

Approximation for the Inverse Speed of Sound in Seawater, Suitable for Assimilating Acoustic Tomography Data into Numerical Models*

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ABSTRACT

An approximate formula for the reciprocal speed of sound in seawater is obtained in the form of a polynomial that is cubic in potential temperature, and quadratic in pressure and salinity. The expression provides a reformulation of Del Grosso's empirical formula for the speed of sound in seawater in terms of salinity, pressure, and potential temperature, which are the basic thermodynamic parameters describing oceanic state in numerical models. This makes the proposed approximation convenient for constraining the acoustic tomography data by dynamics.

1. Introduction

Recent progress in computer technologies and numerical modeling of the ocean has put forward the necessity for computationally efficient algorithms relating seawater properties with thermodynamic fields describing oceanic state in numerical models. In a number of recent studies (Levitus and Isayev 1992; Jackett and McDougall 1995; Brydon et al. 1999) efficient approximations have been found to the United Nations Educational, Scientific and Cultural Organization (UNESCO) equation of state for seawater. These approximations simplified the original formula of Millero et al. (1980) and were accurate enough to be consistent with the error level imposed by limitations of high-end GCMs.

Speed of sound is another important characteristic of seawater that is used to monitor oceanic state by means of acoustic tomography (AT). Until now, most of the AT data inversions were done by purely statistical meth-

ods, which are not so computationally intensive as data assimilation into GCMs. Recently, however, a few attempts were made to synthesize AT data with dynamical models (Cornuelle and Worcester 1996; ATOC Group 1998; Menemenlis and Chechelnitsky 2000). Processing the output of long-term AT arrays by means of high-resolution GCMs may put forward the same question of computational efficiency for algorithms relating the travel times of acoustic signals with the basic seawater parameters. The issue becomes even more important in view of high temporal resolution of the acoustic tomography (5–20 min) and the nonlocal structure of the operators that map model fields onto travel times.

In the AT data inversions, the empirical formula of Del Grosso (1974, hereafter DG74) for the speed of sound in seawater c is the most widely used. This relationship expresses c in terms of the third-order polynomials of in situ temperature T , salinity S , and pressure p . In most of the models, however, potential temperature θ is used instead of T because of its entropy-conserving properties. Therefore, direct application of the DG74 formula is complicated by the necessity to invert an empirical nonlinear relationship between T and θ . Besides, acoustic tomography measures integrals of the inverse sound speed $\sigma = 1/c$ in seawater along the ray paths, so that AT data are related to the sound speed distribution in a nonlinear manner. The reciprocal speed of sound, on the other hand, is linearly related to to-

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TABLE 1. Polynomial coefficients for the “wide” range of variables.

n	c_{1n}	c_{2n}	c_{3n}
1	$7.124717 \times 10^{+2}$	-8.115607×10^{-1}	-2.396404×10^{-4}
2	$-2.488983 \times 10^{+0}$	-2.479911×10^{-2}	1.028096×10^{-4}
3	-6.382621×10^{-1}	1.048549×10^{-3}	6.878128×10^{-6}
4	3.400505×10^{-2}	8.422523×10^{-6}	-1.918726×10^{-6}
5	8.281208×10^{-3}	1.518041×10^{-3}	-2.010478×10^{-6}
6	-1.952940×10^{-4}	1.553815×10^{-6}	9.077217×10^{-9}
7	-1.047685×10^{-4}	-4.336037×10^{-7}	5.782815×10^{-9}
8	2.199894×10^{-5}	-2.153050×10^{-5}	2.453841×10^{-8}

mographic data in the traditional approximation. In this respect, it turns out to be the key acoustic parameter of the seawater.

In this note we obtain an expression for σ in terms of potential temperature, salinity, and pressure. We utilize the approach of Brydon et al. (1999) to generate a polynomial in θ , S , and p that provides a best fit to the DG74 formula. The polynomial gives a relationship (cubic in θ and quadratic in S) between the travel time and the model fields. It is more suitable for direct differentiation of the misfit between the model and AT data with respect to the model variables. The rms error of the fit lies well below the error bars of the DG74 formula itself, so that the fit can be treated as a reformulation of the DG74 formula in terms of potential temperature.

2. A polynomial fit to the DG74 formula

Speed of sound in seawater was determined with a precision of several centimeters per second ($\delta c/c \sim 10^{-5}$). This accuracy is sufficient for the purposes of AT, due to the presence of much larger uncertainties typical for real travel time measurements in the ocean. The DG74 formula, the estimated accuracy of which is $5\text{--}10 \text{ cm s}^{-1}$, has the form of a 19-term polynomial $P(S, T, p)$ that is cubic in pressure and in situ temperature and quadratic in salinity.

