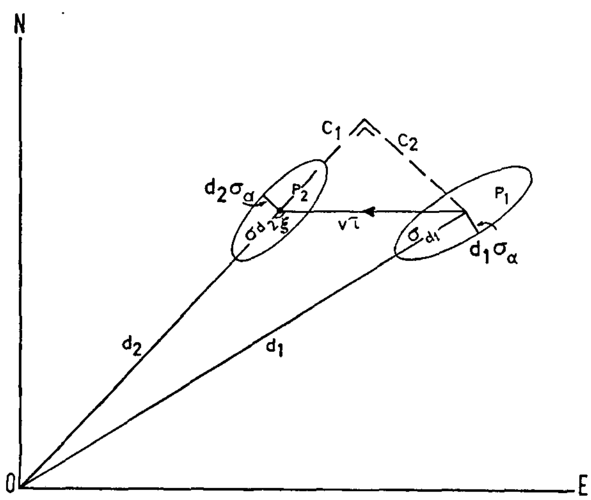

FIG. 13. Error ellipses for consecutive projected balloon positions.

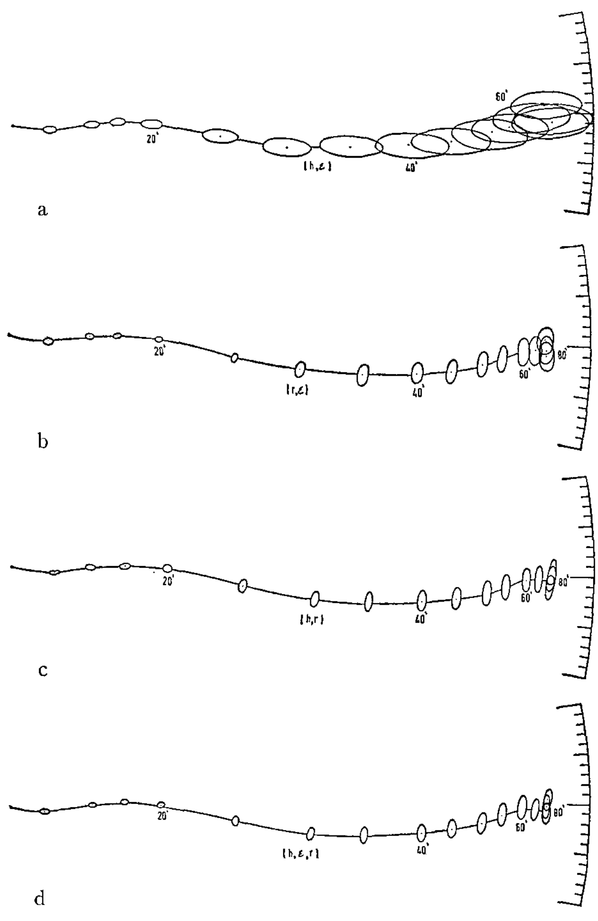

FIG. 14. Projection of an experimental balloon track as computed in the (h,ϵ), (r,ϵ), (h,r) and (h,ϵ,r) modes, a–d, respectively. Positions are shown 5 min apart with associated error ellipses for 5 times the standard deviation.

