Sampling Strategies for Tornado and Mesocyclone Detection Using Dynamically Adaptive Doppler Radars: A Simulation Study

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ABSTRACT

Increasing tornado and severe storm warning lead time (lead time is defined here as the elapsed time between the issuance of a watch or warning and the time at which the anticipated weather event first impacts the specified region) through the use of radar observations has long been a challenge for researchers and operational forecasters. To improve lead time and the probability of detecting tornadoes while decreasing the false alarm ratio, a greater understanding, obtained in part by more complete observations, is needed about the region of storms within which tornadoes form and persist. Driven in large part by this need, but also by the goal of using numerical models to explicitly predict intense local weather such as thunderstorms, the National Science Foundation established, in fall 2003, the Engineering Research Center for Collaborative Adaptive Sensing of the Atmosphere (CASA). CASA is developing a revolutionary new paradigm of using a network of small, closely spaced, inexpensive, low-power dual-polarization Doppler weather radars to overcome the inability of widely spaced, high-power radars to sample large regions of the lower atmosphere owing to the curvature of earth given that zero or negative beam elevation angles are not allowed. Also, current radar technology operates mostly independently of the weather and end-user needs, thus producing valuable information on storms as a whole but not focused on any specific phenomenon or need. Conversely, CASA utilizes a dynamically adaptive sensing paradigm to identify, and optimally sample, multiple targets based upon their observed characteristics in order to meet a variety of often competing end-user needs.

The goal of this study is to evaluate a variety of adaptive sampling strategies for CASA radars to assess their effectiveness in identifying intense low-altitude vortices. Such identification, for the purposes of this study, is defined as achieving a best fit of simulated observations to an analytic model of a tornado or mesocyclone. Several parameters are varied in this study including the size of the vortex, azimuthal sampling interval, distance of the vortex from the radar, and radar beamwidth.

Results show that, in the case of small vortices, adaptively decreasing the azimuthal sampling interval (i.e., overlapping beams) is beneficial in comparison to conventional azimuthal sampling that is approximately equal to the beamwidth. However, the benefit is limited to factors of 2 in overlapping. When simulating the performance of a CASA radar in comparison to that of a Weather Surveillance Radar-1988 Doppler (WSR-88D) at close range, with both operating in the conventional nonoverlapping mode, the WSR-88D (with a beamwidth about half that of a CASA radar) performs better. However, when overlapping is applied to the CASA radar, for which little additional processing time is required, the results are comparable. In effect, the sampling resolution of a radar can be increased simply by decreasing the azimuthal sampling interval as opposed to installing a larger antenna.

1. Introduction

The current National Weather Service (NWS) Weather Surveillance Radar-1988 Doppler (WSR-88D; Crum and Alberty 1993) radar network is the principal tool used for detecting severe storms and tornadoes and for issuing warnings. This network has been critical to the forecasting of severe weather and has saved...
countless lives since its formal commissioning in 1994 (Simmons and Sutter 2005). To improve lead time and the probability of detecting tornadoes while decreasing false alarms (Fig. 1), a greater understanding, obtained in part by more complete observations, is needed about the region of storms within which tornadoes form and persist; that is, the region within a few kilometers of the ground. Driven in large part by this need, but also by the goal of using numerical models to explicitly predict intense local weather such as thunderstorms, the National Science Foundation (NSF) established in fall 2003 the Engineering Research Center for Collaborative Adaptive Sensing of the Atmosphere (CASA).

CASA is led by the University of Massachusetts at Amherst with several academic partners, including the University of Oklahoma, Colorado State University, and the University of Puerto Rico at Mayaguez, along with industrial, educational, and end-user partners. It is developing five radar test beds, the first of which is located in southwestern Oklahoma and consists of four radars and emphasizes the detection of storms and tornadoes. This test bed collected numerous datasets during the spring 2007 and 2008 severe weather seasons, and the data are now being analyzed (e.g., Kong et al. 2007; Weiss et al. 2007; Kain et al. 2009). The second test bed, which is being developed in downtown Houston, emphasizes quantitative precipitation estimation for hydrology. Located in Puerto Rico, the third test bed is being developed entirely by students and utilizes solar power and wireless communication technologies to study quantitative precipitation estimation and forecasts in mountainous terrain. The fourth and fifth test beds focus, respectively, on clear-air sensing and phased array technology, and are now being planned.

