Model Studies of the Beam-Filling Error for Rain-Rate Retrieval with Microwave Radiometers

EUNHO HA* AND GERALD R. NORTH

Climate System Research Program, Texas A&M University, College Station, Texas

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ABSTRACT

Low-frequency (<20 GHz) single-channel microwave retrievals of rain rate encounter the problem of beam-filling error. This error stems from the fact that the relationship between microwave brightness temperature and rain rate is nonlinear, coupled with the fact that the field of view is large or comparable to important scales of variability of the rain field. This means that one may not simply insert the area average of the brightness temperature into the formula for rain rate without incurring both bias and random error. The statistical heterogeneity of the rain-rate field in the footprint of the instrument is key to determining the nature of these errors. This paper makes use of a series of random rain-rate fields to study the size of the bias and random error associated with beam filling. A number of examples are analyzed in detail: the binomially distributed field, the gamma, the Gaussian, the mixed gamma, the lognormal, and the mixed lognormal ("mixed" here means there is a finite probability of no rain rate at a point of space-time). Of particular interest are the applicability of a simple error formula due to Chiu and collaborators and a formula that might hold in the large field of view limit. It is found that the simple formula holds for Gaussian rain-rate fields but begins to fail for highly skewed fields such as the mixed lognormal. While not conclusively demonstrated here, it is suggested that the notion of climatologically adjusting the retrievals to remove the beam-filling bias is a reasonable proposition.

1. Introduction

The use of satellites in the estimation of rain rates has become an important endeavor in atmospheric science. The reason for resorting to such an exotic and seemingly expensive technology is the importance of latent heat release and its effect upon the general circulation of the atmosphere. For example, it is well known that the sea surface temperature changes in the tropical Pacific cause rain concentrations to move from the East Indies region well into the mid-Pacific region under the influence of El Niño. This change of location of intense precipitation, which can persist over periods of months to years, leads to well-documented changes of weather patterns over North America and indeed throughout the world. Unfortunately, no simple and inexpensive in situ method exists for measuring these tropical oceanic precipitation rates over a sufficient period and with sufficient spatial and temporal resolution. For example, for the checking of general circulation models used in climate studies, it is suggested that a minimal data requirement be for a precipitation dataset that is smoothed over 1-month-by-500-km boxes. The importance of small time and space scales of tropical rain precludes the exclusive use of tropical island stations with radars supplemented with ship observations in providing a useful time series for model checking. The coverage by these conventional observing systems is simply inadequate. A review of these issues has been presented by Simpson et al. (1988).

The satellite techniques for estimating rain rate include infrared algorithms based upon measurements from geosynchronous operational satellites. Such techniques have pioneered the way toward our appreciation of the importance of tropical precipitation. The temporal resolution of these measurements (few hours) is excellent, but the retrieval algorithms are unfortunately very crude, being based upon inferences that make use of the areal extent of cloud below a critical temperature (Arkin 1979). This method is entirely empirical, and there is insufficient data to ascertain the geographical and seasonal dependencies of the phenomenological coefficients except in a few isolated instances when and where massive field experiments have been conducted. A more satisfactory arrangement would be to make measurements directly upon the airborne hydrometeors.

To directly measure some aspect of the hydrometeor population (e.g., density, size distribution, rain rate as

* Current affiliation: Inje University, Department of Statistics, Kimhae, Republic of Korea.

Corresponding author address: Dr. Gerald R. North, Climate System Research Program, College of Geosciences, Texas A&M University, College Station, TX 77843-3150.

function of altitude, etc.) requires the use of microwave electromagnetic radiation, since radiation at these wavelengths approximately 0.1–10 cm penetrates typical volume distributions, ordinary cloud droplets, and ice crystals with relatively little attenuation. On the other hand, significant absorption occurs in the nearly spherical water drops that are raindrop size (>1 mm). The emissivity of these raindrops tends to be close to that of a blackbody in the microwave range. This is to be contrasted with the emissivity of the sea surface, which is only about 0.4. Hence, the sea surface as viewed through a clear sky by an overflying microwave radiometer will have an apparent temperature of the order of 150 K. When rain is present over the sea surface, a larger apparent temperature will be observed, since for an optically thick column of rain the origin of upwelling radiation will be mainly from the airborne raindrops. For rain rates up to 10–20 mm h^{-1}, there is a one-to-one correspondence between the observed apparent microwave temperature and the rain rate. This assertion is based upon a fairly robust theory and is supported by aircraft and other types of measurements and intercomparisons (Wilheit et al. 1977; Wilheit 1986; Shin et al. 1990; Short and North 1990). Above 10–20 mm h^{-1} scattering of the microwave radiation by the hydrometeors becomes important, and this causes the relationship between rain rate and microwave temperature to become double valued. Since most rain falls at rates well below this threshold, we ignore this aspect of the problem in the present paper, although it is a subject worthy of future study.

The problem we pose to study here is that of the retrieval of rain-rate statistics from a satellite-based single-channel microwave radiometer that has a finite resolution. That is to say, the instrument measures an apparent microwave temperature averaged over a horizontal area usually of the order of 20–50 km across. If the relationship between microwave temperature and rain rate were linear, there would be no problem for this simple case of retrieval using a single microwave channel (taken here nominally to be the 19.6-GHz channel, since this is the one used on ESMR-5 and similarly on most modern radiometers). The problem is that the relationship, although one to one, is nonlinear and has the shape of a saturating exponential that can written approximately as

$$T(R) = A + Be^{-cR}$$

or

$$R(T) = -\frac{1}{c}\ln\frac{(T-A)}{B}, \qquad (1)$$

where $T(R)$ is microwave temperature, and R is the rain rate averaged over a vertical column or perhaps a slanted column. The vertical or slant vertical averaging tends to smooth the corresponding field horizontally as well, hence the horizontal statistics of the field actually being measured are not strictly those observed at the ground but are slightly smoothed in both space and time.

Since the radiometer measures an area average of T, referred to here as $[T(R)]$, the inversion to obtain an area average of R, $[R]$, is not so straightforward since

$$[R] \neq R([T]). \qquad (2)$$

In fact, it can be shown that heterogeneities in the field of view (FOV) cause $[R] \geqslant R([T])$. It is easy to see that the retrieval is not unique; that is there are many values of $[R]$ that can correspond to a given measured value of $[T]$. This leads us to a pair of questions: 1) What is the bias,

$$\beta = E(\delta R) = E\{[R] - R([T])\} \qquad (3)$$

due to the finite footprint and its effect on lowering (incorrectly) our estimate of $[R]$? 2) What is the distribution of random errors associated with the estimate? In particular, what is the variance associated with the distribution

$$\mathrm{var}(\delta R) = E\{(\delta R - \beta)^2\}? \qquad (4)$$

The quantity δR is referred to as the beam-filling error. It can be modeled as a random variable taking on different values with each realization (snapshot). Given the $T(R)$ relationship, the problem reduces to a study of the statistics of the rain field within restricted areas the size of the field of view of the instrument. For example, what is the role of spatial autocorrelation in the rain-rate field? Does the non-Gaussian probability density function of rain rates enter in an important way (after all, most of the time it is not raining)?

The present paper undertakes to study a sequence of very simple model rainrate fields to see how the beam-filling error depends upon these field statistics. The hope is that one can eventually find some fairly robust rules for correcting the retrievals by making use of the climatological statistics characteristic of precipitation in a given region of the world and at a given time of the year. In principle, such statistics could be gathered by routine surface radar measurements at representative locations. Earlier studies based upon tropical Atlantic data suggest that this may be the case (Chiu et al. 1990; Short and North 1990; Graves 1993).