We approximate the reciprocal of P by the polynomial function of the form

$$\begin{aligned} \sigma(\theta, S, p) = & C_1(p) + C_2(p)\theta + C_3(p)S + C_4(p)\theta^2 \\ & + C_5(p)S\theta + C_6(p)\theta^3 + C_7(p)S\theta^2 \\ & + C_8(p)S^2\theta. \end{aligned} \quad (1)$$

$$C_n(p) = c_{1n} + c_{2n}p + c_{3n}p^2; \quad n = 1, \dots, 8.$$

The fit was obtained in a manner closely following Brydon et al. (1999). In the first stage, a least squares fit of $\sigma(\theta, S, p)$ to the reciprocal of the DG74 formula was carried out over the prescribed range of θ and S for discrete values of p ranging from 0 to 50 MPa in increments of 1 MPa. The fit was performed by minimizing the integrals of the squared difference between the exact and approximate functions. The integrals were taken at fixed pressure values p_k , $k = 1, \dots, 51$:

TABLE 2. Polynomial coefficients for the “narrow” range of variables.

n	c_{1n}	c_{2n}	c_{3n}
1	$7.124305 \times 10^{+2}$	-8.117177×10^{-1}	-2.290479×10^{-4}
2	$-2.482139 \times 10^{+0}$	-2.481029×10^{-2}	9.934918×10^{-5}
3	-6.371984×10^{-1}	1.052861×10^{-3}	6.595177×10^{-6}
4	3.431766×10^{-2}	1.921658×10^{-6}	-1.924687×10^{-6}
5	7.885921×10^{-3}	1.522837×10^{-3}	-1.838321×10^{-6}
6	-2.028645×10^{-4}	1.559289×10^{-6}	9.796547×10^{-9}
7	-1.086373×10^{-4}	-2.617763×10^{-7}	5.502352×10^{-9}
8	2.734109×10^{-5}	-2.165234×10^{-5}	2.244996×10^{-8}

$$\int_{S_0}^{S_1} \int_{\theta_0}^{\theta_1} w(\theta) \left\{ \sigma(\theta, S, p_k) - \frac{10^6}{c[S, T(\theta, S, p_k), p_k]} \right\}^2 d\theta dS. \quad (2)$$

After the integrals are minimized with respect to C_n at all 51 values of p , quadratic parabolas are fitted to each set of 51 coefficients C_n . As a result, the values of c_{kn} are obtained. The weighting function $w(\theta)$ is introduced to increase accuracy of the fit in the region of the θ – S space, where the AT rays are usually located. In most oceanographic experiments, transceivers are positioned at the depths of the sound channel (500–1500 m). Correspondingly, most of the acoustic rays sample the ocean within the depth range of 100–4000 m (approximate temperature range $-1^\circ < \theta < 25^\circ\text{C}$).

The integration in (2) requires an algorithm to compute in situ temperature T for given values of θ , S , and p . For that purpose we utilized the formula based upon Bryden’s (1973) polynomial for adiabatic lapse rate and the fourth-order Runge–Kutta integration scheme (Fofonoff 1977).

3. Results

We performed two major fits of the DG74 formula. They were made for two ranges of parameters in (θ, S, p) space. The “narrow” range is defined by $\theta_0 = -1^\circ\text{C}$, $\theta_1 = 25^\circ\text{C}$, $S_0 = 33 \text{ psu}$, $S_1 = 37 \text{ psu}$, and the pressure range of 0–50 MPa. The larger domain is located within the limits of $-2^\circ\text{--}30^\circ\text{C}$ and 30–38 psu. Both fits were performed with the weighting function $w(\theta) = 1 + 5 \exp[-(\theta - 4^\circ)^2/100]$. The corresponding polynomial coefficients for the fits are shown in Tables 1–2.

To give an idea of the accuracy of these approximations, we show in Figs. 1 and 2 the difference $\delta\sigma$ between σ and $10^6/c$. The square of $\delta\sigma$ has been minimized in (2) and it has units of milliseconds (ms) per 1000 km (Mm). Thus, the patterns give errors in travel time between two transceivers separated by 1000 km in a homogeneous ocean. In “sound speed units,” a $\delta\sigma$ value of 5 ms Mm^{-1} roughly corresponds to a sound speed approximation error $\delta c \sim c^2\delta\sigma \sim 1.1 \text{ cm s}^{-1}$. The standard deviation of the DG74 polynomial fit to experimental data is 5 cm s^{-1} , so that our approximation

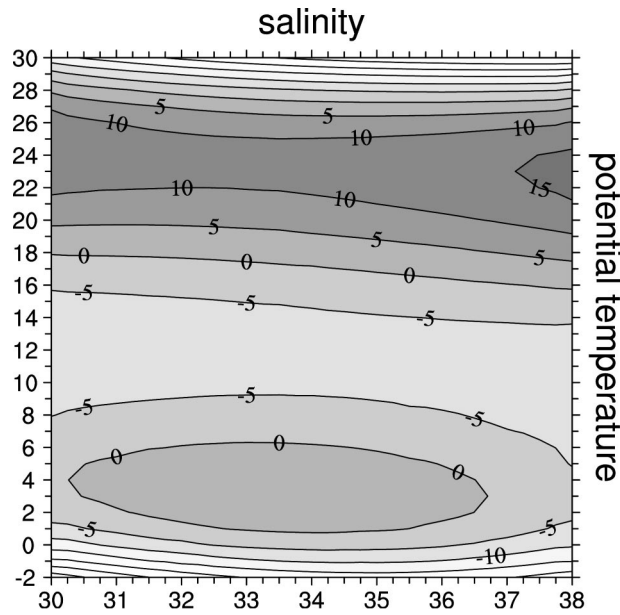


FIG. 1. Error distribution at 7 MPa for the fit with a “wide” range of parameters. Contour interval is 5 ms Mm⁻¹.