In contrast to most current weather radars, which operate in a “sit and spin” mode independent of evolving weather, CASA radars are designed to operate in a distributed, collaborative, adaptive framework (DCAS; McLaughlin et al. 2005). “Distributed” refers to placing radars in clusters with a spacing much closer (~25 km) than that of the WSR-88D network so as to overcome sampling problems (e.g., azimuthal resolution degradation or beam location above ground) that occur at a long range owing to beam spreading and the earth’s curvature. “Collaborative” connotes the coordinated targeting of multiple radar beams based on atmospheric and hydrologic analysis tools, such as detection, predicting, and tracking algorithms (McLaughlin et al. 2005). By utilizing this collaboration, the system allocates resources such as radiated power, beam position, and polarization diversity for optimally sampling regions of the atmosphere where a particular threat exists. The term “adaptive” refers to the ability of the CASA radars and the associated infrastructure to quickly reconfigure (e.g., begin using a smaller azimuthal sampling interval) in response to changing weather and to meet various end-user needs.

Another unique adaptive characteristic of CASA radars is the ability to change from sampling a large area to much smaller sector scans, say of a tornadic region, independently or collaboratively with other CASA radars. In contrast, WSR-88D radars scan independently of one other and continuously surveil the entire volume largely independent of weather occurring within it. Given these features, a fundamental research challenge for CASA concerns understanding which modes of radar operation will yield the maximum amount of useful information about a particular weather phenomenon while minimizing the use of available resources (so they are available for other phenomena or use by other neighboring radars).

As a first step toward addressing this and related challenges, the goal of the present study is to use pseudo-observations of an idealized vertical vortex to evaluate a variety of sampling strategies for CASA radars in order to determine which might be most effective for real tornadoes and mesocyclones. Here, effectiveness is defined as the best fit of the pseudo-observations to an analytic model of tornadoes and mesocyclones.

1 The Terminal Doppler Weather Radar (TDWR; e.g., Wieler and Shrader 1991), operates in a sector scan mode to detect microbursts and other aviation hazards. The WSR-88D utilizes a variety of volume coverage patterns (e.g., Klazura and Imy 1993; Brown et al. 2005) that are chosen based upon weather characteristics; however, none involve sector scanning or other modes of adaptation to specific weather features.

Several parameters are tested, including the radius of the vortex, azimuthal sampling interval, and the distance of the vortex center from the radar. It may seem logical to sample a vortex or other atmospheric phenomena with as much spatial and temporal resolution as possible, but doing so may actually waste available radar resources\(^2\) while providing no useful added information.

The allocation of resources is especially important in DCAS, where multiple, frequently competing end-user goals must be met in an optimal manner. In the current radar test bed located in Oklahoma, this allocation is performed via a policy mediation framework in which detection algorithms run in real time on radar moment data categorize observed signatures and store their attributes (e.g., feature type, time of development, location) in a so-called feature repository. This information is combined with user-specified priorities (Philips et al. 2007) to produce scanning pattern commands that are communicated to the radars by the Meteorological Command and Control (MC&C) system. The frequency of this communication is determined by the radar “heartbeat,” which presently is 60 s. That is, every 60 s, the radars can be retasked to scan different regions at different rates. For more information on CASA scanning strategies, see Brotzge et al. (2005, 2008), and Gagne et al. (2008). The results of the present study are expected to help optimize this process for identifying potentially hazardous atmospheric vortices early in their lifetime.

Section 2 describes the methodology used, including a detailed explanation of the vortex model, radar emulator, and retrieval technique, along with parameters varied to study various sampling strategies. The results are discussed in section 3, and a summary and suggestions for future work are presented in section 4.