There are many different types of errors that can interfere with the rain-rate estimation process besides beam filling. However, Short and North (1990) showed that the errors made by ESMR-5 in flying over an array of surface radars during the GATE experiment (cf. Patterson et al. 1979) were entirely consistent with their being dominated by the beam-filling error.

Our study is admittedly simplistic in that precipitation fields are undoubtedly more complex than those we have chosen to single out for our study, but we feel there is sufficient confusion about this problem that some simple examples will prove useful to the community learning process. Before entering the body of the study, we point out that beam filling enters in many other fields, in particular in the rain-rate retrieval by radar attenuation, a situation that was examined in a few very simple cases by Nakamura (1991).

2. Definitions

Suppose the radiometer measures brightness temperature over a rectangular area $L \times L$, which is called the FOV. Figure 1 shows a schematic diagram of such an FOV of L km on a side. Let $R(\mathbf{x})$ be the rain rate at the location $\mathbf{x} = (x_1, x_2)$. The random rain-rate field on an L km $\times$ L km FOV consists of the rain rate $R(\mathbf{x})$, which follows a certain probability distribution and a prescribed spatial correlation between any two points $\mathbf{x}$ and $\mathbf{x}'$. Strictly speaking, higher moments are required for a complete specification of non-Gaussian fields. By partitioning the FOV we effectively treat the random field as a multivariate vector. We assume that the statistics of the random rain-rate field are homogeneous and isotropic. This assumption amounts to

$$\rho\{R(\mathbf{x}), R(\mathbf{x}')\} = f(|\mathbf{x} - \mathbf{x}'|),$$

where ρ denotes the spatial correlation between any two points x and x' in the field.

The area-average rain rate $[R]$ is defined by

$$[R] = \frac{1}{L^2} \sum_{x \in A} R(\mathbf{x}) \Delta A, \tag{5}$$

where the sum is over all grid points within the area $A = L^2$, and ΔA is the area of a grid box. We can rewrite (5) as

$$[R] = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} R_{ij}, \tag{6}$$

where $N = L/s$ is the number of subdivisions that have been chosen for the partitioning of the FOV. Occasionally in our study we will present numerical results in which case s is given the same spacing (4 km) as the gridded GATE [GARP (Global Atmospheric Research Program) Atlantic Tropical Experiment] data, and R_{ij} represents the area-average rain rate in an s km $\times$ s km grid square, which we call a *tile,* and thus the number of tiles in FOV is N^2.

Several studies (Wilheit et al. 1977; Shin et al. 1990) have shown that the 19.6-GHz microwave temperature T is related to the rain rate R nonlinearly, and its empirical relationship in the low-rain-rate regime can be modeled as in (1), where T is the brightness temperature in kelvins, R is rain rate in millimeters per hour, and A, B, and c are parameters that depend on the environmental conditions. Shin et al. (1990) discuss the range of conditions under which this simple formula is valid. The parameter A is typically 280 K, and B is about 130 K. A typical value of c is 0.19 h mm^{-1} (Wilheit et al. 1977) for tropical rainfall measurements. For our study in this section, we assume that A, B, and c are constant over the FOV of the satellite.

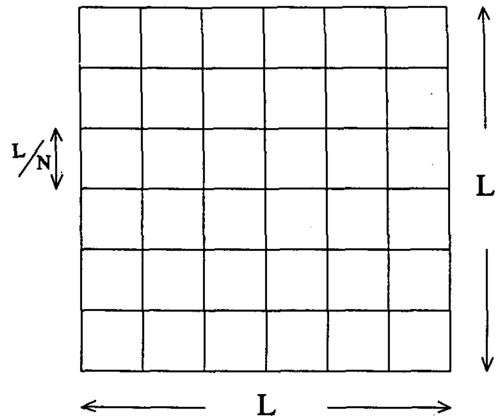


FIG. 1. Schematic diagram for FOV of L km on a side.

We can form an estimate of the area-average rain rate $[R]$ as

$$R_E = R([T]) = -\frac{1}{c} \log\left(\frac{[T] - A}{B}\right) = -\frac{1}{c} \log\{[\exp(-cR)]\},$$

where square brackets again denote averages over an area L^2, the FOV. The difference between the area-average rain rate $[R]$ and the estimate of area-average rain rate R_E is the *beam-filling error* δR and is written as

$$\delta R = [R] - R_E = [R] + \frac{1}{c} \log\{[\exp(-cR)]\}. \tag{7}$$

Note that only the parameter c remains, A and B do not contribute to the beam-filling error.

Chiu et al. (1990) presented an approximate formula for the beam-filling error denoted by δR_{FO}, which has the advantage of simplicity and easy interpretation. The subscript FO denotes the fact that it is only a first-order approximation. Their formula reads

$$\delta R_{\mathrm{FO}} = -\frac{1}{2}\left\{(R - [R])^2 \frac{T''([R])}{T'([R])}\right\}, \tag{8}$$

where as before square brackets denote FOV average. In the special case that the R–T relation is exponential (3) as we are taking here, we have $T''([R])/T'([R]) = -c$ and

$$\delta R_{\mathrm{FO}} = ([R^2] - [R]^2)\frac{c}{2}. \tag{9}$$

Note that the first-order formula for the exponential R–T relation can be derived by retaining only the first two terms in a Taylor expansion of (4). The last formula shows that the bias is neatly factored into two parts, $([R^2] - [R]^2)$ and $c/2$. The first is a property of the rain field only, namely, the variance with respect to the averaging area. The second is dependent only on the local curvature of the $R - T$ relation. The formula (6) also shows that R_E always underestimates $[R]$ because the term $T''/T' = -c$ is strictly negative. Hereafter, the true bias of the beam-filling error δR is denoted by β and the first-order bias δR_{FO} is denoted by β_{FO}.

3. The bias of the beam-filling error based on the first-order formula: White noise case

We derive the general formula for bias of the beam-filling error using the first-order formula when the rain rate is spatial white noise. A rain-rate field is said to be a white noise field if

$$\rho\{R(\mathbf{x}), R(\mathbf{x}')\} = \begin{cases} 1, & \mathbf{x} = \mathbf{x}' \\ 0, & \mathbf{x} \neq \mathbf{x}', \end{cases}$$

where ρ is the spatial correlation between two points $\mathbf{x}$ and $\mathbf{x}'$ in the field. Let's assume that the rain rate R is white noise with mean $E(R)$ and variance $\mathrm{var}(R)$. The bias of the beam-filling error based on the first-order formula is written as

$$\beta_{\mathrm{FO}} = E(\delta R_{\mathrm{FO}}) = \frac{c}{2} E([R^2] - [R]^2).$$

We have

$$E([R^2]) = E(R^2),$$

$$E([R]^2) = \frac{\mathrm{var}(R)}{N^2} + E^2(R).$$

From these last formulas we obtain the bias of the beam-filling error for the general white noise case $\beta_{\mathrm{FO}}(\mathrm{WN})$,

$$\beta_{\mathrm{FO}}(\mathrm{WN}) = \frac{c}{2}\left(1 - \frac{1}{N^2}\right)\mathrm{var}(R).$$

We can infer the following things from the last formula, which is the first-order formula for bias when the rain rate is white noise (referring to the tile partitioning of Fig. 1).

1) The bias increases as the R–T relation curvature coefficient c increases.