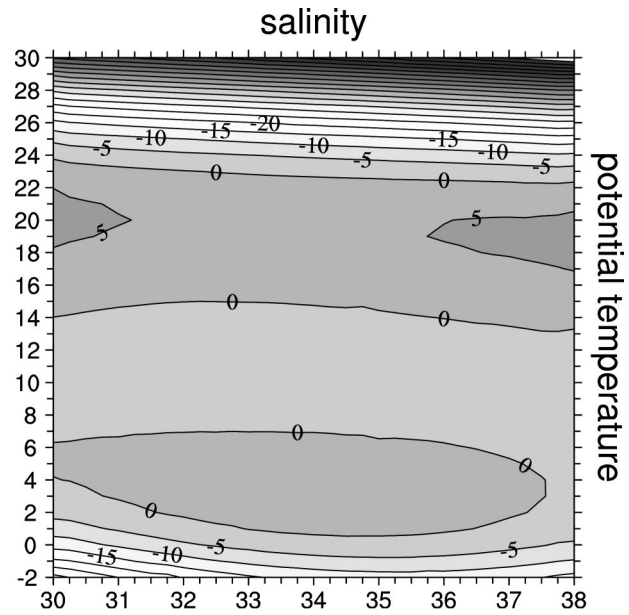


FIG. 2. Error distribution at 25 MPa for the fit with a “narrow” range of parameters. Contour interval is 5 ms Mm⁻¹.

can be considered as a reformulation of the DG74 formula in terms of potential temperature.

Figure 3 demonstrates the θ - S -averaged deviations from the DG74 formula as a function of pressure. For the “narrow” range of parameters the pressure-averaged standard deviation is 7 ms Mm⁻¹, or 1.5 cm s⁻¹ in sound speed units. Errors of the fit defined by the “wide” range of parameters are approximately 1.5 times larger than those obtained for the narrow range. Nevertheless, their typical values are still consistent with the accuracy of the DG74 formula.

We checked the obtained approximations against simulated AT data, which have been generated by the θ - S profiles taken from the World Ocean Circulation Experiment (WOCE) climatology. Twenty profiles were extracted from the various parts of the World Ocean and the eigenray systems were computed for 20 pairs of hypothetical transceivers. The transceivers were located at sound channel axes, separated by 1 Mm, and horizontal homogeneity of the potential temperature and salinity fields was assumed. The mean difference in travel times computed by the DG74 formula and our approximation was 0.12 ms with the standard deviation of 2.7 ms ($\delta c \sim 0.6$ cm s⁻¹). These numbers are well below the travel time errors usually adopted for tomographic measurements (sound speed equation errors, linearization errors, travel time biases due to internal waves, etc.). Computation of travel time using our approximation was nearly 3 times faster than with the DG74 formula. This gain in CPU time arises due to the necessity to convert θ into in situ temperature before applying the DG74 formula.

Check values for inverse speed of sound computed using (1) and Tables 1, 2 are $\sigma(S = 35$ psu, $\theta = 1^\circ\text{C}$,

$p = 30$ MPa) = 664.8589 s Mm⁻¹ and $\sigma(S = 34$ psu, $\theta = 7^\circ\text{C}$, $p = 5$ MPa) = 673.1178 s Mm⁻¹, respectively.

The obtained polynomial approximations of the reciprocal speed of sound may be useful in global and medium-range inversions of AT data that employ potential temperature as one of the seawater parameters.

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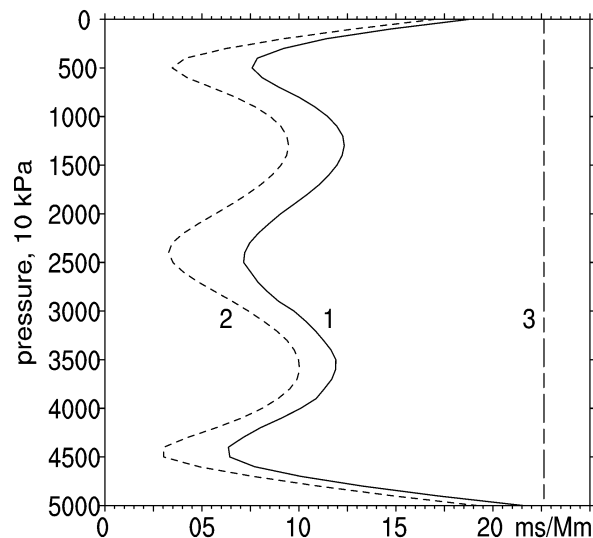


FIG. 3. The θ - S -averaged errors plotted against pressure. Solid curve 1 corresponds to the wide fit, averaged over the entire wide θ - S range. Dashed curve 2 is obtained through averaging of the narrow domain fit. Dashed line 3 gives the error level for the approximation of experimental data by the formula of Del Grosso (1974).

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