2. Methodology

a. Vortex model

To create a representative tornadic or mesocyclonic flow field appropriate for sampling by a virtual CASA radar, an analytic vortex model is used to generate an idealized one-dimensional (horizontally through the center of the vortex) azimuthal velocity profile. An analytic vortex model, for which an exact solution is available, is appropriate for use in assessing the potential value of various adaptive strategies prior to their application with real data. One familiar model is the Rankine (1901, 574–578) combined vortex (RCV), which is given by

\[
 v_t = V_s f(r, R_x),
\]

where \( v_t \) is the tangential velocity, \( V_s \) is the maximum tangential velocity, and \( f(r, R_x) \) is the dimensionless velocity profile given by

\[
f(r, R_x) = \begin{cases} \frac{r}{R_x}, & 0 \leq r < R_x \\ \frac{R_x}{r}, & R_x \leq r \end{cases}.
\]

Here, \( r \) is the radial distance from the vortex center and \( R_x \) is the core radius at which \( V_s \) occurs. The “core” of the vortex (\( r \leq R_x \)) is characterized by solid body rotation, while for \( r \geq R_x \), potential flow is assumed.

The RCV profile was not used in the present work owing to the discontinuity in velocity at \( r = R_x \) and its effects on the minimization algorithm used here. The observed Doppler core radius is usually larger than \( R_x \) (which needs to be known to make an initial guess) because of smearing effects within the radar beam. When applying the minimization algorithm, an updated \( R_x \) must be known before the velocity profile is scanned again, and it is difficult for the algorithm to determine which part of the velocity profile (core or potential flow) the radar is sampling. This can cause instability in the algorithm and a lack of convergence to a solution. Also, according to high-resolution tornado observations (e.g., Wurman and Gill 2000), this discontinuity does not exist in real tornadoes, and the decay of tangential wind speed with distance outside the core of solid body rotation occurs less rapidly than with the RCV.

For this reason, we use a more flexible analytic model (developed by L. White, University of Oklahoma, personal communication 2005) that does not contain a discontinuity and that decays more slowly than the RCV. Referred to as the three-parameter vortex model (TPVM), it is characterized by a function that has no discontinuities yet retains the principal features of and provides greater flexibility than the RCV:\(^3\)

\[
 V_n(r, R_x) = V_s \Phi_n,
\]

where \( \Phi_n = 2nR_x^{2n-1}/[(2n-1)R_x^{2n} + r^{2n}] \) is the velocity profile of the vortex, \( V_s \) is tangential velocity, \( r \) the radius from the vortex center, and \( n \) a nonzero integer value that controls the velocity profile beyond the core radius. After testing various combinations of (3) with

\(^2\) Note that maximizing resources, in a framework such as CASA in real time, is inherently a property of dynamically adaptive systems.

\(^3\) Unlike the RCV, the TPVM does not conserve angular momentum. However, that property is unimportant for the purposes of this study.
modified the emulator to accommodate CASA radar characteristics using velocity fields based upon the TPVM.

The three-dimensional Doppler velocity ($V_d$) can be expressed in terms of the radial ($v_r$), tangential ($v_t$), and vertical ($w + V_T$) components of hydrometeor motion and is given by

$$V_d = v_r \sin \gamma \cos \theta_d' + v_t \cos \gamma \cos \theta_d' + (w + V_T) \sin \theta_d',$$

(5)

where $\gamma$ is the angle between the radar viewing direction $\phi_d$ at a point $(R_d, \phi_d, \theta_d)$ and the tangential velocity component, $V_T$ is the terminal fall speed of hydrometeors (a negative quantity), and $\theta_d'$ is the beam elevation angle plus the angle between the vertical axis at the radar and the vertical axis at the location of the vortex is given by Doviak and Zrnić (1993, p. 307):

$$\theta_d' = \theta_d + \tan^{-1}\left(\frac{R_d \cos \theta_d}{a_e + R_d \sin \theta_d}\right),$$

(6)

where $a_e$ is the effective earth radius (1.33 times the mean earth radius) that is used to account for beam curvature due to beam refraction.

The azimuthal width of a radar beam increases linearly with increasing distance from the radar; however, in addition to this broadening with range, an effective azimuthal broadening also occurs for a horizontally rotating beam, defining the effective half-power beamwidth. That is, when a radar antenna scans azimuthally through a finite sampling interval while transmitting and receiving pulses, the feature being sampled is smeared in the azimuthal direction, as if the antenna were wider. Three radar parameters determine the degree broadening (Doviak and Zrnić 1993): 1) antenna rotation rate, 2) number of pulses transmitted and received, and 3) time interval between pulses. Effective half-power beamwidth can be reduced by decreasing one or more of these three parameters.