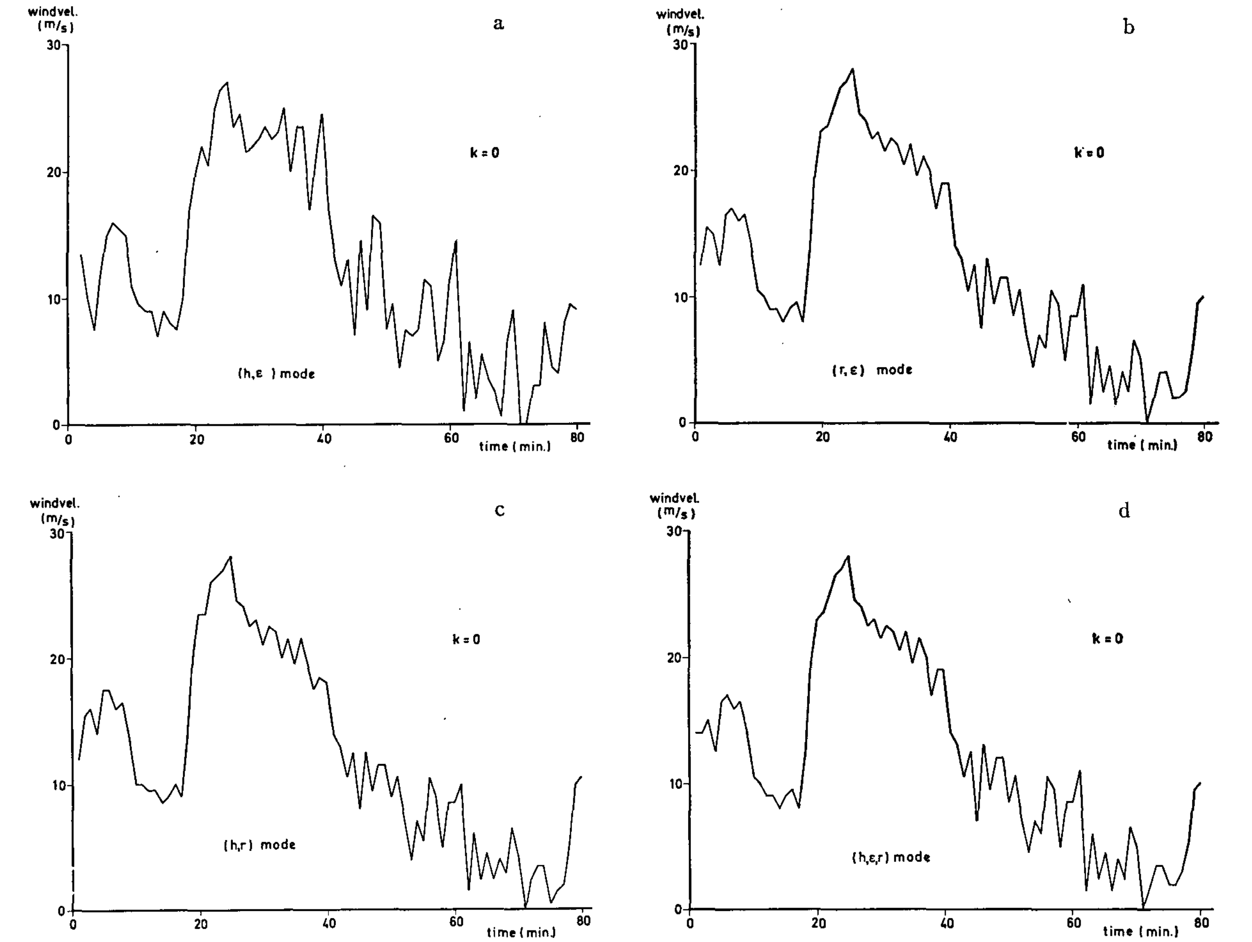

FIG. 15. Wind speed in m sec^{-1} vs. time in min for an experimental balloon flight without smoothing ($k=0$): a, wind profile in the (h,ϵ) mode; b, wind profile in the (r,ϵ) mode; c, wind profile in the (r,h) mode; d, wind profile in the (h,ϵ,r) mode.

the noise. This may be accomplished by a process of *conditional smoothing*, i.e., a smoothing which depends on the scatter of points due to instrumental errors only. Such an error dependent smoothing can easily be inserted in the computer program, the principles of which will now be described.

We consider the balloon projection, the true positions being unknown. Under the assumption that the location is normally distributed with respect to the true position, each projection point is surrounded by an ellipse in which the true position will be found with a certain probability (see Fig. 13). The "error ellipse" is nothing but the intersection of a cone about the radar beam and the horizontal plane through the balloon's real location in space. The principal axes of the ellipses are directed towards and normal to the radar site, respectively, and the size and eccentricity are dependent on the elements which describe the location of the target.