2) The bias increases as the number of tiles N^2 increases but reaches a finite upper bound. This is consistent with the result of Chiu et al. (1990).

3) The bias increases as the variance of rain rate increases.

4) The three factors c, N^2, and $\mathrm{var}(R)$ affect the bias multiplicatively.

5) The bias computed from the first-order formula is independent of the rain-rate distribution. It depends only on second moments of rain rate. This will not hold in general.

a. Binomial model

We compute the bias of the beam-filling error using the exact formula and using the first-order formula when the rain field is binomial. Consider subdividing the FOV into N^2 tiles. Let X of the N^2 tiles be raining with rain rate r, and in the rest of the $N^2 - X$ tiles there is no rain. If the probability of rain in an individual tile is p and one tile is independent of the other, we can adopt a binomial model for the rain field. For a given realization of the model, the area-average rain rate in the FOV is

$$[R] = \frac{Xr}{N^2}$$

and

$$[R^2] = \frac{Xr^2}{N^2}.$$

Since X is binomial with parameters N^2 and p, the bias of the beam-filling error is

$$\beta = E(\delta R) = rp + \frac{1}{c}\sum_{x=0}^{N^2} \log\left\{\left(1 - \frac{x}{N^2}\right) + \frac{x}{N^2}\exp(-cr)\right\}P(x), \quad (10)$$

where $P(x) = \binom{N^2}{x}p^x(1-p)^{N^2-x}$. When we use the first-order formula of the beam-filling error, we can derive the bias

$$\beta_{\mathrm{FO}} = \frac{c}{2} r^2 p(1-p)\left(1 - \frac{1}{N^2}\right). \quad (11)$$

When p is small and N^2 is large, we can use the Poisson model as the approximation of the binomial model. Then we have the bias formula of the beam-filling error as

$$\beta_{\mathrm{FO}} = \frac{c}{2}pr^2\left(1 - p - \frac{1}{N^2}\right).$$

b. Mixed gamma model

Rain rates are inherently patchy. That is there is a finite probability that at a point of space–time there is no rain. This leads to a probability density function with a delta function spike at zero rain rate. Formally, we say the rain rate R has a positive probability $1 - p$ for the event $\{R = 0\}$ and $\Pr(R = r) = 0$, $r > 0$. More precisely, let G be the cumulative distribution function of R, $G(r) = P(R \leqslant r)$. Then it can be represented as a convex combination of two increasing functions H and F:

$$G(r) = (1-p)H(r) + pF(r),$$

where

$$H(r) = \begin{cases} 0, & r < 0 \\ 1, & r \geqslant 0, \end{cases}$$

and F is a continuous distribution function such that $F(r) = 0$, $r \leqslant 0$, with a density $f(r) = F'(r)$, $r > 0$. The moment generating function of this distribution is given by

$$M_R(t) = \int_{-\infty}^{\infty} \exp(rt) dG(r)$$
$$= (1-p) + p \int_0^{\infty} \exp(tr) f(r) dr,$$

and the kth moment of this distribution is given by

$$E(R^k) = \int_{-\infty}^{\infty} r^k dG(r) = p \int_0^{\infty} r^k f(r) dr.$$

Some authors such as Neyman and Scott (1967) report very good fit of the gamma distribution to precipitation amounts. In this section, we will investigate the bias of the beam-filling error assuming that the rain-rate distribution is (mixed) gamma. Recall that the gamma density is given by

$$f(r, \theta) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} r^{\alpha-1} \exp(-\lambda r), & r > 0 \\ 0, & r \leqslant 0, \end{cases}$$

where $\theta = (\alpha, \lambda)$, $\alpha, \lambda > 0$.

We can compute the following equations using the moment generating function:

$$E([R]) = p \frac{\alpha}{\lambda},$$

$$E([R^2]) = p \frac{\alpha(\alpha+1)}{\lambda^2}.$$

Therefore, the bias of the beam-filling error using the first-order formula is

$$\beta_{\mathrm{FO}} = \frac{c}{2}\left(1 - \frac{1}{N^2}\right) \frac{p}{\lambda^2} [(1-p)\alpha^2 + \alpha]. \quad (12)$$

When we consider only the rain rate conditional on $R > 0$, that is, $p = 1$, the rain rate follows the usual gamma distribution, and the bias of the beam-filling error is given by

$$\beta_{\mathrm{FO}} = \frac{c}{2}\left(1 - \frac{1}{N^2}\right) \frac{\alpha}{\lambda^2}. \quad (13)$$

From (13), we see that the bias increases as α and $1/\lambda$ increase.

c. *Mixed lognormal model*

There are many studies that suggest that the rain rate follows the mixed lognormal distribution (e.g., Kedem et al. 1990). In this section, we will investigate the bias of the beam-filling error assuming that the rain-rate distribution is mixed lognormal and lognormal based on the first-order formula. Recall that the lognormal density is given by

$$f(r, \theta) = \begin{cases} \dfrac{1}{r\sigma(2\pi)^{1/2}} \exp\left[-\dfrac{1}{2\sigma^2}(\log r - \mu)^2\right], & r > 0 \\ 0, & r \leqslant 0. \end{cases}$$

Since we assumed that the rain rate has a mixed lognormal distribution, we have

$$\mathrm{var}(R) = p \exp(2\mu + \sigma^2)[\exp(\sigma^2) - p]. \quad (14)$$

Therefore, the bias of the beam-filling error using the first-order formula is

$$\beta_{\mathrm{FO}} = \frac{c}{2}\left(1 - \frac{1}{N^2} p \exp(2\mu + \sigma^2)[\exp(\sigma^2) - p]\right). \quad (15)$$

When we consider only the rain rate conditional on the $R > 0$, that is, $p = 1$, the rain rate follows the usual lognormal distribution, and the bias of the beam-filling error is given by

$$\beta_{\mathrm{FO}} = \frac{c}{2}\left(1 - \frac{1}{N^2}\right) \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

4. The bias and variability of the beam-filling error: Gaussian random field

We derive some statistical properties of the beam-filling error using the first-order formula when the rain rate is an homogeneous and isotropic Gaussian random field. Let the rain rate be a homogeneous and isotropic Gaussian random field with mean μ and covariance function $K(i,j) = E\{(R_{s,t} - \mu)(R_{s+i,t+j} - \mu)\} = K\{(i^2 + j^2)^{1/2}\}$.

a. *The bias of the beam-filling error*

By the statistical properties of the Gaussian random field, we can derive

$$E([R^2]) = \sigma^2 + \mu^2, \tag{16}$$

$$\begin{aligned}E([R]^2) &= \frac{1}{N^4}\sum_{i=-N}^{N}\sum_{j=-N}^{N}(N-|i|)\\&\quad\times(N-|j|)K(i,j)+\mu^2\\&=\frac{1}{N^4}\{N^2\sigma^2-4\sum_{i=1}^{N}\sum_{j=1}^{N}(N-|i|)\\&\quad\times(N-|j|)K(i,j)\}+\mu^2,\end{aligned} \tag{17}$$

where σ^2 is the variance of the rain-rate field evaluated on any given tile. The bias of the beam-filling error for the Gaussian random field is

$$\begin{aligned}\beta_{FO} &= \frac{c}{2}\left\{\sigma^2-\frac{1}{N^4}\sum_{i=-N}^{N}\sum_{j=-N}^{N}(N-|i|)\right.\\&\quad\left.\times(N-|j|)K(i,j)\right\}\\&=\frac{c}{2}\left(1-\frac{1}{N^2}\right)\sigma^2-\frac{c}{2}\left\{\frac{1}{N^4}\sum_{\substack{i=-N\\(i,j)\neq(0,0)}}^{N}\sum_{j=-N}^{N}(N-|i|)\right.\\&\quad\left.\times(N-|j|)K(i,j)\right\}.\end{aligned} \tag{18}$$