The mean Doppler velocity component $\nabla_d(\phi_o, R_o, \theta_o)$ at the center azimuthal angle $\phi_o$, center range $R_o$, and the center elevation angle $\theta_o$ of the effective resolution volume of the radar beam is given by

$$\nabla_d(\phi_o, R_o, \theta_o) = \sum_i \sum_j \sum_k \frac{V_d G(\phi, \theta_d)|W(R_d)|^2 Z}{\sum_i \sum_j \sum_k G(\phi, \theta_d)|W(R_d)|^2 Z},$$

(7)
where \(i, j, k\) are the azimuth, range, and elevation directions, respectively; \(\phi\) is the azimuthal angle; and \(\theta\) is the elevation angle. Reflectivity \((Z)\) is assumed uniform across the vortex; \(G(\phi, \theta_j)\) is the two-way Gaussian pattern weighting function used to weight Doppler velocity at the \((\phi_n, \theta_k)\) data point and is given by

\[
G(\phi, \theta_j) = \exp \left[ -\frac{(\phi_j - \phi)^2}{2\sigma_{\phi}^2} - \frac{(\theta_j - \theta)^2}{2\sigma_{\theta}^2} \right].
\]

(8)

In (8), \(\sigma_{\phi}^2\) and \(\sigma_{\theta}^2\) are the standard deviations of the Gaussian density in the \(\phi\) and \(\theta\) directions, respectively, and are specified by

\[
\sigma_{\phi}^2 = \frac{\phi_c^2}{16 \ln 2}.
\]

(9)

\[
\sigma_{\theta}^2 = \frac{\theta_1^2}{16 \ln 2}.
\]

(10)

where \(\phi_c\) is the effective half-power beamwidth in the azimuthal direction and \(\theta_1\) is the vertical half-power beamwidth in the elevation direction. Another term in (7) is the Gaussian-shaped range weighting function, \(|W(R_d)|^2\), used to weight Doppler velocity in range and is given by

\[
|W(R_d)|^2 = \exp \left[ -\frac{(R_d - R_o)^2}{2\sigma_R^2} \right],
\]

(11)

where

\[
\sigma_R^2 = \frac{0.35 c \tau}{2}.
\]

(12)

and where \(R_d\) is a given range, \(R_o\) is the center range, \(c\) is the speed of light, and \(\tau\) is the pulse width (Doviak and Zrnić 1993).

The expression in (7) is a general equation for mean Doppler velocity. In this work, a simplified two-dimensional geometry in the \(x-y\) plane is employed using assumptions shown in Table 1. In this case, (7) reduces to a two-dimensional problem with the aid of (1), (3), and (5) and is given by

\[
\dot{V}_d(\phi_o, R_o) = \frac{\sum_{i} \sum_{j} V_n(R_{max}, r) \cos \gamma G(\phi)|W(R_d)|^2}{\sum_{i} \sum_{j} G(\phi)|W(R_d)|^2},
\]

(13)

where \(V_n\) is the tangential velocity profile being used where \(n = 1, 2,\) or 12 and \(\cos \gamma\) is defined as

\[
\cos \gamma = \frac{R_c \sin(\phi - \phi_c)}{r} = \sin \psi,
\]

(14)

and where \(\phi_d\) is the radar viewing direction, \(\phi_c\) is the azimuth of the model vortex center from the radar, and \(\psi\) is the angle between \(\phi_d\) and the radial velocity component (via the laws of sines).

c. Retrieval technique

The retrieval technique used in this study is based on a variational method similar to that employed by Wood (1997) and estimates vortex radius \((R_c)\), and maximum wind speed \((V_c)\), from pseudo-observations of the specified axisymmetric vortex. Other retrieval methods, such as the principal component analysis (PCA) method (Harasti and List 2005), could be employed as well. The technique involves first developing initial guesses for \(V_c\) and \(R_c\) that are used to solve a set of nonlinear equations. Their solution yields a set of linear equations from which the retrieved values of \(V_c\) and \(R_c\) are obtained. Details about the retrieval technique are discussed in the appendix.

d. Input parameters

Several input parameters are held constant in our experiments, including antenna elevation angle and the angle between the vortex center and the closest data point to the vortex center (both were 0°). Holding these two parameters constant allows us to sample the vortex close to the ground and at its center. The parameters varied (Table 2) include the radius of maximum azimuthal wind \((R_c)\) to encompass an array of vortices ranging from a small tornado to an average-sized mesocyclone.

