The Role of Initial Uncertainties in Prediction[1,2]

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ABSTRACT

When the initial values, or the parameters, of prognostic equations are not known with certainty, there must also be errors in the solution. The initial conditions may be represented by an ensemble, each member of which is consistent with all available knowledge. The mean of this ensemble is a reasonable "best" solution to the prognostic equation. Following Gleeson, we have examined the behavior of the error in the forecast, as represented by the rms deviation of the ensemble members from their mean, for a few simple equations. We have further examined the time-dependent behavior of the ensemble mean, as opposed to the solution obtained by applying the prognostic equation to the original mean values. These are, in general, different. It is concluded that optimum procedures for forecasting, i.e., solving prognostic equations, require including terms in the equations to represent the influence of the initial uncertainties. Since the nature of these uncertainties may also have profound influences on the error of the forecast, this aspect, too, must be taken into consideration.

1. Introduction

While it has long been recognized that the initial atmospheric conditions upon which meteorological forecasts are based are subject to considerable error, little if any explicit use of this fact has been made. Operational forecasting consists of applying deterministic hydrodynamic equations to a single "best" initial condition and producing a single forecast value for each parameter. While there are always ready admissions that a "perfect" or errorless forecast is not possible, no effort is made under ordinary circumstances to ascertain the manner in which the errors in the initial conditions propagate through the forecast procedure.

Recently, Gleeson (1966, 1967, 1968) has indicated how we can begin to study the influence of initial error on the error in the prediction. Using a few simple prognostic equations and assuming there were errors in the parameters and initial conditions required for the forecast, he made a study of the time variation of the uncertainty of the prediction.

One of the questions which has been entirely ignored by the forecasters, and one which Gleeson, too, chose to neglect, is whether or not one gets the "best" forecast by applying the deterministic equations to the "best" values of the initial conditions and relevant parameters. For the purpose of this discussion we may define "best" as the mean of all possible values consistent with the observations. As Gleeson (1968) clearly points out, one cannot know a uniquely valid starting point for each forecast. There is instead an ensemble of possible starting points, but the identification of the one and only one of these which represents the "true" atmosphere is not possible. The deterministic forecast equations apply equally to each member of the ensemble. The question we will pose is whether applying such equations to any single member is the proper forecast procedure under realistic conditions.

Gleeson sought to examine how the spread of the points that make up the ensemble changes with time. He points out, for example, that the variance of any parameter increases or decreases in time according to whether the correlation between the current value of the parameter and its time derivative is positive or negative. By studying several simple mathematical-physical systems, Gleeson (1968) was able to show that a wide variety of possibilities existed, and that as a result the more complex meteorological problems would require considerable study before the time behavior of errors introduced by uncertainties in the initial conditions and parameters could be understood. The significance of this is that it gets to the heart of the problem of the predictability of the atmosphere. One is able to generate useful predictions, even given perfect mathematical-physical relationships, only to the extent that errors introduced by uncertainties in initial conditions, parameters, and numerical approximations remain within acceptable bounds.

In the present study I have re-examined the same four simple examples used by Gleeson (1968). A method of greater generality is used making it possible to evaluate critically some of the assumptions which Gleeson found convenient. In addition to studying the spread of

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points, however, I have specifically examined the behavior of the mean of the ensemble, as opposed the behavior of the individual member of the ensemble whose location may happen to coincide with the mean. It turns out that there are, in certain cases, distinct differences between these two. This means that, in general, the "best" forecast will not be obtained by applying deterministic equations to the "best" initial conditions.

2. Analyses of examples used by Gleeson

a. Constant rate of change

The first example to be treated is represented by the simple differential equation

$$\frac{dq}{dt}=C=\text{constant}, \tag{1}$$

which has the solution

$$q=Q+Ct. \tag{2}$$

Both Q and C are subject to uncertainty and it is not unreasonable to expect the errors in the estimation of both of these parameters to be Gaussian. We may follow Gleeson in assuming that Q and C are normally distributed with means (unknown) of $\bar{Q}$ and $\bar{C}$ and standard deviations of σ_Q and σ_C. While Gleeson assumed Q and C independent, let us assume their joint distribution is bivariate normal with covariance $\rho\sigma_Q\sigma_C$.

The mean value of q at any time t may be evaluated by taking the expected value of (2); thus,

$$E(q)=\bar{q}-E(Q+Ct)=\bar{Q}+\bar{C}t.$$

The variance of q is

$$\begin{aligned}\operatorname{var}(q)&=E(q^2)=[E(q)]^2,\\ &=E(Q^2+2QCt+C^2t^2)-(\bar{Q}+\bar{C}t)^2,\\ &=\operatorname{var}(Q)+t^2\operatorname{var}(C)+2t\operatorname{cov}(Q,C),\end{aligned}$$

or

$$\sigma_q^2=\sigma_Q^2+t^2\sigma_C^2+2t\rho\sigma_Q\sigma_C.$$

Taking $\rho=0$ gives the situation Gleeson considered, and the same result, i.e., that $d\sigma_q^2/dt=2t\sigma_C^2$. In that case the variance (or the standard deviation) of q increases with time at a rate governed solely by the uncertainty in C. In the more general case, however, one can only place bounds on σ_q corresponding to $-1\leq\rho\leq1$. These bounds are illustrated in Fig. 1. Note that for $\rho<0$ and $t<-\rho\sigma_Q/\sigma_C$, the rate of change of σ_q will be negative. Also, for $\rho<0$, σ_q remains smaller than σ_Q for $t<-2\rho\sigma_Q/\sigma_C$.