When the rain rate is white noise Gaussian, we have $K(i,j)=0$ for $(i,j)\neq(0,0)$. Thus, the bias of the beam-filling error is obtained from (18):

$$\beta_{FO}=\frac{c}{2}\left(1-\frac{1}{N^2}\right)\sigma^2. \tag{19}$$

We discussed the general properties of the bias of the beam-filling error for white noise rain rate. We additionally infer that a nonzero mean rain rate μ does not affect the bias for white noise Gaussian rain rate. The relationship between the bias for Gaussian rain-rate random field $\beta_{FO}(\mathrm{GR})$ and that of white noise Gaussian rain rate $\beta_{FO}(\mathrm{WGR})$ is obtained from (18) and (19),

$$\begin{aligned}\beta_{FO}(\mathrm{GR}) &= \beta_{FO}(\mathrm{WGR})-\frac{c}{2}\left\{\frac{1}{N^4}\sum_{\substack{i=-N\\(i,j)\neq(0,0)}}^{N}\sum_{j=-N}^{N}(N-|i|)\right.\\&\quad\left.\times(N-|j|)K(i,j)\right\}.\end{aligned} \tag{20}$$

From (20), we see that when the rain rate is positively correlated, the bias of Gaussian rain-rate random field is less than the bias of white noise. Also note that this is consistent with the N^2 behavior in the white noise case; that is, the correlations lead to a smaller "effective" value of N. We add that the Gaussian property was not needed in the derivation of the above results, but it is necessary in obtaining the results of the next subsection.

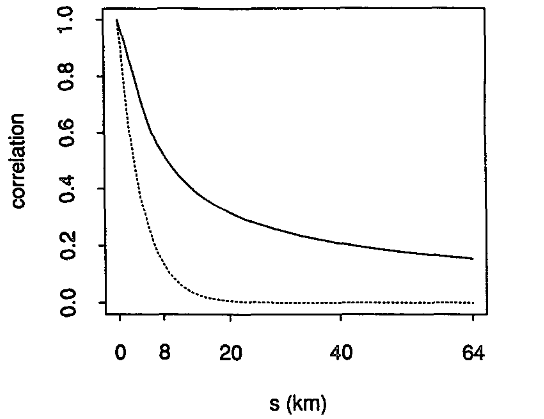


FIG. 2. Two spatial correlations employed in the simulation. Solid curve shows the spatial correlation given in Eq. (23) and dotted curve shows the spatial correlation given in Eq. (24).

b. The variability of the beam-filling error

The variance of the beam-filling error based on the first-order formula is

$$\mathrm{var}(\delta R_{FO})=\frac{c^2}{4}\mathrm{var}\left\{\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}R_{ij}^2-\frac{1}{N^4}(\sum_{i=1}^{N}\sum_{j=1}^{N}R_{ij})^2\right\}.$$

We can evaluate this for the Gaussian case

$$\begin{aligned}&\mathrm{var}(\delta R_{FO})\\&=\frac{c^2}{2N^4}\left(\sum_{i=-N}^{N}\sum_{j=-N}^{N}(N-|i|)(N-|j|)K^2(i,j)\right.\\&\quad+\frac{1}{N^4}\{\sum_{i=-N}^{N}\sum_{j=-N}^{N}(N-|i|)(N-|j|)K(i,j)\}^2\\&\quad\left.-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{\sum_{k=1}^{N}\sum_{l=1}^{N}K(i-k,j-l)\}^2\right).\end{aligned} \tag{21}$$

We also can derive the variability of the beam-filling error for the white noise Gaussian rain rate from (21)

$$\begin{aligned}\mathrm{var}(\delta R_{FO})&=\frac{c^2}{4}\left\{\sigma^4\left(1-\frac{1}{N^4}\right)-\sigma^4\left(1-\frac{1}{N^2}\right)^2\right\}\\&=\frac{c}{N^2}\left\{\frac{c}{2}\sigma^4\left(1-\frac{1}{N^2}\right)\right\}=\frac{c}{N^2}\sigma^2\beta_{FO}.\end{aligned}$$

We can say the same things regarding c, μ, and σ^2 about the variability of the beam-filling error as we did about the bias. The relationship between variability and c, σ^2 is quadratic, while the relationship between bias and c, σ^2 is linear. However, as N^2 increases the bias increases, but the variability decreases because the denominator has N^4 and the numerator has N^2. Short (1990) calls this the "ensemble field of view" case since it fully contains the ensemble distribution of rain rates.