The analytic profile used to generate the pseudo-observations is different from that used for the control profile. Analytic azimuthal velocity profile \(V_1\) typically

<table>
<thead>
<tr>
<th>Assumption No.</th>
<th>Assumption Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tangential velocity field is uniform with height</td>
</tr>
<tr>
<td>2</td>
<td>Vortex is steady state and does not translate</td>
</tr>
<tr>
<td>3</td>
<td>Radial and vertical velocity components of the vortex in (5) are zero</td>
</tr>
<tr>
<td>4</td>
<td>Radar beam pattern is Gaussian shaped and there is no attenuation</td>
</tr>
<tr>
<td>5</td>
<td>Effective half-power beamwidth varies for each azimuthal sampling interval</td>
</tr>
<tr>
<td>6</td>
<td>Uniform reflectivity across the vortex</td>
</tr>
<tr>
<td>7</td>
<td>Beam axis is horizontal so (\theta_j) is approximately zero</td>
</tr>
<tr>
<td>8</td>
<td>CASA radars have a constant rotation rate</td>
</tr>
<tr>
<td>9</td>
<td>No Nyquist velocity limit</td>
</tr>
</tbody>
</table>
is used to generate pseudo-observations (Fig. 2), and is stronger everywhere than the control profile $V_{12}$ beyond the core radius, whereas the opposite is true for $V_2$. Profile $V_2$ has such a steep gradient beyond $R_x$ that convergence is rarely achievable in the retrieval or a very poor estimate is attained. The true tangential velocity profile $V_{12}$ is used in only a few experiments to test the algorithm because it yields overly optimistic results for obvious reasons.

The azimuthal sampling interval is defined as the angular distance from the center of one beam in the scan pattern to the center of the next adjacent beam. Because the half-power beamwidth of the CASA radar antenna is $2^\circ$, creating a scan with no missing or overlapping radials requires an azimuthal sampling interval of $2^\circ$ (Fig. 3, top); in reality, the effective beamwidth of the scanning antenna is greater than $2^\circ$ (owing to the factors discussed above), but to simplify the current discussion we assume that it is $2^\circ$. However, because the CASA radars can scan adaptively, for example, by changing the azimuthal sampling interval based upon the phenomena being sampled, examining impacts associated with variations in this interval is an important part of the present study. For this reason, smaller azimuthal sampling intervals of $1^\circ$ (Fig. 3, middle) and $0.5^\circ$ (Fig. 3, bottom) were tested; that is, for an azimuthal sampling interval of $1^\circ$ ($0.5^\circ$), $1^\circ$ ($0.5^\circ$) separates the centers of two adjacent $2^\circ$ beamwidths. Using a $1^\circ$ or $0.5^\circ$ azimuthal sampling interval is referred to here as overlapping because the beam moves less than one beamwidth from one azimuthal sampling interval to the next. A limitation of overlapping is that data fields may appear to be noisier than the more heavily smoothed fields, as the overlapped beams provide less smoothing and better resolution of smaller-scale features. If overlapping were achieved by using fewer samples (pulses) instead of slowing down the antenna, the data fields would appear slightly noisier.

### Table 2. Parameters used throughout study to test various sampling strategies for the CASA radars. The bottom row gives values for a typical WSR-88D.