b. Nocturnal cooling

Gleeson's second example concerns the equation

$$T=T_0-At^{\frac{1}{2}}, \tag{3}$$

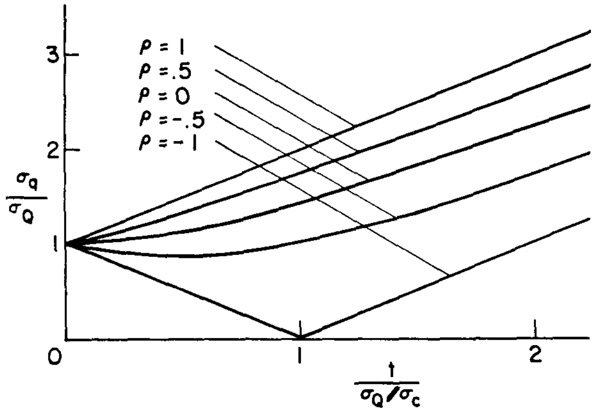

FIG. 1. Ensemble standard deviation of q for various values of the coefficient of correlation ρ between the initial value Q and the constant rate of change C.

which has on occasion been used to represent radiational cooling at night. A is an empirical parameter and T_0 is the initial temperature; both are subject to uncertainty. If one makes the same assumptions about their joint distribution as in the previous example, i.e., that they are bivariate normal with parameters $\bar{T}_0$, σ_{T_0}, $\bar{A}$, σ_A, ρ, then

$$E(T)=\bar{T}=\bar{T}_0-\bar{A}t^{\frac{1}{2}},$$

and

$$\operatorname{var}(T)=\sigma_{T_0}^2+t\sigma_A^2-2t^{\frac{1}{2}}\sigma_{T_0}\sigma_A\rho.$$

Gleeson's result that

$$\frac{\partial\sigma_T^2}{\partial t}=2E\left[(T-\bar{T})\left(\frac{\partial T}{\partial t}-\frac{\partial\bar{T}}{\partial t}\right)\right]=\sigma_A^2,$$

is thus confirmed for the case $\rho=0$. This special case, plus the bounds on σ_T^2 in the more general case, are shown in Fig. 2.

The same comments apply here as did in the first example. The variance may decrease initially if the

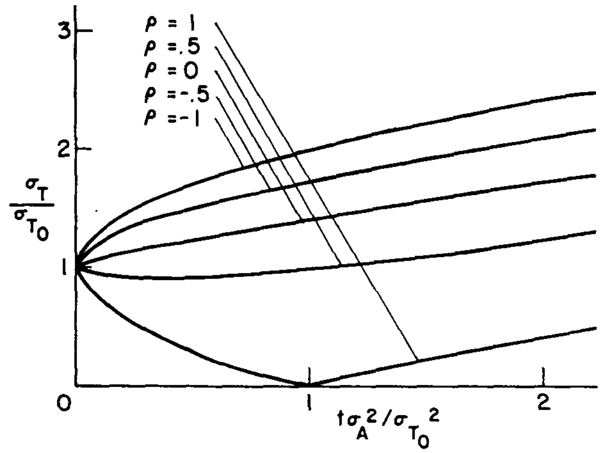

FIG. 2. Ensemble standard deviation of T for various values of the coefficient of correlation ρ between initial temperature T_0 and the cooling coefficient A.

covariance between the two parameters is of the proper sign. In this case, if $\rho>0$, then σ_T is less than its initial value for $t<(2\rho\sigma_{T_0}/\sigma_A)^2$. In addition, when applied to the initial mean value, the equation gives the same results as the expected value of the ensemble.

The similarity of these two first cases lies in the linearity of (2) and (3) as concerns the statistically variable components. The next two cases will be nonlinear and one may therefore expect different results. Note also, in these first two cases, that the assumptions of bivariate normality were excessively stringent. Any joint distributions with the same first and second moments would have given identical results.

c. Frictional retardation

Let us now examine the simplest type of friction, expressed by the differential equation

$$\frac{dv}{dt}=-kv,$$

with the solution

$$v=v_0\exp(-kt), \tag{4}$$

where v is a velocity and k a coefficient of friction. The two parameters subject to error in this case are v_0, the initial velocity, and k. For this example Gleeson noted that when $t=0$,

$$\frac{1}{2}\frac{d\sigma_v^2}{dt}=E\left[(v-\bar{v})\left(\frac{dv}{dt}-\frac{d\bar{v}}{dt}\right)\right]=-E[k(v_0-\bar{v}_0)^2],$$

which is certainly less than zero. He concluded that as time passes the uncertainty of v would continue to decrease, and at the same time the mean value of v would approach ever closer to zero. In time, Gleeson concluded, a very precise forecast of zero velocity is possible.

We now proceed to examine the problem more formally. If we make the assumption that v_0 and k are independent, then we wish to evaluate

$$\bar{v}=E(v_0)E(e^{-kt})=\bar{v}_0E(e^{-kt}), \tag{5}$$

and

$$\sigma_v^2=E(v^2)-[E(v)]^2=(\bar{v}_0^2+\sigma_{v_0}^2)E(e^{-2kt})-\bar{v}^2. \tag{6}$$

In this case, although only the first two moments of the distribution of v_0 are required, evaluation of (5) and (6) requires that we know the moment generating function, i.e., all the moments of the distribution of k. While one is tempted to make the "obvious" simple assumption that k is normally distributed, closer study reveals that this would be a physically unrealistic choice. The normal distribution implies that all values between $\pm\infty$ can occur, but in the present example only positive values of k have physical meaning. Indeed, if one assumes that k is normal, one obtains the result that $E(v)$ becomes infinite for large t (curve a in Fig. 4). It is necessary to avoid unreasonable stochastic assumptions if one seeks physically meaningful results. (Note that the same argument could have been applied to the distribution of A in the previous example. In that case, however, only the first two moments of the distribution were pertinent. The first

