Relaxation toward Observations in Level and Isopycnic Models

PETER D. KILLWORTH

Southampton Oceanography Centre, Southampton, United Kingdom

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ABSTRACT

Relaxation toward observed values is frequently undertaken in ocean models for numerous reasons. In level models, relaxation of some quantity takes the form of a linear ''nudging'' term proportional to the difference between observed and computed value of that quantity. In isopycnic models, relaxation of tracers and/or layer depth toward observed values is often employed as an equivalent. This note shows that relaxation of temperature and salinity—and hence density—in a level model is *not* equivalent to relaxation either of those tracers or of layer thickness in an isopycnic model. Comparison of layer thickness tendencies in the two model types shows that these differ by the ratio of observed vertical density gradient to model vertical density gradient. Only in the special case where the model remains close to observations are the two methods the same to leading order. It is not obvious whether isopycnic or level relaxation is to be preferred.

1. Introduction

Ocean models seldom, if ever, perform as those running them would desire. One way of adjusting the response of a model is to use relaxation toward observations within the model, as well as providing surface forcing conditions. Limited area models frequently have recourse to this approach to handle open boundary conditions. A good example was the DYNAMO intercomparison (Dynamo Group 1997). Three models were used (level, isopycnic, and sigma coordinate models) to describe the North Atlantic Ocean at a resolution of ⅓°. In order to maintain water mass structure near the boundaries of the models, relaxation toward observations was employed at the north and south boundaries, and also near the Mediterranean outflow, since Gibraltar was closed in these models. The models all relaxed the appropriate variables toward (smoothed) observed fields with spatially varying timescales.

The word ''appropriate'' in the last sentence is apposite. What variables should be relaxed? The observations, under all but very unusual circumstances, consist of tracer variables, that is, temperature and salinity. In level or sigma coordinate models, it seems natural to relax precisely these variables toward their observed values. In isopycnic models, however, two different approaches are needed depending on how the model is organized: either both temperature and salinity are maintained as independent variables (Oberhuber 1993) and layer thicknesses adjusted each time step to maintain their density values, or one tracer only is maintained (Bleck et al. 1992) with the other determined diagnostically. In the former case, the natural approach would be to relax temperature and salinity as before. In the latter case, normal practice is to relax the active tracer and layer elevations to observed values for that density. Consider the relaxation of two elevations that neighbor vertically. The difference between these two, except for possible differences at the top and bottom of a water column, is identical to the relaxation of the layer thickness toward its observed value. Henceforth, then, we shall use layer thickness relaxation as a model (but the differences at surface and floor may not be trivial). The system will be discussed in terms of a continuously stratified model, to avoid extra differences which inevitably occur due to numerical truncation; these are thus neglected in what follows.

This note shows that neither of these practices is equivalent to relaxation of temperature and salinity in level models because of an unbalanced treatment of layer thickness. It is not clear, however, whether isopycnic or level model treatments are to be preferred in general.

2. Relaxation in level and isopycnic models

Consider the following subset of the equations in a level, or z-coordinate model, where fields are being relaxed toward observations:

Corresponding author address: Dr. Peter D. Killworth, James Rennell Division for Ocean Circulation, Southampton Oceanography Centre, Empress Dock, Southampton S014 3ZH, United Kingdom.
E-mail: peter.d.killworth@soc.soton.ac.uk

$$\frac{DT}{Dt} = T_t + \mathbf{u} \cdot \nabla T = \frac{1}{\tau}[T_o(x, y, z) - T] \quad \text{(L1)}$$

$$\frac{DS}{Dt} = S_t + \mathbf{u} \cdot \nabla S = \frac{1}{\tau}[S_o(x, y, z) - S], \quad \text{(L2)}$$

where the equation lettering L represents levels; I will be used for isopycnic equations. Here T and S represent temperature and salinity (or indeed any other tracer), $\mathbf{u}$ the velocity field, and τ is a relaxation time. Suffix ''o'' refers to the observed values of the fields. An explicit reference to location is included for later comparison. Assuming a linear equation of state, these combine to an equation for density ρ:

$$\frac{D\rho}{Dt} = \rho_t + \mathbf{u} \cdot \nabla \rho = \frac{1}{\tau}[\rho_o(x, y, z) - \rho] \equiv Q \quad \text{(L3)}$$

(the expression for the rhs would be more complicated with a nonlinear equation of state, making the point of this note still stronger).