<table>
<thead>
<tr>
<th>Parameter name</th>
<th>Values tested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum tangential velocity ($V_x$)</td>
<td>40 m s(^{-1}) for majority of tests. Also tested 60 and 80 m s(^{-1}).</td>
</tr>
<tr>
<td>Radius of maximum winds ($R_x$)</td>
<td>0.1, 0.5, 1, 1.5, 2, 2.5 km. A few tests used 0.05 km.</td>
</tr>
<tr>
<td>Pseudo-observations generated by: $V_1$, $V_2$, or $V_{12}$</td>
<td>$2.5$–$30$ km (increments of $2.5$ km) for majority of tests. $40$–$100$ km was tested also.</td>
</tr>
<tr>
<td>Range</td>
<td>$2^\circ$ for majority of tests; $1^\circ$ was tested also.</td>
</tr>
<tr>
<td>Beamwidth (bw)</td>
<td>$0.5^\circ$, $1^\circ$, and $2^\circ$ (based on beamwidth used)</td>
</tr>
<tr>
<td>Effective half-power beamwidth (ebw)</td>
<td>For $bw = 1^\circ$, $daz = 0.5^\circ$, $ebw = 1.12^\circ$. $bw = 2^\circ$, $daz = 0.5^\circ$, $ebw = 2.07^\circ$. $bw = 2^\circ$, $daz = 1.0^\circ$, $ebw = 2.23^\circ$. $bw = 2^\circ$, $daz = 2.0^\circ$, $ebw = 2.90^\circ$.</td>
</tr>
<tr>
<td>Range sampling interval (drg)</td>
<td>$0.01$ km for majority of tests; $0.25$ km was tested also.</td>
</tr>
<tr>
<td>Noise</td>
<td>Specified standard deviation of white noise for most tests as $1$ m s(^{-1}); also tried $0$, $2$, $4$, $6$, $8$, and $10$ m s(^{-1}).</td>
</tr>
<tr>
<td>WSR-88D</td>
<td>$bw = 0.89^\circ$, $daz = 1.0^\circ$, $ebw = 1.39^\circ$, elevation angle $= 0.5^\circ$. $drg = 0.25$ km.</td>
</tr>
</tbody>
</table>

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4 Some studies use the term “oversampling,” but this also relates to the number of pulses transmitted within a given timeframe, thus confusing the issue.
3. Results

In this section we present results of various simulated sampling strategies for CASA radars, including azimuthal overlapping with varying beamwidths and vortex sizes. Only results for $V_x$ are shown because the results for $V_y$ are qualitatively similar. We also compare the performance of a single CASA radar with a WSR-88D radar.

a. Azimuthal overlapping with varying beamwidths

The results presented in this subsection utilize four different sampling strategies applied to three differently sized vortices. Two sampling strategies use the CASA conventional beamwidth of $2^\circ$ while the other two use a $1^\circ$ beamwidth (note that the beamwidth of CASA radars currently is $2^\circ$). For each pair of sampling strategies, one uses an azimuthal sampling interval equal to the half-power beamwidth (referred to as conventional sampling), while the other uses an interval half as large (referred to azimuthal overlapping). The goal is to evaluate any improvement achieved by azimuthal overlapping.

Figure 4 shows percent error in $V_x$ for a moderate-sized tornado-scale vortex having a core radius ($R_v$) of 0.1 km. Percent error is defined as the difference between the retrieved tangential wind $V_x$ and the true value (obtained from pseudo-observations based upon the $V_{12}$ model profile) divided by the true value. Note that the error is positive near the radar, indicating that the retrieved profile has slightly stronger velocities than the true profile. Conversely, because the vortex is sampled poorly with increasing range, the associated error becomes increasingly negative.

The conventional sampling strategy, in which both the beamwidth and azimuthal sampling interval are $2^\circ$, has the largest overall error while a $1^\circ$ beamwidth and overlapped sampling interval of $0.5^\circ$ has the smallest (Fig. 4). One might notice that the two middle curves are essentially identical. The azimuthal sampling interval is the same ($1^\circ$), but the beamwidths differ by a factor of 2. It is curious that two different beamwidths would produce nearly the same results. So, to determine whether or not this situation is typical, we reran the computations for the same vortex without noise and with various combinations of beamwidth ($0.5^\circ$, $1^\circ$, $2^\circ$) and azimuthal sampling interval ($0.5^\circ$, $1^\circ$, $2^\circ$). We found that the tendency was for curves with the same azimuthal sampling interval to cluster together, regardless of beamwidth (not shown). This finding indicates that, for a given-sized vortex, the azimuthal spacing of data points is more important than the beamwidth in resolving the signature of a vortex whose core diameter is less than the beamwidth.

An adaptive strategy of overlapping provides notable improvement, thereby illustrating that adaptive sampling would be useful for the CASA radars when sampling tornado-scale vortices. However, to maintain the same number of samples for computing Doppler moments and thus maintain data quality, the antenna rotation rate must be reduced with overlapping, thus increasing data collection time. Shifting a CASA radar into an overlapping mode is an important capability of the radar, particularly when a conventional surveillance mode is only marginally able to detect the signature of a small vortex.