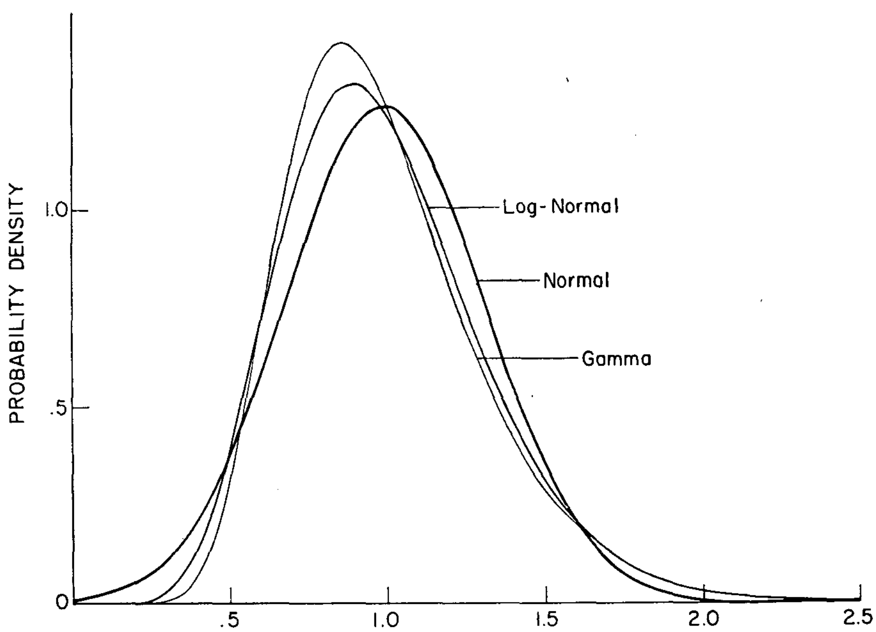

FIG. 3. Normal, log-normal and gamma probability distribution functions for $E(k)=1$ sec^{-1}, var$(k)=0.1$ sec^{-2}, if the abscissa is interpreted as k (sec^{-1}) [Eq. (4)]. Taking the abscissa as units of $k\times10^{-4}$ sec^{-1}, the curves correspond to $E(k)=10^{-4}$ sec^{-1}, var$(k)=10^{-9}$ sec^{-2}.

two moments were not sufficient to imply whether negative values of A $(\bar{A}>0)$ exist. In the present example *all* moments of the distribution influence the results.)

Two frequently encountered distributions for random variables which take on only positive values are the gamma distribution

$$g(k)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}k^{\alpha+1}e^{-k\beta}, \qquad (7)$$

and the log-normal distribution

$$h(k)=\frac{1}{sk\sqrt{2\pi}}\exp\left[-\frac{(\ln k-m)^2}{2s^2}\right]. \qquad (8)$$

Both distributions contain two parameters which are uniquely determined by specifying the mean and variance. Specifically, $\alpha=(\bar{k}/\sigma_k)^2$, $\beta=\sigma_k^2/\bar{k}$, $m=\frac{1}{2}\ln[\bar{k}^4/(\bar{k}^2+\sigma_k^2)]$ and $s^2=\ln(1+\sigma_k^2/\bar{k}^2)$. Fig. 3 is a plot of the normal, gamma and log-normal distributions for $\bar{k}=1$ sec^{-1} and $\sigma_k^2=0.1$ sec^{-2}. For the present we will choose the gamma distribution to represent the uncertainty in k since its moment generating function is known in closed form. Eq. (7) implies that $E(e^{-kt})=(1+\beta t)^{-\alpha}$. Substituting this into (5) and (6) gives

$$\bar{v}=\bar{v}_0\left[1+\frac{\sigma_k^2 t}{\bar{k}}\right]^{-\bar{k}^2/\sigma_k^2}, \qquad (9)$$

and

$$\sigma_v^2=(\sigma_{v_0}^2+\bar{v}_0^2)\left[1+\frac{2\sigma_k^2 t}{\bar{k}}\right]^{-\bar{k}^2/\sigma_k^2} -\bar{v}_0^2\left[1+\frac{\sigma_k^2 t}{\bar{k}}\right]^{-2\bar{k}^2/\sigma_k^2}. \qquad (10)$$

Eq. (9) is plotted as curve b in Fig. 4. The values used in the calculations are $\bar{v}_0=10$ m sec^{-1}, $\sigma_{v_0}=1$ m sec^{-1}, $\bar{k}=10^{-4}$ sec^{-1} and $\sigma_k=3\times10^{-5}$ sec^{-1}. Curve c in Fig. 4 is $v_0e^{-\bar{k}t}$, which is how one would estimate v if the deterministic equation (4) were applied directly to the initial mean values. In this case such a procedure would always underestimate the ensemble mean.

Curve b in Fig. 5 is based on (10). We find, in agreement with Gleeson, that $d\sigma_v^2/dt=-2\bar{k}\sigma_{v_0}^2$ at $t=0$, but we also find, after a relatively short period of time ($\sim10^3$ sec), that the variance begins to increase. After about 2500 sec, σ_v becomes larger than its initial value. Only after about 10^4 sec does it show the expected decrease toward zero. Curve c in Fig. 5 is a plot of $\sigma_{v_0}e^{-\bar{k}t}$, the limiting case of (10) when σ_k approaches zero (i.e., when the only uncertainty is that in the determination of v_0). Curve a is the result one obtains by assuming k is normal with the same mean and variance.