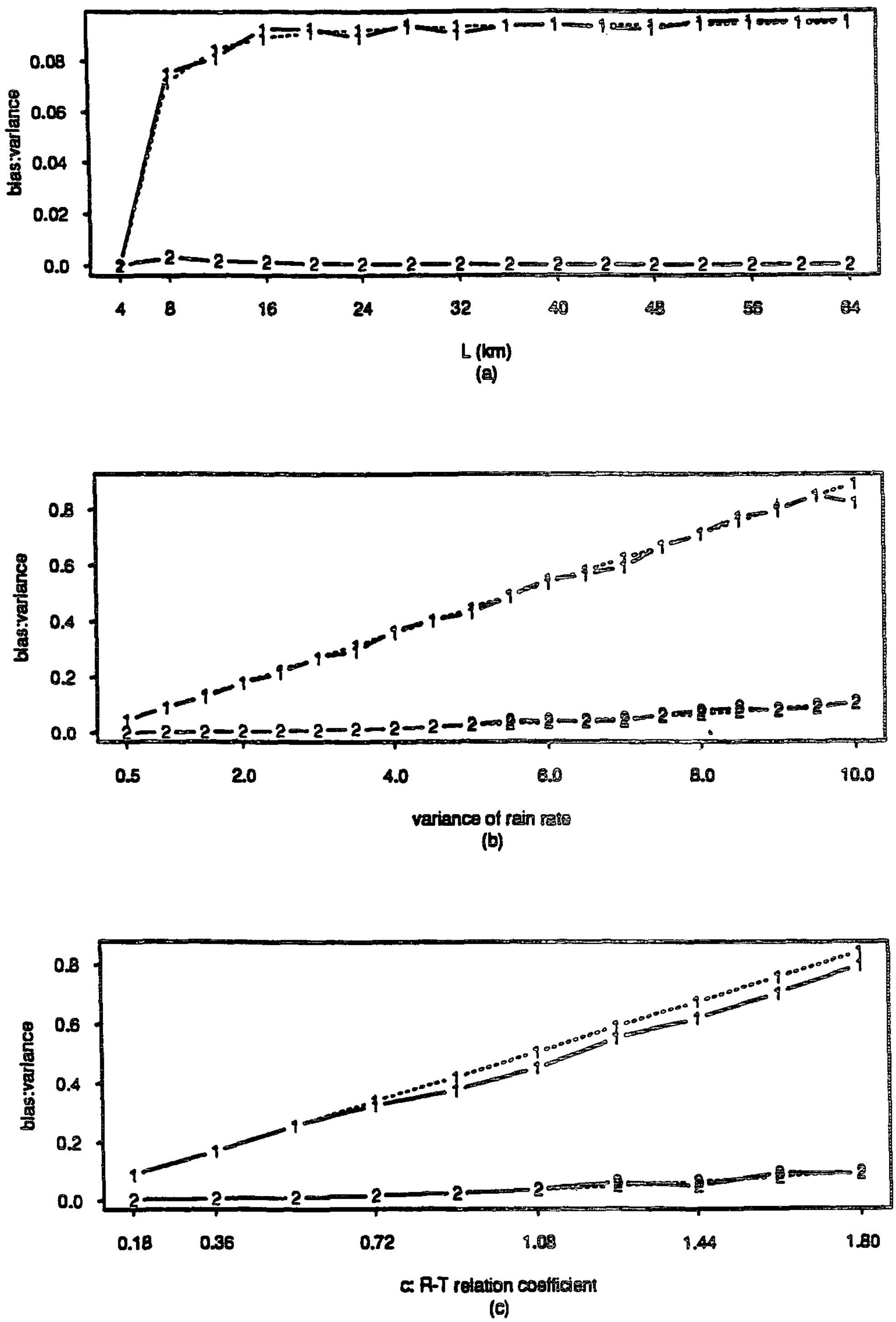


FIG. 3. Bias and variance of the beam-filling error when rain rate is Gaussian white noise. Bias and variance as a function of (a) L; (b) variance of the rain rate; (c) R–T relation coefficient. Key: bias, 1; variance, 2; approximation, dotted line; simulated value, solid line.

c. Upper bound of bias for Gaussian random field

The upper bound of bias of the beam-filling error can be obtained by an application of Jensen's inequality (Rao 1973). Let the rain rate R be a Gaussian random field with mean μ, variance σ^2, and covariance function K. Then, $\exp(-cR_{ij})$ is lognormal with mean $\exp\{-\mu c + (c^2\sigma^2/2)\}$. Thus, we have

$$E([R]) = \mu,$$

$$E(R_E) = -\frac{1}{c} E\{\log[\exp(-cR)]\}.$$

By applying Jensen's inequality to R_E, the bias β of the beam-filling error for a Gaussian random field is

$$\beta = E([R]) - E(R_E) \leqslant \mu - \mu + \frac{c\sigma^2}{2} = \frac{c}{2}\sigma^2. \quad (22)$$

By (19) and (22), we can see that the upper bound on the bias for a Gaussian random field is equivalent to the bias computed by the first-order formula for white noise Gaussian random fields as N goes to infinity.

5. Monte Carlo simulation study

We conduct a Monte Carlo simulation to investigate how well the first-order formula explains the actual beam-filling error when the rain rate is Gaussian, mixed gamma, or mixed lognormal. We also introduce some other methods of calculating the bias of the beam-filling error. For the binomial model, we compute the bias of the actual beam-filling error and compute them with the bias of the beam-filling error computed through the first-order formula. We replicate $N \times N$ random numbers for given parameters of the assumed distribution. Generally, we restrict N to be less than or equal to 16 and $c = 0.19$, which is the R–T relation coefficients. Figures 3–5 compare the results of our simulations with the results computed from the first-order formula. Figures 3–5 use a dotted line to denote the bias or variance of the beam-filling error computed by the first-order formula; the solid line denotes the bias or variance computed from simulated data. For the binomial model, a solid line is used for the actual value of bias of the beam-filling error. We derived the formula of the variability of the beam-filling error for binomial model and mixed gamma model based on the first-order formula, but we do not present them here because they are so complicated. We examine the variability of the beam-filling error via the simulation study. Here we show a few figures illustrating the bias and variance of the beam-filling error, but every statement in this section is actually based on simulations.

a. First-order formula

1) GAUSSIAN MODEL

We take zero as the mean rain rate for a Gaussian random field in our simulation, but we can easily show that the results hold for nonzero mean rain rate. Of course, rain rates are hardly Gaussian with zero mean, but the results we are about to show hold equally for the mean shifted to some positive value ($\geqslant \sigma$). Generally, we use $\sigma^2 = 1$ and $c = 0.19$. First, we replicate $N \times N$ Gaussian white noise numbers 100 times. We also replicate an $N \times N$ Gaussian random field 100 times. We generate the random field only at $N = 1, 2, 4, 8$, and 16 to take advantage of the fast Fourier transform. When we generate the Gaussian random field, we adopt the method of generating Gaussian random fields used in Bell (1987). When we generate the Gaussian random field and investigate the effect of spatial correlation in the bias and variability of the beam-filling error, we use the following two different spatial correlation functions $\rho_1(s)$ and $\rho_2(s)$,

$$\rho_1(s) = \left(\frac{s}{4} + 0.63682\right)^{-2/3}, \quad s \geqslant 4 \text{ km}, \quad (23)$$

$$\rho_2(s) = \exp\left(\frac{-s}{4}\right), \quad s \geqslant 4 \text{ km}. \quad (24)$$

The functions $\rho_1(s)$ and $\rho_2(s)$ are frequently used in analyses of the GATE data (Bell 1987). The shape of these spatial correlation functions is in Fig. 2.

We can see from Figs. 3 and 4 that the first-order formula of the beam-filling error we used fits quite well. We also can conclude the following from Fig. 3 about the bias and variability of the beam-filling error from the simulation when the rain rate is Gaussian white noise.

1) The bias increases as the R–T relation coefficient c, the number of tiles N^2, and the variability σ^2 of rain rate increases.

2) The three factors c, N^2, and σ^2 affect the bias multiplicatively.

3) The variability increases as c, σ^2 increases, and the relationship between variability and c, σ^2 is quadratic, not linear.

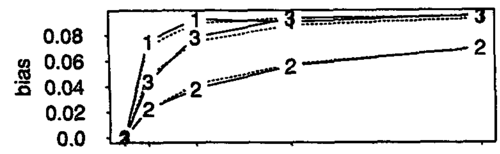
4) The bias increases as N^2 increases; but the variability increases at first then decreases.

5) The bias approaches the upper bound of the bias we derived in the section 4c.

We can also say the following about the bias and variability of the beam-filling error from the simulation when the rain rate is a Gaussian random field (Fig. 4).

1) The bias increases and the variability decreases as N^2 increases.

2) Since the spatial correlation we considered in the simulation is positively correlated, the bias of the Gaussian random field is always less than that of Gaussian white noise. Moreover, since the magnitude of ρ_2 is less than that of ρ_1, using ρ_1 gives smaller bias than that of ρ_2.



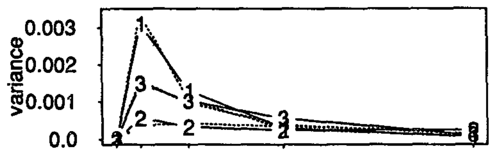


FIG. 4. Comparison of bias and variance of the beam-filling error when rain rate is Gaussian random field and Gaussian white noise. (a) Bias of Gaussian random field and Gaussian white noise as a function of L; (b) variance of Gaussian random field and Gaussian white noise as a function of L. Key: white noise, 1; Gaussian random field with spatial correlation given in Eq. (23), 2; Gaussian random field with spatial correlation given in Eq. (24), 3; approximation, dotted line; simulated value, solid line.

3) The variability of the Gaussian random field is always less than that of Gaussian white noise. Moreover, using ρ_1 gives smaller variability than that of ρ_2.