The retrieval error is smaller when using overlapping because the density of azimuthal data points is greater, as illustrated in Fig. 5. Also, the effective half-power beamwidth is smaller with decreased azimuthal sampling and, thus, less smoothing/smearing of the true velocity profile occurs. In both the top and bottom images, none of the data points used to determine the Doppler velocity peak occurred within the vortex core (shaded band). This in part explains why the overall retrieval for this small vortex exhibits significant error regardless of the azimuthal sampling interval. However, for the smaller azimuthal (overlapped) interval, the data density is greater and thus the profile is closer to the model Doppler velocity peaks (dashed black line), thereby allowing for a better retrieval of $V_x$ and $R_v$.

Figure 6 shows percent error in $V_x$ for a large tornado-scale vortex having a core radius of 0.25 km. As in Fig. 4, conventional sampling produces the greatest error of the four sampling strategies tested with overlapping.
showing a great deal of improvement. As expected, the 1° beamwidth, coupled with a 0.5° azimuthal sampling interval, produces the smallest error; however, all strategies except for conventional 2° sampling begin to exhibit similar errors with this larger vortex. Compared to the vortex shown in Fig. 5, Fig. 7 illustrates that a larger vortex results in an improved retrieval.

As $R_v$ increases, the error approaches a minimum value of +3% to +5% for velocity profile $V_1$ (Fig. 8) for a mesocyclone-sized vortex having a core radius of 2 km. The small and progressively smaller errors with increasing vortex radius result from an increased number of data points within and beyond the core of the vortex (Figs. 5, 7, and 9). The data points in Fig. 9 have the same azimuthal spacing ($\Delta AZ$) as in Fig. 5, but the greater number of points across the larger vortex results in a better overall retrieval. Not only do the errors become progressively smaller with increasing vortex radius, they also become increasingly positive. Had the pseudo-observations been generated using control profile $V_{12}$ instead of profile $V_1$, the errors would have approached zero. However, with the $V_1$ profile having stronger velocities than the $V_{12}$ profile beyond $R_v$, the retrieved profile $V_{RET}$ has stronger velocities than the control profile and therefore the errors are positive.

b. Azimuthal overlapping with a constant beamwidth

The results described previously show that overlapping is indeed a beneficial adaptive sampling strategy, especially for small vortices that might be only marginally detected using conventional scanning. Therefore, the goal of experiments in this subsection is to determine the degree of overlapping needed to show significant improvement in the retrieved results. A beamwidth of 2° is used for all cases and overlapping factors of 2 and 4 are tested. Only one plot, for $R_v = 0.1$ km, is shown because overlapping has the biggest impact on this size vortex. For larger vortices, all of the sampling strategies produce nearly equal results. Figure 10 shows the percent error in $V_x$ for $R_v = 0.1$ km using three sampling strategies. Any factor of overlapping yields an improvement over conventional sampling. A much larger improvement in error is evident when the azimuthal sampling interval is reduced from 2° to 1° than from 1° to 0.5°. Resolution improvements owing to overlapping can be calculated by taking the ratio of the effective half-power beamwidths (e.g., Brown et al. 2002). Therefore, when reducing the azimuthal sampling interval from 2° to 1°, the relative increase in azimuthal resolution is given by

$$\text{resolution improvement} = \frac{\text{ebw for } \text{daz = } 2°}{\text{ebw for } \text{daz = } 1°} = \frac{2.90°}{2.23°} = 1.30 = 30\%.$$

Although we focus here on vortices, CASA radars can surveil other features within a storm (e.g., gust fronts, heavy precipitation cores) in a temporally interleaved manner.
Similarly, when reducing from $1^\circ$ to $0.5^\circ$, the increase is

\[\text{resolution improvement} = \frac{\text{ebw for daz} = 1^\circ}{\text{ebw for daz} = 0.5^\circ} = \frac{2.23^\circ}{2.07^\circ} = 1.08 = 8\% . \quad (16)\]

Consistent with our results, this suggests that a factor of more than 2 in overlapping would most likely be a waste of resources.