It is relatively simple to evaluate these error ellipses. The axis towards the origin for an ellipse of 68 per cent probability is given by σ_d and the axis normal to it by $d\sigma_\alpha$. It is obvious that the ellipses differ in orientation and size depending on which of the (h,ϵ), (r,ϵ), (h,r) or (h,ϵ,r) modes of operation are used. Fig. 14 shows the horizontal tracks for a pilot balloon ascent as determined from all four computational methods where 5-min positions are given with associated error ellipses for 5 times the standard error. If a process of smoothing is developed, a comprehensive criterion of smoothing will not only depend on the size of the ellipses but also on their orientation with respect to the wind displacement. Where the error ellipses in terms of 1-min distances are very small, smoothing makes no sense. Before establishing a criterion it is recommended that either the azimuth angle or the horizontal distance, or both, be smoothed. As the ellipses are orientated towards the origin, the criterion can be split up into two parts, one for overlapping in the tangential direction and the other at right angles. Then taking the displacement $v\tau$, where v denotes wind speed and τ the time increment, the criterion can be based on the components c_1 and c_2 of this displacement in the radial and tangential directions, i.e.,

$$c_1 = v_T |\cos\xi|,$$
$$c_2 = v_T |\sin\xi|,$$

where ξ denotes the angle between the balloon track and the radius vector towards the origin. These components are compared with the magnitude of the radial and tangential axes a_1 and a_2 of the error ellipses associated with neighboring positions in such a way that overlapping takes place for 68, 95, 99, $\cdots$ per cent probability limits, i.e.,

$$a_1 = k(\sigma_{d_1} + \sigma_{d_2}),$$
$$a_2 = k(d_1\sigma_\alpha + d_2\sigma_\alpha),$$

where the *smoothing factor* $k = 1, 2, 3, \cdots$ refers to the above probability limits. The introduction of a smoothing factor enables one to bring some flexibility into the criterion.

The criterion will be

for $c_1 \geqslant a_1$ and $c_2 \geqslant a_2$, no smoothing at all;
for $c_1 \geqslant a_1$ and $c_2 < a_2$, smoothing of the azimuthal angle;
for $c_1 > a_1$ and $c_2 \geqslant a_2$, smoothing of the horizontal distance;
for $c_1 > a_1$ and $c_2 < a_2$, smoothing of both azimuth and distance.

The method of smoothing to be advocated is a matter of taste. Observations in the (α,t) plane as well as in the (d,t) plane are equidistant but nothing is known of the path of the functions. One method consists in drawing a straight line through 3, 5, $\cdots$ neighboring points and determining a new value at the midpoint. If the curvature changes rapidly we use parabolas, for example, instead of approximations with line segments. It seems reasonable to take five neighboring points and if we let the points be labelled d_{-2}, d_{-1}, d_0, d_1, d_2, then the least square solution of a parabola gives

$$d = \frac{1}{35}(-3d_{-2} + 12d_{-1} + 17d_0 + 12d_{+1} - 3d_{+2})$$

at the midpoint.

The scheme of smoothing proposed here is rooted in some elementary concepts but one should bear in mind that the ways of tackling the problem are varied [see Schoenberg (1953)].

9. Practical results

This new approach to the calculation of upper winds has been programmed in ALGOL-code with inclusion of both adjustment and smoothing. In its most general form the scheme has the advantage that the program is adaptable to any of the simple modes of operation in use at present. This can be realized by merely putting one of the elements of the **S**-matrix, i.e., σ_r, σ_ϵ, or σ_h equal to infinity. Moreover, one can regulate the smoothing factor k.