We now consider an isopycnic model that at time t has exactly the same velocity, pressure, and tracer fields as the z-coordinate model, so that tendency terms can be compared sensibly. In an isopycnic model, relaxation is usually carried out on the predicted tracer (e.g., temperature) and layer thickness

$$\frac{DT}{Dt} = T_t + \mathbf{u} \cdot \nabla T = \frac{1}{\tau}[T_o(x, y, \rho) - T] \quad \text{(I1)}$$

$$\frac{Dh}{Dt} = h_t + \nabla \cdot (\mathbf{u}h) = \frac{1}{\tau}[h_o(x, y, \rho) - h], \quad \text{(I2)}$$

where the layer thickness h is z_ρ, with z being the height of an interface. The operator D/Dt measures the same quantity in both coordinate systems. Note that here the relaxation is toward the observed value at that density, not at the local depth, so that the rhs of (I1) will not be the same as that in (L1). Using two active tracers (Oberhuber 1993), one might prefer to relax both and not to relax layer thickness. The question then arises: is the relaxation of density (L3) equivalent to that of layer thickness (I2) and/or that of tracer (I1)?

To proceed, convert (L3) to isopycnic coordinates. This is standard algebra; we have

$$\left.\frac{\partial}{\partial t}\right|_z \rightarrow \left.\frac{\partial}{\partial t}\right|_\rho - \frac{z_t}{z_\rho}\frac{\partial}{\partial \rho}; \qquad \frac{\partial}{\partial z} \rightarrow \frac{1}{z_\rho}\frac{\partial}{\partial \rho}.$$

Applying this to (L3) gives an expression for vertical velocity

$$w = z_t + \mathbf{u} \cdot \nabla z + Q z_\rho, \quad \text{(C1)}$$

(using C for converted equation numbers) and this, in turn, into the divergence equation ($\nabla \cdot \mathbf{u} = 0$), yields

$$z_{\rho t} + \nabla \cdot (\mathbf{u} z_\rho) + (Q z_\rho)_\rho = h_t + \nabla \cdot (\mathbf{u}h) + (Qh)_\rho = 0. \quad \text{(C2)}$$

Both (C1) and (C2) show clearly that the density relaxation Q plays the role of a diapycnic flux when converted to isopycnic coordinates.

Equation (C2) is to be compared with (I2). It is clear that in general the two results are, simply, different: there is no reason that diapycnal fluxes and relaxation terms should be similar. This holds whether (I2) has a right-hand side (when thickness *is* relaxed) or whether it does not (if both active tracers, but not thickness, are being relaxed).

3. How the forcings differ

For simple cases we can evaluate the forcing terms $-(Qz_\rho)_\rho$ and $(h_o - h)/\tau$. We suppose that

$$\left.\begin{aligned} \rho_o &= A + Bz \\ \rho &= C + Dz \end{aligned}\right\}, \quad \text{(C3)}$$

so that, inverting,

$$\left.\begin{aligned} z_o &= \frac{\rho - A}{B} \\ z &= \frac{\rho - C}{D} \end{aligned}\right\}. \quad \text{(C4)}$$

Then

$$Q = \frac{\rho_o - \rho}{\tau} = \frac{A - C + (B - D)\dfrac{(\rho - C)}{D}}{\tau}$$

$$z_\rho = \frac{1}{D},$$

so that the thickness forcing converted from level coordinates is

$$-(Qz_\rho)_\rho = \frac{D - B}{\tau D^2}. \quad \text{(C5)}$$

However, the thickness forcing in the isopycnal system is

$$\frac{z_{o\rho} - z_\rho}{\tau} = \frac{D - B}{\tau DB}. \quad \text{(C6)}$$

Thus the ratio

$$\frac{\text{thickness forcing in level model}}{\text{thickness forcing in isopycnic model}} = \frac{B}{D} = \frac{\rho_{oz}}{\rho_z}, \quad \text{(C7)}$$

so that the forcings differ by the ratio of the vertical gradients of observed and model density fields.