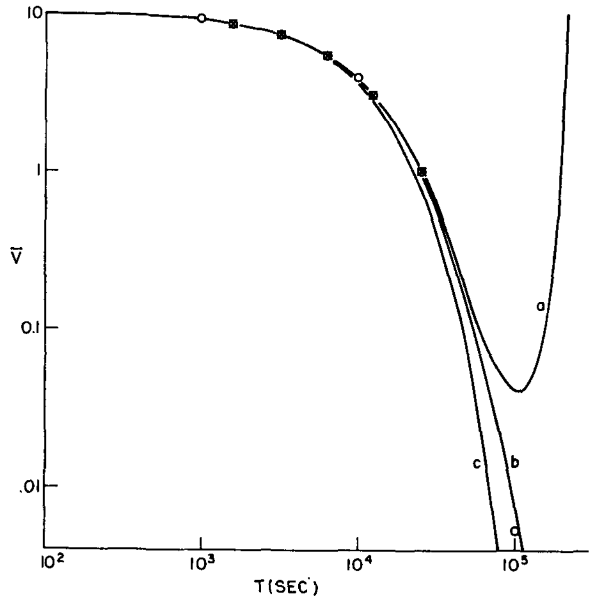

Fig. 4. Expected value of v as a function of time. Curves a, b and c are cases for which the friction coefficient k has a normal distribution, a gamma distribution, and is known with certainty, respectively. In each case $\bar{v}_0=10$ m sec^{-1} and $\sigma_{v0}=1$ m sec^{-1}. For curves a and b, $\bar{k}=10^{-4}$ sec^{-1} and $\sigma_k=3\times10^{-5}$ sec^{-1}. Plotted points are based on samples of 1000, using a log-normal distribution for k.

It is possible to explain the behavior illustrated in curve b in terms of the changes which must occur in the distribution of v, if σ_k is sufficiently large. At time $t=0$, the distribution of v is normal and the initial decrease in variance is due to the general shrinkage of all members of the ensemble toward smaller v. However, as this proceeds, the cases of small k (especially if linked with a large v_0) will lag behind, and soon will be even further from the modal value of v than they were initially; ensemble members with large k will emphasize the left-hand tail of the distribution. This will tend to increase the variance. Finally, as t continues to increase, the bulk of the ensemble members are near $v=0$, having caught up with those on the left-hand tail, and are no longer decreasing; while the ones on the right-hand tail of the distribution are cases of small k, and approach the origin, slowly but consistently. Thus, the variance eventually decreases toward zero. (Note that this description fits the case of the normal distribution of k until large t when the very few cases of $k<0$ make their presence felt.)

To evaluate and test the ideas behind the sequence described in the preceding paragraph, the following experiment was carried out. Two sets of pairs of random numbers (1000 pairs in each set) were generated such that one member of each pair was normal [v_0: mean 10, standard error 1] and one was independently log-normal [k: $m=\frac{1}{2}\ln(1.09\times10^{-8})$, $s^2=\ln(1.09)$; cf. Eq. (8)]. Eq. (4) was applied separately to each pair of values, i.e., to each member of the ensemble, and means and standard deviations for the entire ensemble

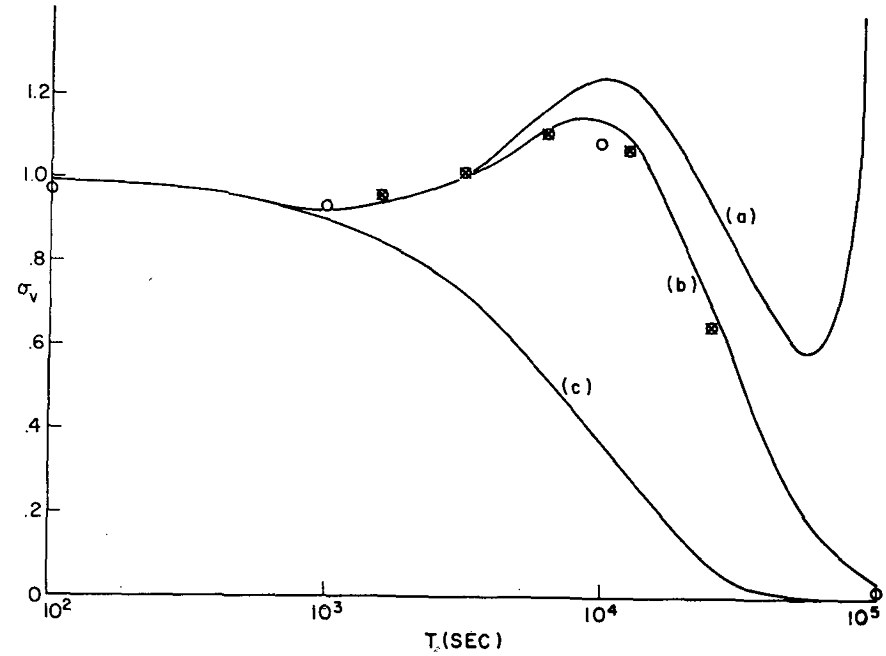

FIG. 5. Standard deviation of v as a function of time. The curves are labeled as in Fig. 4 and the points are again based on samples of 1000 pairs with k having a log-normal distribution.

were calculated for several values of t. These values are the points plotted in Figs. 4 and 5. In addition, it was possible to examine the frequency distributions[4] of v at different values of t, some of which are plotted in Fig. 6. Just as the gamma distribution was used earlier because it provided analytic simplicity, the log-normal distribution was used in this experiment because of computational simplicity. Comparison of the results allows some added insight into the significance of the shape of the distributions (Fig. 3).

[4] The same end could have been accomplished by solving numerically Gleeson's continuity equation for probability density. If $\psi(k,v,t)$ is the time dependent joint probability density of k and v, the continuity equation here takes the form $\partial\psi/\partial t = k(\psi + v\partial\psi/\partial v)$.