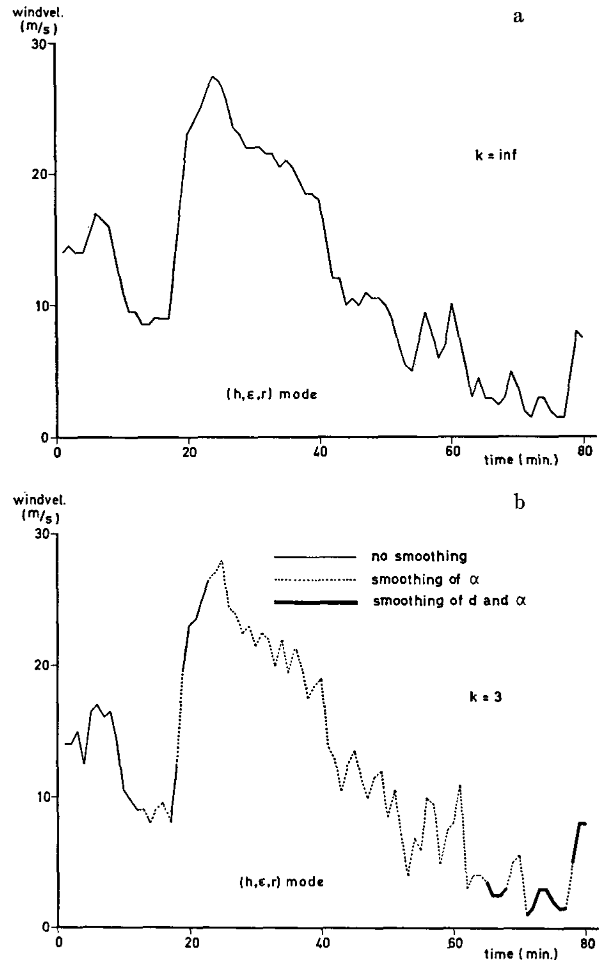

Fig. 16. Wind speed vs. time for an experimental balloon flight in the optimal (h,ϵ,r) operation mode: a, wind profile with smoothing in all points $(k \rightarrow \infty)$; b, wind profile with error dependent smoothing for $k=3$ (99 per cent). Dotted curves, smoothing in azimuth angle; solid curve, smoothing in azimuth and distance; thin curve, no smoothing.

In Fig. 15 the wind profiles associated with the tracks of Fig. 14 are shown as a function of time using the (h,ϵ), (r,ϵ), (h,r) and (h,ϵ,r) modes without smoothing $(k=0)$. These were obtained by means of the X-1 Electrologica computer. The effect of the smoothing process in the (h,ϵ,r) profiles is shown in the two parts of Fig. 16 where k was first set equal to infinity, resulting in smoothing of the whole track and then set equal to 3 (99 per cent) to show the effect of error dependent smoothing. In the last part of this figure the profile shows parts with no smoothing at all, parts with azimuth smoothing (dotted) and parts with both azimuth and distance smoothing (solid).

10. Conclusions

In the overdetermined system of radar/radio tracking of radiosondes and pilot balloons, an optimizing method can be developed which, in contrast with the simple methods in use at present, discards no information. As can be seen in the scheme of Section 5, the optimal operation mode differs from the simple modes of operation in two significant respects. It involves *all* observed variables and it comprises the associated measures of precision of these variables. The last feature is typical for the problem. Implementation of the method requires the knowledge of the covariance matrix for the observed variables. The determination of the content of this matrix **S** which must be specified separately for each system of observation may, under certain circumstances, be a difficult task. In the case of radar, estimates of the standard errors σ_r, σ_α and σ_ϵ are well established, but for radiosondes, σ_h and $\sigma_h{}^*$ estimates lack a firm base. Much depends on the type of measuring elements with which the sonde is equipped. The situation is better for pilot balloons, where it is easy to find σ_h in a collection of balloon ascents of the same type and same payload. In the case of radio direction finding there is some doubt about the independency of measuring elevation and azimuth. Here the **S** matrix involves non-diagonal elements and the theory, accordingly, must be reviewed along more general lines.