2) BINOMIAL MODEL

We compute the bias and variance of the beam-filling error to investigate how the first-order formula of the beam-filling error explains the real beam-filling error when the rain rate is binomial. We can see from Fig. 5 that the first-order formula for the beam-filling error fits qualitatively for binomial model. We use $c = 0.19$, $p = 0.1$, and $r = 4$ for the binomial model to produce figures. We can conclude the following things about the bias of the beam-filling error when the rain rate is binomial (N^2, p) (Fig. 5).

1) The bias increases and the variability decreases as the number of tiles N^2 increases.

2) The bias and variability increase as the probability p of rain increases.

3) The bias and the variability increase as the rain rate r increases.

3) MIXED GAMMA AND MIXED LOGNORMAL MODEL

First, we replicate gamma (α, λ) random numbers 100 times. We use parameters $\alpha = 0.33$ and $1/\lambda = 12.25$, which are chosen by using the method of moments to match conditional rain-rate statistics from GATE data, having a first moment of 4 and a second moment of 65 (Short et al. 1991). We also replicate lognormal(μ, σ) random numbers with parameters $\mu = 0.685$ and $\sigma = 1.184$ to tune the GATE data (Short et al. 1991). We can conclude that the first-order formula of the beam-filling error we used does *not* fit well for the bias of the beam-filling error from the simulation when the rain rate is gamma (α, λ) and lognormal (μ, σ).

b. *Alternative methods of first-order formula*

We examined in section 5a(3) that the first-order formula is not satisfactory to evaluate the bias of the beam-filling error for (mixed) gamma and lognormal distribution. In this section, we introduce some other methods to compute the bias of the beam-filling error.

1) HIGHER-ORDER APPROXIMATION FORMULA

The first-order formula for the bias of the beam-filling error takes into account only the variability of the random field. We may improve the first-order formula when we consider the higher moments of the random field. We can derive the skewness correction formula based on the third-order Taylor expansion of the T–R relation $T(R)$

$$\begin{aligned}\beta_{SK} &= \frac{c}{2}\left(1 - \frac{1}{N^2}\right)\mathrm{var}(R) \\ &\quad - \frac{c^2}{6}\left(1 - \frac{1}{N^2}\right)\left(1 - \frac{2}{N^2}\right)E(R-\mu)^3 \\ &= \beta_{FO} - \frac{c^2}{6}\left(1 - \frac{1}{N^2}\right)\left(1 - \frac{2}{N^2}\right)E(R-\mu)^3, \quad (25)\end{aligned}$$

where $\mu = E(R)$. We also can derive the kurtosis correction formula, which includes up to fourth moments of the random field. The skewness and kurtosis correction formula do not work well for mixed gamma and lognormal distribution. We need to include the higher moments in order to compute more exact bias formula, but its derivation is complicated.

2) LARGE FOV APPROXIMATION FORMULA

The first-order (skewness and kurtosis correction) approximation formula for the bias of the beam-filling error does not work well for (mixed) gamma and (mixed) lognormal distribution. However, we can expect more precise bias when we have large FOV, that is, N^2 is large, of white noise random field. Note that for the (mixed) gamma or lognormal distribution, the

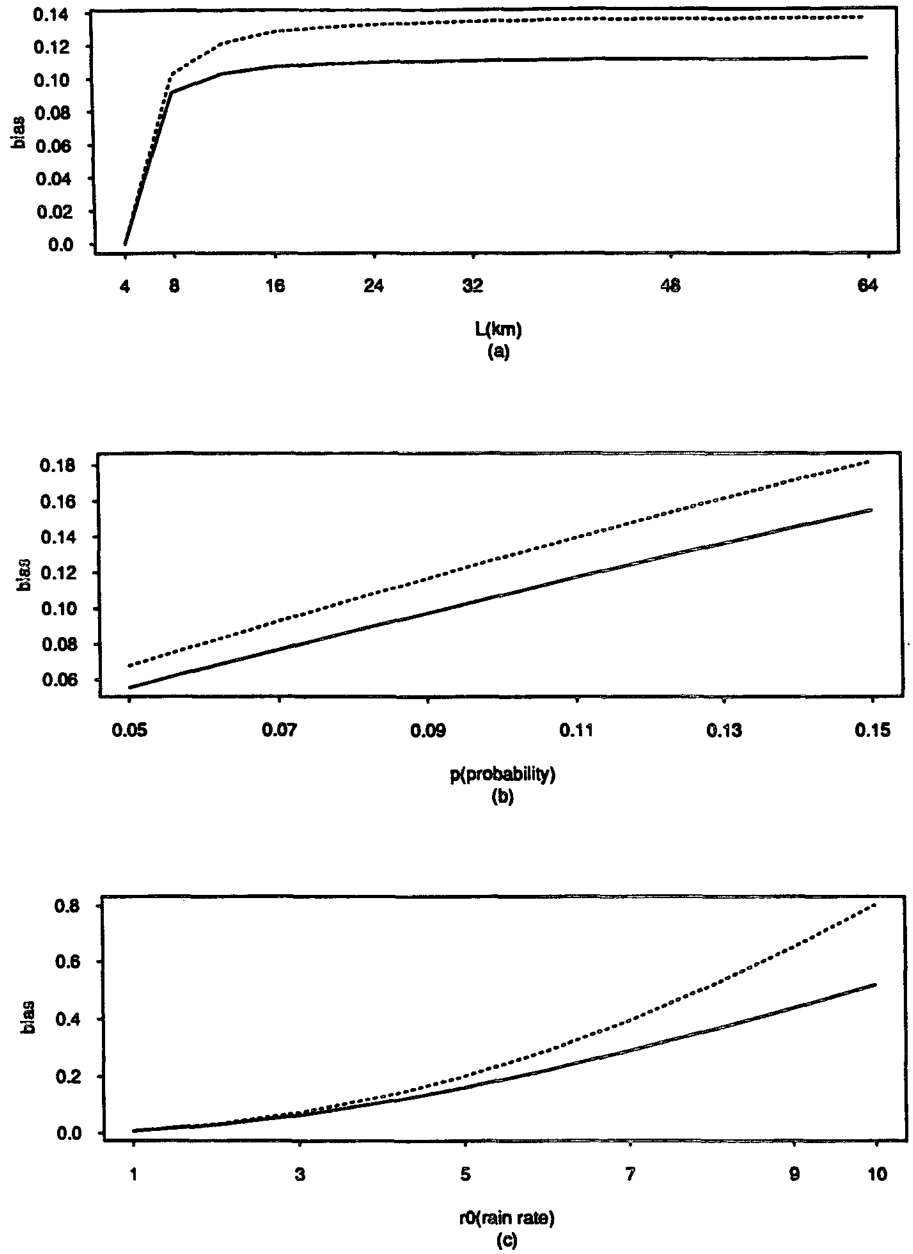


FIG. 5. Comparison of exact bias and the bias based on the first-order formula of the beam-filling error when rain rate is binomial random field. Bias of binomial random field as a function of (a) L with $p = 0.1$ and $r_0 = 4$; (b) p with $L = 16$ km and $r_0 = 4$; (c) r_0 with $p = 0.1$ and $L = 16$ km. Key: approximation, dotted line; exact value, solid line. The variable r_0 is the rain rate when raining.

bias saturates are not for a big N^2. The beam-filling error is

$$\delta R = [R] + \frac{1}{c}\log\{[\exp(-cR)]\}. \qquad (26)$$

Applying the *law of large numbers* (Rao 1973), we have

$$[R] \rightarrow E(R) \quad \text{in probability} \qquad (27)$$

and

$$\log\{[\exp(-cR)]\} \rightarrow \log E\{\exp(-cR)\} \quad \text{in probability.} \qquad (28)$$