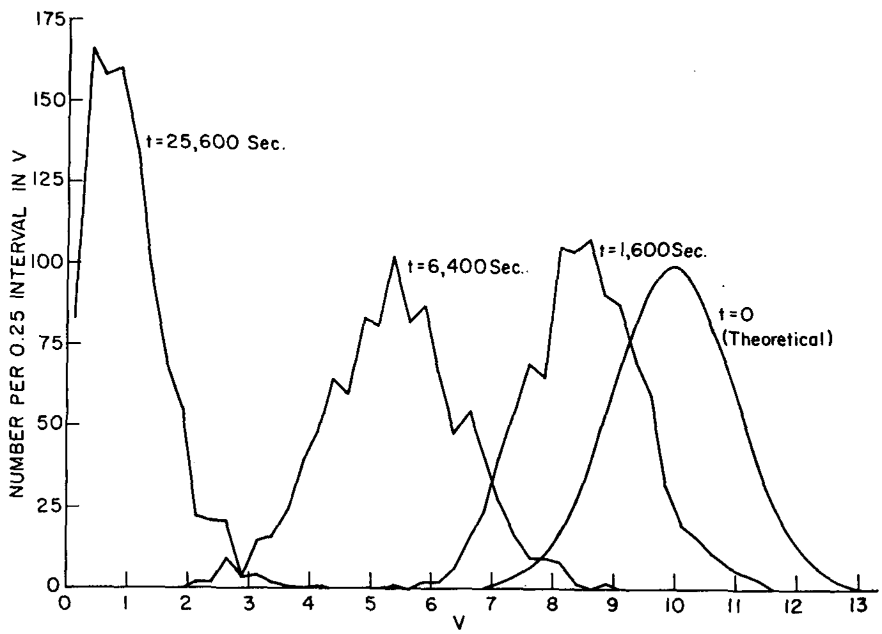

FIG. 6. Frequency distributions of v after various time intervals. The curve labeled $t=0$ is the theoretical initial distribution. Other curves are based on experimental samples of 1000 pairs of numbers for which v_0 is normal with mean 10.0 m sec^{-1} and standard deviation 1.0 m sec^{-1}, and k is log-normal with $\bar{k}=10^{-4}$ sec^{-1} and $\sigma_k = 3\times10^{-5}$ sec^{-1}.

TABLE 1. Calculated values of $\bar{v}$ and σ_v for $\bar{v}_0=10$ m sec^{-1}, $\sigma_{v0}=1$ m sec^{-1}, $\bar{k}=10^{-4}$ sec^{-1} and $\sigma_k=3\times10^{-5}$ sec^{-1}.

t (sec)	$\bar{v}$ $\sigma_k=\infty$	$\bar{v}$ Gamma	$\bar{v}$ Log-normal	σ_v $\sigma_k=\infty$	σ_v Gamma	σ_v Log-normal
1600	8.521	8.531	8.536	0.8521	0.9448	0.9596
3200	7.262	7.295	7.302	0.7262	1.001	1.016
6400	5.273	5.367	5.378	0.5273	1.124	1.125
12800	2.780	2.978	2.988	0.2780	1.106	1.072
25600	0.773	0.999	0.997	0.0773	0.7070	0.6456

Note: The columns headed $\sigma_k=\infty$ and Gamma refer to the distributions of k and the calculations are based on formulae given in the text. The column headed Log-normal is based on an experimental random sample of size 1000.

The points plotted in Figs. 4 and 5 show that the behavior of the gamma and log-normal distributions are remarkably similar. This is emphasized by the tabulation of these results in Table 1.

The distributions plotted in Fig. 6 substantiate the argument given earlier. At $t=1600$ sec the distribution of v is still relatively normal but σ_v is just beginning to increase. The cases on the tails of the distribution are still mostly cases of small and large v_0, but the influence of the variability in k is beginning to be felt. At $t=6400$ sec the standard deviation is near its maximum, but the distribution is still relatively symmetrical. The total range of values of v, 6.00 m sec^{-1} at $t=1600$ sec is now 6.71 m sec^{-1}. Clearly, cases on both tails must be there primarily because of large or small values of k.

At $t=25{,}600$ sec the distribution is very asymmetrical. The values of v_0 are of relatively little weight in determining where an individual case may lie. Note that while the extreme theoretical values of v_0 are in the approximate ratio 13/7, or near 2, the extreme values of v at 25,600 sec are 0.028 and 4.11, a ratio near 150. Thus, for large t the distribution of k becomes increasingly important.

In the foregoing the assumption was made, for simplicity, that k and v_0 were independent.[5] One can relax that assumption by selecting a particular form of dependence. It is relatively simple to examine the case where the marginal distribution of k is gamma, as previously, and the distribution of v_0, given k, has a mean of $ak+b$ and variance $s_{v_0}^2$. One then finds that

$$\bar{v}_0=E(v_0)=b+a\bar{k},$$

$$\sigma_{v_0}^2=\text{var}(v_0)=s_{v_0}^2+a^2\,\text{var}(k),$$

$$\text{cov}(k,v_0)=a^2\,\text{var}(k)=\rho\sigma_k\sigma_{v_0}.$$

[5] A dependence between k and v_0 (like the supposed correlation, in the previous example, between A and T_0) may seem unreasonable if one takes the values of A and k as absolute constants. However, as in many meteorological situations, these "constants" represent parameterizations of processes whose scales or details preclude explicit treatment. It thus seems reasonable to admit the possibility that circumstances giving rise to errors of one sign or the other in measured quantities may be circumstances when somewhat higher or lower values of the "parameter" are applicable. In any case, we include treatment here of the case of intercorrelation, more to illustrate the general features of the results than to imply that they should be applied literally in these elementary examples.