It may further be suggested that in radar/radiosonde systems the ascent rate of the balloon is still another information element. In this case the adjustment procedure must be expanded to account for two conditional constraints. Although such a refinement may, in principle, be easily processed for a computer, it is not to be expected that the additional gain in accuracy will be substantial.

APPENDIX

In general, an experiment is performed to obtain a quantity f which is a function of, say, n variables $x_1, \cdots x_n$.

The value obtained for f will not be exact due to errors in the measurement of $x_1, \cdots x_n$, interpreted as stochastic variables. Taking the standard deviation $\sigma_1, \cdots \sigma_n$ as a measure of precision and assuming that the x_i are stochastically independent, the measure of precision for f is given by the well-known law of propagation of errors due to Gauss, i.e.,

$$\sigma_f = \left\{ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \sigma_i \right)^2 \right\}^{\frac{1}{2}}.$$

The resulting error cannot be reduced unless the measurement is repeated. However, in overdetermined systems, where certain constraints exist between the true or exact values of x_i such as

$$\varphi_i(x_1, \cdots x_n) = 0 \quad (i=1, \cdots M)\; M \leqslant n,$$

an estimation $\mathcal{S}(f)$ may be sought in terms of f to achieve, in a certain sense, an optimal reduction of the variance.

In modern analyses of experiments this is achieved by a process of matrix calculus specified by the following requirements:

a) for true values of x_i one has $\mathcal{S}(f) = f$;

b) $\mathcal{S}(f)$ is such that its associated variance is a minimum.

Assume one has at his disposal a number N of independent functions $f_1, \cdots f_N$ which are equivalent in that they take the same value for true values of x_i. Then the number N cannot exceed $M+1$. This may be seen by observing the fact that the set of equations $f_1 - f_i = 0$ $(i=2, \cdots N)$, for example, constitutes a set of $N-1$ constraints of the right form which is at the most of order M. The question arises whether it will be possible to express an estimation $\mathcal{S}(f)$ as a suitable linear combination of f_i $(i=1, \cdots N)$ which implies an optimal variance reduction.

If we introduce the notations

$$\mathbf{F} = \begin{vmatrix} f_1 \\ \vdots \\ f_N \end{vmatrix} = \begin{vmatrix} f_1(x_i) \\ \vdots \\ f_N(x_i) \end{vmatrix},$$

$$\mathbf{\Lambda} = |\lambda_1, \cdots \lambda_N|,$$

then $\mathcal{S}(f) = \mathbf{\Lambda F}$. In order that $\mathcal{S}(f)$ fulfils requirement a) one has $\sum_{i=1}^{N} \lambda_i = 1$, or in matrix form, $\mathbf{\Lambda e} = \mathbf{1}$, where, by definition,

$$\mathbf{e} = \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}.$$

The weights λ_i have to be further specified in such a way that the variance of $\mathcal{S}(f)$ is minimized.

According to well-known rules

$$\mathrm{var}\mathcal{S}(f) = \mathbf{\Lambda F}_x \mathbf{S F}_x{}^T \mathbf{\Lambda}^T.$$

Here **S** denotes the covariance matrix for the variables x_i, the "exponent" T indicates transposition of a matrix and $\mathbf{F}_x$ denotes the Jacobian of the column matrix **F**.

Minimizing the variance under the subsidiary condition $\sum_{i=1}^{N} \lambda_i = 1$ is obtained by introducing an undetermined multiplier ρ and minimizing the function W,

$$W = \mathbf{\Lambda F}_x \mathbf{S F}_x{}^T \mathbf{\Lambda}^T + \rho(\mathbf{\Lambda e} - 1).$$