Another, and perhaps more intuitive, way to see this is to compute the thickness tendency directly from the two approaches. In level coordinates, we have a density tendency $\rho_{zt} \approx (\rho_{oz} - \rho_z)/\tau$, which converts to a thickness tendency

$$z_{\rho t} \approx \partial/\partial t(1/\rho_z) = -\rho_{zt}/\rho_z^2 = \frac{(\rho_z - \rho_{oz})}{\tau\rho_z^2},$$

while in the isopycnic model, the thickness tendency is

$$z_{\rho t} \approx \frac{1}{\tau}(z_{o\rho} - z_\rho) = \frac{1}{\tau}\left(\frac{1}{\rho_{oz}} - \frac{1}{\rho_z}\right) = \frac{(\rho_z - \rho_{oz})}{\tau\rho_z\rho_{oz}},$$

which leads to the same result as before.

Therefore in general, the two relaxation methods will yield forcings which can differ strongly.

4. Solutions ‘‘close’’ to observations

However, if the assumption that the solutions are not ‘‘far’’ from observations is made, then the two results become closer. Specifically, assume

$$T \approx T_o + T'$$

with similar assumptions for other variables. We can then write, for example,

$$\rho' = -\rho_{oz}z', \qquad z' = -z_{0\rho}\rho', \qquad z_{o\rho} = \frac{1}{\rho_{oz}},$$

connecting the two coordinate systems, where the assumption that the vertical dependence dominates has been made for convenience. (The value of z' will depend on the variable being discussed; we here limit attention to density.) Then, to leading order,

$$Q = \frac{1}{\tau}(\rho_o - \rho) = -\frac{\rho'}{\tau} = \frac{\rho_{oz}}{\tau}z' = \frac{z'}{\tau z_{o\rho}}, \quad \text{(C8)}$$

working in density coordinates. Substitution into (C2) then gives the thickness equation, to leading order, as

$$\begin{aligned} h_t + \nabla\cdot(\mathbf{u}h) + (Qh)_\rho &= h_t + \nabla\cdot(\mathbf{u}h) + \left(\frac{z'}{\tau z_{o\rho}}z_\rho\right)_\rho \\ &\approx h_t + \nabla\cdot(\mathbf{u}h) + \left(\frac{z'}{\tau z_{o\rho}}z_{o\rho}\right)_\rho \\ &= 0, \qquad \text{(C9)} \end{aligned}$$

where we assume that the diabatic term is already small through Q, so that the background $z_{o\rho}$ can be substituted for h (this would not be appropriate in the more general case of the previous section). Using (C8), this becomes

$$\begin{aligned} h_t + \nabla\cdot(\mathbf{u}h) + \frac{1}{\tau}z'_\rho &= h_t + \nabla\cdot(\mathbf{u}h) + \frac{1}{\tau}(z_\rho - z_{o\rho}) \\ &= h_t + \nabla\cdot(\mathbf{u}h) - \frac{1}{\tau}(h_o - h) \\ &= 0, \qquad \text{(C10)} \end{aligned}$$

which is the same as (I2) to this order. Hence relaxation of density in level coordinates becomes relaxation of layer thickness in isopycnic coordinates provided that the model remains ‘‘sufficiently close’’ to observations.