For a suitable comparison, we have calculated v and σ_v, for various values of ρ, choosing in each case values for a, b and s_v such that $\bar{v}_0=10$ m sec^{-1} and $\sigma_{v_0}=1$ m sec^{-1} as before. The differences in $E(v)$ from those values, shown as the middle curve in Fig. 3, amount to variations of only about $\pm5\%$ as ρ varies between -1 and $+1$, for $t<10^5$ sec. The lack of independence has a much more dramatic influence on σ_v, as illustrated in Fig. 7. As one could anticipate, a negative correlation between v_0 and k implies that large initial velocities decrease slowly while slower ones decrease rapidly, implying an increasing standard deviation at small t. For large positive correlation, the large initial values of v_0 decrease rapidly and thus catch up with the ones that are small initially but decrease slowly; at this time a pronounced minimum of the variance occurs.

The important lesson to be learned from the above discussion is that the statistical interaction between two parameters can play a very significant role. In the present case we find that although the velocity of every member of the ensemble is known to decrease (with an average time constant of about 3 hr), the mean velocity 3 hr later has indeed decreased (by an amount we might tend, if not careful, to overestimate) but the absolute error in our specification of that small mean wind may be greater than the error in the original values of v_0. This result is clearly dependent on our choice of distributions and parameters, but there is the suggestion that if the distribution is *reasonable* then the results will be reliable. Unreasonable distributions may lead to wholly fallacious results. The degree of dependence among the various parameters can play a very significant role in the variation of variance with time. Finally, since the distributions of dependent variables generally will change, the special case of

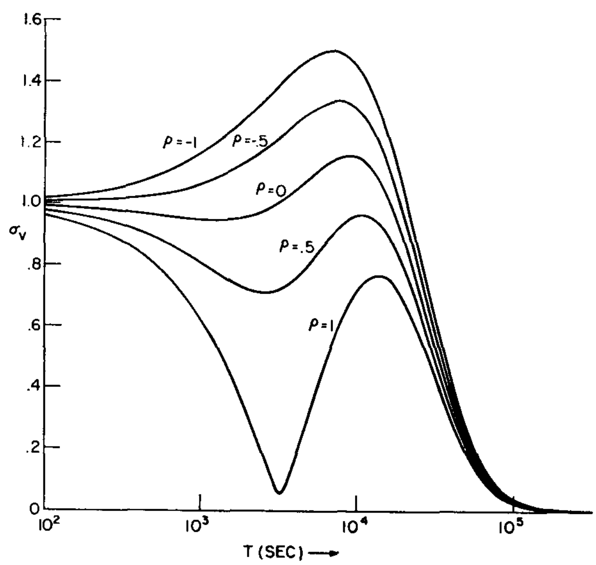

FIG. 7. Standard deviation of v as a function of time for various coefficients of correlation between v_0 and k. The curve labeled $\rho=0$ is identical to curve b in Fig. 5.

Gleeson's (1966, 1968) continuity equation for probability, developed on the assumption of normality, will have little general applicability. The more general statement of that principle which Gleeson gives is, of course, still valid.

d. Cyclical changes

The last example given by Gleeson is represented by

$$q=Q+B\cos(\nu t-\beta), \tag{11}$$

where Q is a time-mean and B the amplitude of some parameter q. The frequency ν, and the phase angle β, as well as Q and B, are all subject to error. This type of expression has particular significance in meteorology since it is so often found convenient to express time series as sums of terms of various amplitudes and phases. Gleeson makes the point that the standard deviation of q will show a behavior that combines cyclical increases and decreases with a superimposed gradual increase. He states that the "true" value of q will be entirely cyclical in its behavior, but he mistakenly identifies this "true" value with $\bar{q}$. He misses the point that the ensemble mean (indeed $\bar{q}$) will have a somewhat different behavior.

In this example, only the first two moments of Q and B are required, but all the moments of ν and β need be known. For simplicity, we will assume that the four parameters are independent of one another, and that the distributions of ν and β are normal. The moment generating function for the normal distribution is

$$E(e^{-\nu t})=\exp(-\bar{\nu}t+\tfrac{1}{2}t^2\sigma_\nu^2).$$

By using the identities

$$\sin\theta=\tfrac{1}{2}i(e^{i\theta}-e^{-i\theta}),$$
$$\cos\theta=\tfrac{1}{2}(e^{i\theta}+e^{-i\theta}),$$

where $i=\sqrt{-1}$, we are able to evaluate such terms as $E(\cos\nu t)$ and show that

$$\bar{q}=\bar{Q}+\bar{B}\exp[-\tfrac{1}{2}(\sigma_\beta^2+t^2\sigma_\nu^2)]\cos(\bar{\nu}t-\bar{\beta}), \tag{12}$$

$$\sigma_q^2=\sigma_Q^2+\tfrac{1}{2}(\bar{B}^2+\sigma_B^2)\{1+\exp[-2(\sigma_\beta^2+t^2\sigma_\nu^2)]\times\cos2(\bar{\nu}t-\bar{\beta})\}-\tfrac{1}{2}\bar{B}^2\exp[-(\sigma_\beta^2+t^2\sigma_\nu^2)]\times[1+\cos2(\bar{\nu}t-\bar{\beta})]. \tag{13}$$

Note, in particular, that the amplitude of the fluctuations of the ensemble mean decreases with time. In the limit, as t gets large, the best estimate for $\bar{q}$ is $\bar{Q}$. This behavior is entirely due to uncertainty in ν, the frequency. The uncertainty of the phase angle β causes the amplitude of the mean of the ensemble to be less than the mean amplitude $\bar{B}$ by a factor $\exp[-\tfrac{1}{2}\sigma_\beta^2]$, but this effect does not change with time. In other words, even at $t=0$, the average departure of q from its time mean, $\bar{q}-\bar{Q}$, is less than the average of the amplitudes. Fig. 8 contains plots of $(\bar{q}-\bar{Q})/\bar{B}$ for several set of values of the relevant parameters. The value of σ_B does not influence $\bar{q}$.