For abbreviation the covariance matrix **C** of **F** is introduced, defined by

$$\mathbf{C} = \mathbf{F}_x \mathbf{S F}_x{}^T,$$

hence

$$W = \mathbf{\Lambda C \Lambda}^T + \rho(\mathbf{\Lambda e} - 1).$$

The condition for minimum variance now reads

$$\frac{\partial W}{\partial \mathbf{\Lambda}} = 2\mathbf{C\Lambda}^T + \rho \mathbf{e} = 0,$$

with the solution

$$\mathbf{\Lambda}^T = -\frac{\rho}{2}\mathbf{C}^{-1}\mathbf{e},$$

or

$$\mathbf{\Lambda} = -\frac{\rho}{2}\mathbf{e}^T\mathbf{C}^{-1}.$$

Substituting for $\mathbf{\Lambda}$ in $\mathbf{\Lambda e}=1$, one finds the equation determining ρ, i.e.,

$$-\frac{\rho}{2}\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}=1,$$

or

$$\rho = -\frac{2}{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}}.$$

Finally $\mathbf{\Lambda}$ becomes

$$\mathbf{\Lambda} = \frac{\mathbf{e}^T\mathbf{C}^{-1}}{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}},$$

and the estimation $\mathcal{S}(f)$ is

$$\mathcal{S}(f) = \frac{\mathbf{e}^T\mathbf{C}^{-1}}{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}}\mathbf{F}.$$

Here $\mathbf{F}$ involves measured values of x_i, while the weights, however, involve true values of x_i.

The minimum variance associated with this optimal estimation is obviously

$$\operatorname{var}\mathcal{S}(f) = \mathbf{\Lambda C \Lambda}^T = \frac{\mathbf{e}^T\mathbf{C}^{-1}}{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}}\mathbf{C}\frac{\mathbf{C}^{-1}\mathbf{e}}{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}} = \frac{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}}{(\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e})^2},$$

or

$$\operatorname{var}\mathcal{S}(f) = \frac{1}{\mathbf{e}^T\mathbf{C}^{-1}\mathbf{e}}.$$

It may be remarked that the denominator of $\operatorname{var}\mathcal{S}(f)$, is nothing but the sum of all elements of the inverse matrix $\mathbf{C}^{-1}$.

It can be shown that the expression for $\mathcal{S}(f)$ and $\operatorname{var}\mathcal{S}(f)$ numerically lead to the same estimation as obtained by the classic method with normal equations, as shown in Bouman and de Jong (1964). Moreover, the remarkable property holds, that the estimation $\mathcal{S}(f)$ is independent of the choice of the elements f_i in the column matrix $\mathbf{F}$.

The scheme to arrive at an optimal estimation $\mathcal{S}(f)$ is to assign a column matrix $\mathbf{F}$, to compute the matrix product $\mathbf{C}=\mathbf{F}_x\mathbf{S}\mathbf{F}_x{}^T$ and, finally, to invert this matrix $\mathbf{C}$. Then in $\mathcal{S}(f)$ the weight factors are proportional to the sum values of all row (and since $\mathbf{C}$ is symmetrical, all column) elements of $\mathbf{C}^{-1}$, the proportionality factor being equal to the minimum variance. The minimum variance proper amounts to the reciprocal sum of all elements of $\mathbf{C}^{-1}$.

REFERENCES

Barbé, G. D., 1958: Données sur le vent en altitude jusqu'à 30 km et au delà. *J. Sci. Météor.*, **10**, 47–62.

Bouman, D. J., and H. M. de Jong, 1964: Generalized theory of adjustment of observations. *Kon. Ned. Meteor. Instituut* (Royal Netherlands Meteorological Institute) *Med. en Verh.*, No. 84, 88 pp.

de Jong, H. M., 1958: Errors in upper-level wind computations. *J. Meteor.*, **15**, 131–137.

Harrison, D. N., 1962: The errors of the Meteorological Office radiosonde, Mark 2B. *Meteorological Office Scientific Paper*, No. 15, London, Her Majesty's Stationery Office, 40 pp.

Linnik, Y. V., 1961: *Method of Least Squares and Principles of the Theory of Observations*. Translated from the Russian. London, Pergamon Press, 360 pp.

Schoenberg, I. J., 1954: On smoothing functions and their generating functions. *Bull. Amer. Math. Soc.*, **59**, 199–230.