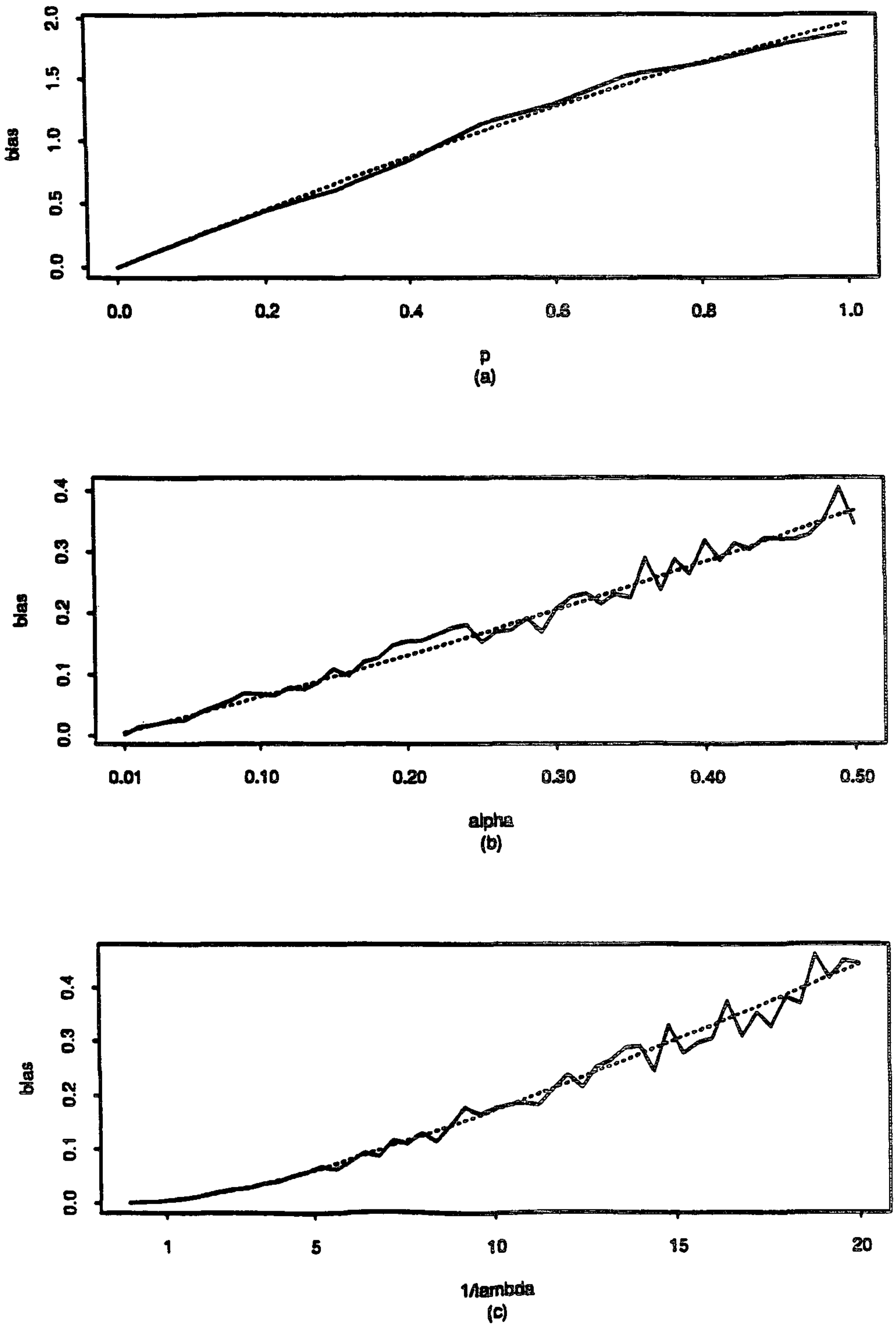


FIG. 6. Comparison of bias and the bias based on the large-FOV formula of the beam-filling error when rain rate is mixed gamma random field for L = 40 km. Bias of the mixed gamma model as a function of (a) p with α = 0.33 and $1/\lambda$ = 12.25; (b) α with p = 0.1 and $1/\lambda$ = 12.25; (c) $1/\lambda$ with p = 0.1 and α = 0.33. Key: approximation, dotted line; simulated value, solid line.

Thus, the beam-filling error converges in probability

$$\delta R = [R] + \frac{1}{c}\log\{[\exp(-cR)]\} \rightarrow E(R) + \frac{1}{c}\log E\{\exp(-cR)\}. \quad (29)$$

For the mixed gamma distribution, we can derive

$$\delta R \rightarrow p\frac{\alpha}{\lambda} + \frac{1}{c}\log\left\{(1-p) + p\left(\frac{\lambda}{\lambda + c}\right)^{\alpha}\right\}. \quad (30)$$

We generate Fig. 6 for the mixed gamma distribution with $L = 40$ km, that is, $N = 10$ FOV. Figure 6 shows that this large FOV approximation works well for mixed gamma distribution with the generated random numbers. Note that for the (mixed) lognormal distribution, the quantity $E\{\exp(-cR)\}$ is not available as an analytical form, so we cannot use the large sample approximation. Computation of the bias is still problematic for the lognormal distribution with this method.

3) Regression method

We may compute the bias of the beam-filling error using the relationship between the bias and some moment (for example, mean or variance) of the rain rate. The empirical result (Chiu et al. 1990) shows that the bias is a linear function of the mean rain rate. We also can use the fact that the bias is a linear function of the variance of the rain rate that was shown generally in section 2 based on the first-order formula for the beam-filling error. We also examine the linear relationship between the variability of the beam-filling error and the mean/variance of the rain field.

In order to apply the regression method, we generate mixed gamma random numbers 100 times for ten equally spaced values for both α and λ. This choice of parameter values results in a mean rain rate, conditional on rain, between 0.5 and 7.5. The parameter value α was chosen between 0.1 and 0.5, and $1/\lambda$ takes values from 5 to 15. For generating the mixed gamma random variables, we use $p = 0.1$ generally. We use $N = 10$ in this simulation to compare the empirical results in Chiu et al. (1990), where they gave the result of using $N = 10$. After we generate the random numbers, we fit the least-squares regression of bias/variability on mean rain rate and the least-squares regression of bias/variability on variance of the rain rate. We show the least-squares regression line and the correlation in the figures. We also put the sample correlation $\hat{\rho}$ of relevant variables we are describing in parentheses. We can say the following things about the bias and variability of the beam-filling error (Figs. 6 and 7).

1) The bias increases as p increases. We can see that the first-order formula does fit well for mixed gamma distribution.