How close is sufficiently close? We have already seen that the vertical density gradients of the model and of the observations must be similar, both in order for the models to appear close as well as for the above analysis to work. In addition, the single-term Taylor series is what gives the approximate equality between (C10) and (I2), so that nonnegligible higher-order terms would mean that the two relaxations diverge. This neglect means that in the expansion

$$\rho' = \rho_{oz}z' + \rho_{ozz}\frac{z'^2}{2} + \cdots,$$

we must have a negligible second term, or, converting back to density

$$\rho' \ll \frac{\rho_{oz}^2}{\rho_{ozz}}.$$

If density is dominated by temperature, we could replace density by temperature in the above. Taking a crude fit to the Levitus and Boyer (1994) data, globally averaged, we have $T \approx 18\exp(z/500\ \text{m})$ in the upper waters. Thus equality of the two relaxations would require

$$T' \ll 18\exp(z/500\ \text{m}).$$

Near surface, this is easily achieved, but at middepths (say, 1500 m), equivalence of the two relaxation methods would need $T' \ll 0.9°\text{C}$ in addition to similarity of gradients. Deviations from observations can be several degrees, so that in general the relaxation schemes would indeed have different effects.

5. Tracers

The situation for tracers needs a little care. Converting the T equation (L1) to isopycnic coordinates gives eventually

$$T_t + \mathbf{u}\cdot\nabla T + QT_\rho = \frac{1}{\tau}[T_o(x, y, z) - T], \quad \text{(C11)}$$

where again Q takes the role of a diapycnic velocity. It is clear that in general (C11) and (I1) are different. Linear gradient approximations, as used in section 3, are not enlightening.

When the model is close to observations, in the sense of the previous section, we can estimate the extra term. We rewrite (C6) as

$$T_t + \mathbf{u}\cdot\nabla T = \frac{1}{\tau}[T_o(x, y, z) - T] - QT_\rho \quad \text{(C12)}$$

and then estimate the rhs. We have

$$QT_\rho \approx \frac{z'}{\tau z_{o\rho}}T_{oz}z_{o\rho} = \frac{z'T_{oz}}{\tau}, \quad \text{(C13)}$$

so that (C12) becomes

$$T_t + \mathbf{u} \cdot \boldsymbol{\nabla} T = \frac{1}{\tau}[T_o(z) - T - z' T_{oz}]. \quad \text{(C14)}$$

In the isopycnic version, we must evaluate $T_o(\rho)$. This is

$$T_o(\rho) = T_o(z) + \rho' T_{o\rho}(\rho) = T_o(z) + \rho' T_{oz} z_{o\rho},$$
$$= T_o(z) - z' T_{oz}, \quad \text{(C15)}$$

so that the isopycnic tracer equation simplifies to

$$T_t + \mathbf{u} \cdot \boldsymbol{\nabla} T = \frac{1}{\tau}[T_o(z) - T - z' T_{oz}], \quad \text{(C16)}$$

which is identical to (C9). So the induced diapycnal velocity Q advects tracer pseudovertically just far enough to position it at the ‘‘correct’’ density level for isopycnic relaxation.

6. Discussion

It is not clear what should be the ‘‘best’’ way to relax toward observations. The point of this article is to show that current practices differ intrinsically between level and isopycnic models, not that one is more or less ‘‘correct’’ than the other. Neither of the methods above, for example, attempt to conserve water mass structure; there are locations where this would not be wise, for example, the Mediterranean outflow. However, what is clear is that current practice in models, which (formally, at least) should tend toward each other as grid spacing becomes ever finer, will give results which differ. At finite resolution, of course, other model aspects are likely to dominate any differences discussed here.

It might be argued that differences in the tracer equation above—ignoring differences in the thickness equation—will have little effect on the dynamics of an isopycnic model, since these feel predominantly density rather than temperature or salinity directly. On some timescale this cannot be the case, since an isopycnic model feels both tracers directly through its surface mixed layer. On the advective–diffusive scale for the model under consideration, the surface dynamics will act to cause level and isopycnic models to have a different response.

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