The influences of the several parameters on the

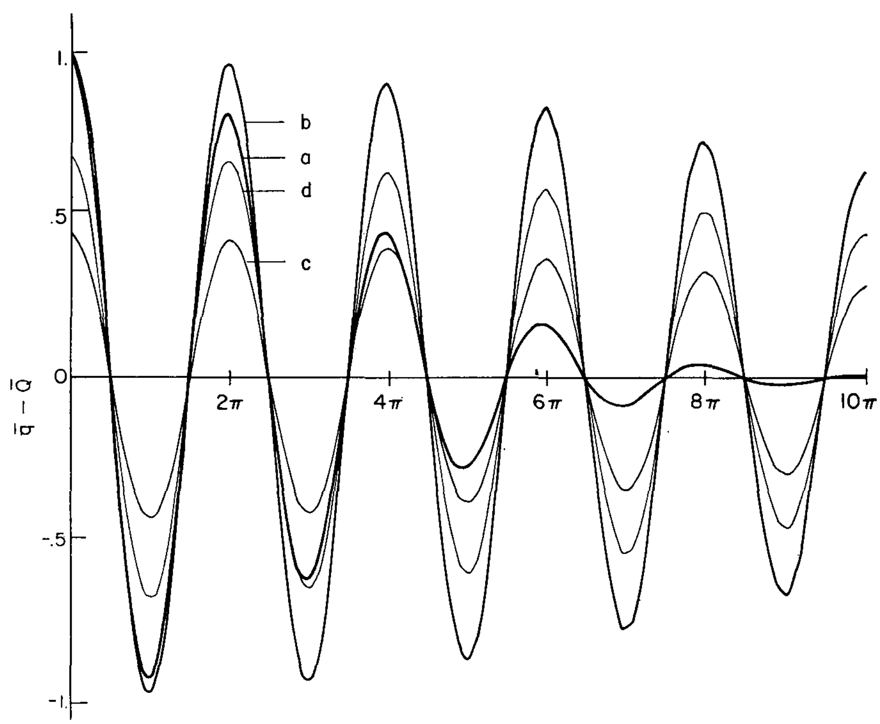

FIG. 8. The ensemble mean of the cyclical variable. The ordinate is $(\bar{q}-\bar{Q})$ for $\bar{B}=1$, and the abscissa is $\bar{\nu}t$. The pertinent parameters are: a) $\sigma_\nu=0.1$, $\sigma_\beta=0.0$; b) $\sigma_\nu=0.03$, $\sigma_\beta=0.0$; c) $\sigma_\nu=0.03$, $\sigma_\beta=1.27$; d) $\sigma_\nu=0.03$, $\sigma_\beta=0.85$.

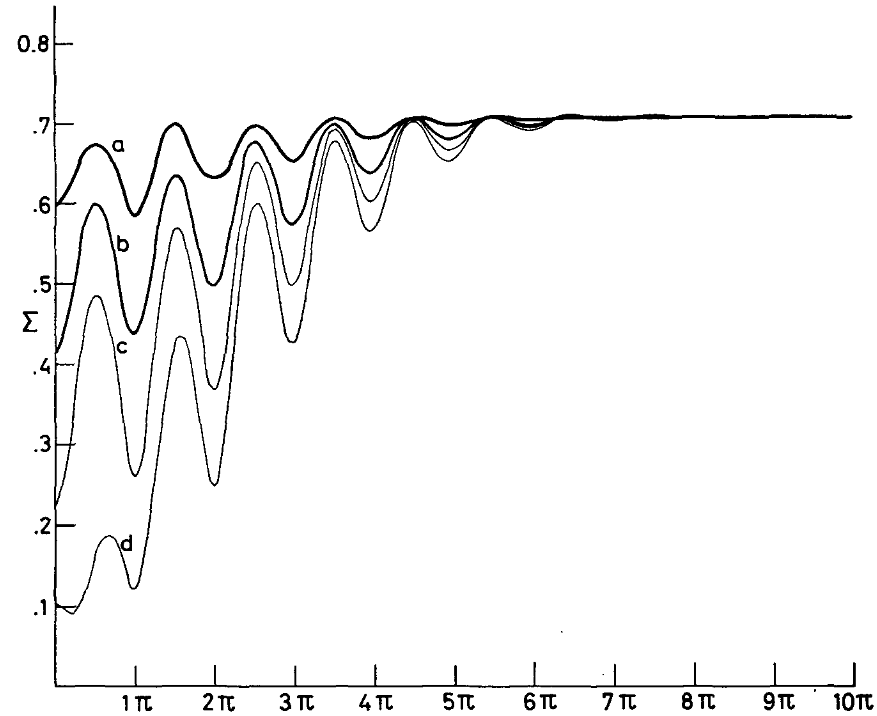

FIG. 9. Plots of the quantity $\Sigma=(\sigma_q{}^2-\sigma_Q{}^2)^{\frac{1}{2}}/\bar{B}$. The abscissa is $\bar{\nu}t$. All curves were calculated for $\sigma_\nu=0.1$ and $\sigma_B=0.1$. For curves labeled a, b, c and d, $\sigma_\beta=1.27$, 0.85, 0.54 and 0.00, respectively.