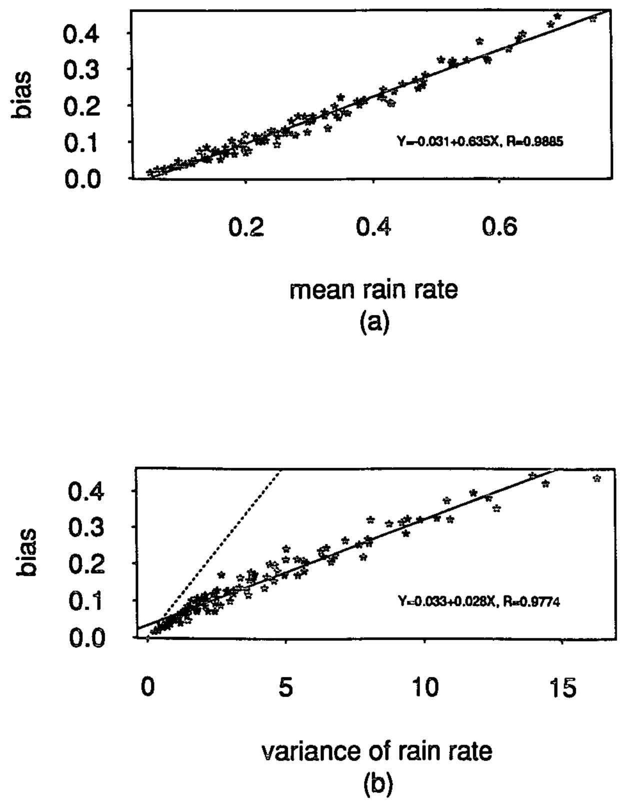


FIG. 7. Regression of bias on mean (variance) of rain rate with the simulated data when rain rate is mixed gamma random field. In simulation, $0.1 < \alpha < 0.5$, $5 < 1/\lambda < 15$, $p = 0.1$, $L = 40$ km, and $c = 0.19$ were used. The least-squares line and the correlation between bias and mean (variance) of rain rate are given in the plot. (a) Regression of bias on the mean of rain rate; (b) Regression of bias on the variance of the rain rate. In this figure R is the correlation coefficient.

2) The regression line of the bias on mean rain rate ($\hat{\rho} = 0.98$) fits very well. This result is very similar to the empirical result shown in Chiu et al. (1990).

3) The regression line of the bias on variance of the rain rate ($\hat{\rho} = 0.98$) fits well as we expected from the first-order formula of the beam-filling error.

4) The regression line of the variability of the beam-filling error on the variance of the rain rate ($\hat{\rho} = 0.95$) fits well and gives a little better result than the regression line of the variability on the mean of the rain rate ($\hat{\rho} = 0.91$).

We conduct the same simulation study with the (mixed) lognormal distribution as we did with the (mixed) gamma distribution. To see the relationship between bias of the beam-filling error and mean (variance) of the rain rate for fixed $N = 10$, we generate mixed lognormal random numbers 100 times for ten equally spaced values for both μ and λ. This choice of parameter values results in a mean rain rate, conditional on rain, between 1 and 7.5 mm h^{-1}. The parameter value μ was chosen between 0.01 and

1, and σ takes values from 0.01 to 1.5 variability of the beam-filling error. For generating the mixed lognormal random variables, we use $p = 0.1$ and $N = 10$ generally. We can see the following things about the bias and variability of the beam-filling error (Fig. 8).

1) The bias increases as p increases. We can see that the first-order formula does not fit well for the mixed lognormal distribution, especially for large p.

2) The regression line of the bias on mean rain rate ($\hat{\rho} = 0.95$) fits very well. This result is very close to the empirical result shown in Chiu et al. (1990).

3) The regression line of the bias on variance of the rain rate ($\hat{\rho} = 0.86$) fits well as we expected from the first-order formula for the beam-filling error, but it is worse than that of the mixed gamma distribution ($\hat{\rho} = 0.97$).

4) The regression line of the variability of the beam-filling error on the variance of the rain rate ($\hat{\rho} = 0.89$) fits well and gives a little better result than the regression line of the variability on the rain rate ($\hat{\rho} = 0.68$). The regression line of the variability of the beam-filling error on the variance (mean) of the rain rate for the mixed gamma distribution gives a better result than for the mixed lognormal distribution.

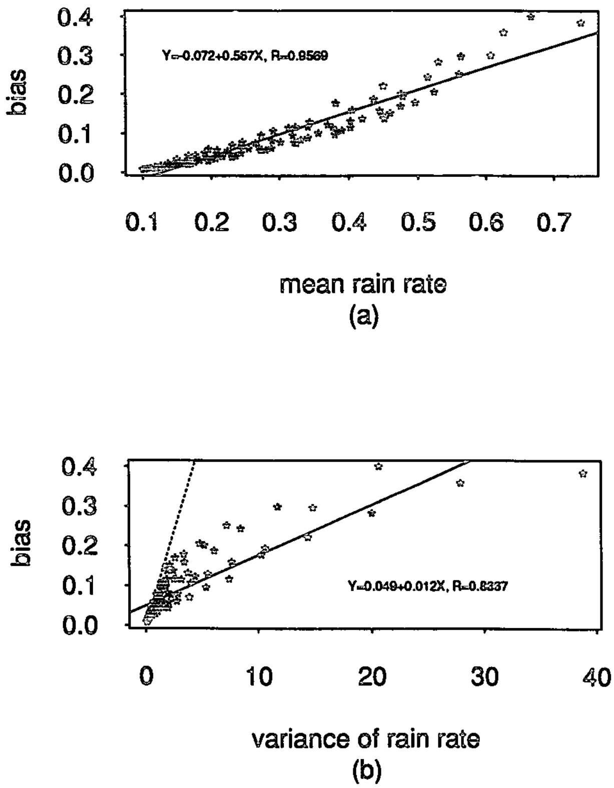


FIG. 8. Regression of bias on mean (variance) of rain rate with the simulated data when rain rate is mixed lognormal random field. In simulation, $0.01 < \mu < 1$, $0.01 < \sigma < 1.5$, $p = 0.1$, $L = 40$ km, and $c = 0.19$ were used. The least-squares line and the correlation between bias and mean (variance) of rain rate are given in the plot. (a) Regression of bias on the mean of rain rate. (b) Regression of bias on the variance of the rain rate. In this figure R is the correlation coefficient.

6. Summary and conclusions

In this paper we have considered a variety of stochastic rain field models in connection with the beam-filling problem. In all cases we subdivide the square FOV into square tiles whose dimensions induce a minimum length scale in the rain fields. The fields considered include a variety (Gaussian, binomial, mixed gamma, mixed lognormal), leading up to some cases that appear to be close to those likely to be encountered in nature. We tested a first-order formula introduced by Chiu et al. (1990) over all the field types. The simple formula holds remarkably well for Gaussian and binomial fields, but begins to fail for distributions that are very skewed such as the mixed lognormal. We introduced some other alternatives of the first-order formula to compute the bias of the beam-filling error for mixed gamma and lognormal random field.

The failure of the highly non-Gaussian fields to satisfy the first-order formula seems to be related to the fact that the beam-filling error depends upon rather high moments of the rain field. The first-order formula makes use only of second moment properties of the field. For Gaussian rain fields the higher moments are essentially already described by the second-order moments.

Several rather important results emerge throughout the study. The beam-filling error tends to saturate at a value of about 50% of the rain rate as the FOV tends to large values compared to the tile size. The variability or random error associated with beam filling tends to zero as the FOV becomes larger. Both of these results were anticipated by Chiu et al. (1990), Short (1990), and Short and North (1990) in their work with GATE and ESMR-5 data. This is a result that is rather insensitive to the probability density function of the rain-rate field. We reassert here that the conventional wisdom stating that the smallest footprint (which leads to small bias) is most desirable, but may not hold in practice. We say this because even though the small footprint leads to a smaller bias, it may not be a predictable bias, whereas the large footprint has a large but rather predictable bias that in principle can be removed. Furthermore, it appears that the variance of the random error is smaller with larger footprint. While much remains to be done in the beam-filling problem, including the effects of multiple radiometer channels and the gathering and evaluation of more rain field data, we can infer that the method of climatological removal of the bias is a reasonable approach to take.

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