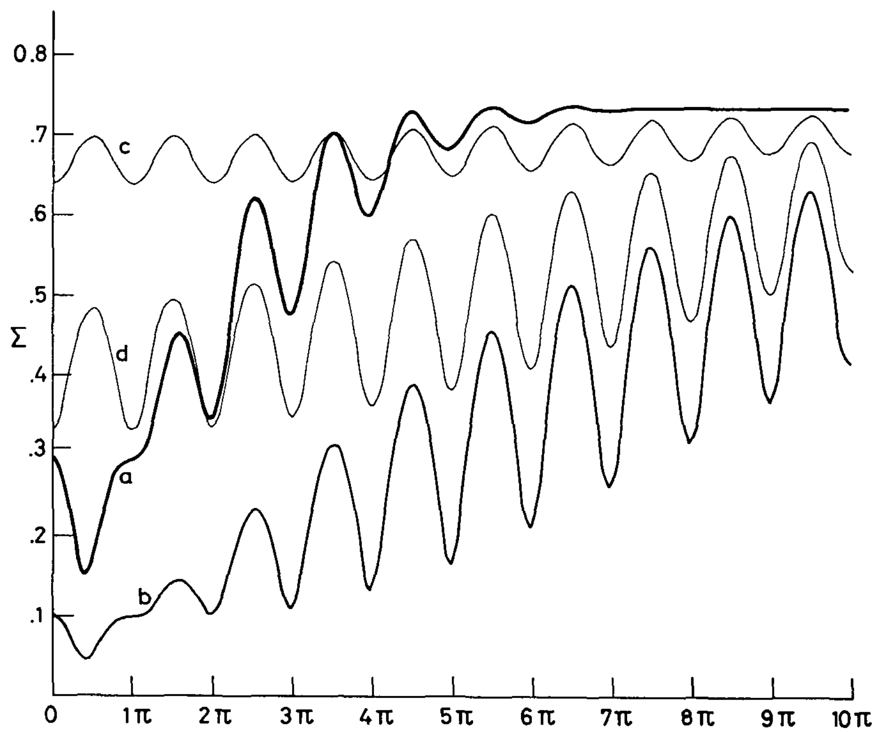

FIG. 10. Plots of the quantity $\Sigma=(\sigma_q{}^2-\sigma_Q{}^2)^{\frac{1}{2}}/\bar{B}$. The abscissa is $\bar{\nu}t$. The parameters used in the calculations were a) $\sigma_\nu=0.1$, $\sigma_B=0.3$, $\sigma_\beta=0.0$; b) $\sigma_\nu=0.3$, $\sigma_B=0.1$, $\sigma_\beta=0.0$; c) $\sigma_\nu=0.3$, $\sigma_B=0.3$, $\sigma_\beta=1.27$; d) $\sigma_\nu=0.3$, $\sigma_B=0.3$, $\sigma_\beta=0.54$.

ensemble standard deviation of q is more involved. It is easier to discuss the quantity $\Sigma=(\sigma_q^2-\sigma_Q^2)^{\frac{1}{2}}/\bar{B}$, the standard deviation of the departure of q from its time-averaged value, normalized to unit mean amplitude. This quantity is the ordinate in Figs. 9 and 10, and in the limit as $t\rightarrow\infty$, approaches $[\frac{1}{2}+\frac{1}{2}(\sigma_B/\bar{B})^2]^{\frac{1}{2}}$, if $\sigma_\nu\neq 0$. This limiting value is also the absolute maximum of Σ.

The influence of σ_β on Σ is emphasized in Fig. 9. If σ_β is large, so also is the initial value of Σ, but the amplitude of the fluctuations of Σ are reduced. One may discern from Fig. 10 that smaller values of σ_ν, while permitting the fluctuations in Σ to persist, do not retard substantially the growth of its time-averaged value. However, the ratio of the amplitude of the fluctuations of $\bar{q}$ (Fig. 8) to the time-averaged standard deviation of q does remain considerably larger if σ_ν is small.

3. Conclusions

Uncertainties in the initial conditions and/or parameters of time-dependent processes manifest themselves in a variety of manners. By examining only a few simple cases we have been able to demonstrate that these effects must not be neglected. We have also shown that care must be taken that assumptions pertaining to the nature of the errors be reasonable in terms of any physical constraints on the problem.

Any uncertainties at all in the parameters of a time-dependent problem may influence all the statistical properties of the ensemble of cases which constitute the population of possible "true" solutions. We have only examined the mean and the variance of a few simple time-dependent variables. The history of the variance is clearly a complex phenomenon since the variance is always nonlinear, even when the basic prognostic equation is linear, and will thus be influenced by any statistical interactions that may be present.

Our most significant conclusion concerns the behavior of the ensemble mean. In general, the ensemble mean value of a variable will follow a different course than that of any single member of the ensemble. For this reason it is clearly not an optimum procedure to forecast the atmosphere by applying deterministic hydrodynamic equations to any single initial condition, no matter how well it fits the available, but nevertheless finite and fallible, set of observations.

One cannot draw any conclusions, from the examples shown, of the relative importance of various effects in the much more complex hydrodynamic equations which describe the atmosphere. Certainly, though, the effects we have described will be present. One cannot help but note, for example, that in the examples shown correlations among the various parameters played a particularly large role in determining the eventual ensemble variance. Doesn't this suggest that one should design objective analysis schemes so as to introduce the right kinds of interdependences among the initial conditions? We may not want statistically independent or uncorrelated sets of initial values.

This being the case, how should we forecast the weather? To answer this question we will have to learn a great deal more about how uncertain our meteorological parameters and observations are, and how these uncertainties are propagated through the differential equations and numerical procedures of physical prediction.

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