c. Azimuthal overlapping for a small vortex

In addition to the vortex sizes evaluated previously, a small vortex of radius $R_x = 0.05$ km, which is closer to the average size of a tornado, was tested. Table 3 (Table 4) shows the results for a beamwidth of $2^\circ$ ($1^\circ$). As expected, a $2^\circ$ beamwidth, even with overlapping, does not provide sufficient resolution to retrieve this vortex at most ranges. That is, the tolerance values in the minimization algorithm were not met and thus convergence is not achieved, most likely caused by a lack information caused by the relatively small number of data points available across the vortex. For a beamwidth of $1^\circ$ with no overlapping, values are retrieved but exhibit large errors. When factors of 2 and 4 overlapping are applied, the retrieval fails in most cases, likely because of the small number of samples given the azimuthal sampling intervals. From this we conclude that in the context of our experiment design, the retrieval technique works best for vortices having core radii of at least 0.1 km (i.e., equivalent to large tornadoes). Another minimization algorithm, such as one that uses the conjugate gradient method, may allow for convergence in more cases.

d. WSR-88D and CASA radar comparisons

A secondary goal of these experiments is to determine how the WSR-88D, with a half-power beamwidth of 0.89$^\circ$ and conventional $1^\circ$ azimuthal sampling, compares to the CASA radar with a $2^\circ$ beamwidth and both conventional $2^\circ$ azimuthal sampling as well as overlapping. Figure 11 shows the percent error for a vortex of radius $R_x = 0.1$ km. Conventional sampling by a CASA radar, as expected, exhibits the highest percent error. Both the CASA radar in overlapping mode and the WSR-88D have a lower error. For such a small vortex, the WSR-88D curve and CASA curve with overlapping are nearly equal at most ranges because the overlapped CASA azimuthal sampling interval is the same as the WSR-88D sampling interval (see discussion in section 3a). Figure 12 shows results for a WSR-88D
radar and a CASA radar, both sampling a vortex where $R_x = 2.5$ km. For such large vortices, both radars exhibit similar error at all ranges within the 30-km limit of the CASA system.

e. Comparison of initial guess to retrieved value

Given the possible sensitivity of the retrieval to the initial guess documented in the previous section, we recomputed the results with a different initial guess of $V_x$ (Fig. 13) for three different vortex radii. For vortices with $R_x = 0.1$ km and $R_x = 0.5$ km, the retrieval has a lower percent error than the initial guess, thus showing that the retrieval represents an improvement over “raw” radar observations. However, for the largest vortex size of $R_x = 2.5$ km, the retrieved value is nearly equal to the initial guess at all ranges within 30 km. Because this vortex is large compared to the size of the radar beam, the initial guess with the correction factor applied to it [see Eq. (A6)] is very close to the true value.

4. Discussion

We evaluated strategies for retrieving the core radius and maximum tangential velocity of simulated vertical vortices, as proxies for tornadoes and mesocyclones, to determine which strategies might be most effective for a real CASA radar, which is a dynamically adaptive system. The measure of effectiveness was defined as the best fit of pseudo-observations to an analytic vortex model. The model used here to create the true and pseudo-observations, known as the three-parameter vortex model (TPVM), does not contain a singularity at the core radius (as does the Rankine combined vortex). The TPVM was applied to a Doppler radar emulator that sampled analytic tangential velocity profiles using data generated by the TPVM. A variational retrieval model was employed to optimally estimate two control variables of the vortex: the maximum tangential velocity $V_x$ and its radius $R_x$. Only results for $V_x$ were shown because the $R_x$ results show comparable behavior. Several parameters were varied to evaluate the effectiveness of CASA sampling strategies on various parameters including vortex size, range from the radar, azimuthal sampling interval, and radar beamwidth. Comparisons of CASA and WSR-88D radars sampling at close ranges also were shown.

For all ranges tested (2.5–30 km from the radar) for a single CASA radar, vortices of radius 0.1 km and larger are detectable using its conventional $2^\circ$ beamwidth, and a $2^\circ$ azimuthal sampling interval. Overlapping of radar
beams was shown to be an important adaptive sampling strategy for a CASA radar, especially in capturing the behavior of small vortices of radius 0.1–0.5 km (medium to large tornadoes). Finally, overlapping does not yield any noticeable improvement for vortices of radius 1 km and larger at any ranges tested for a CASA radar.

For the vortex sizes tested, there appears to exist a limit beyond which additional overlapping (a factor greater than 2) yields no considerable improvement in the retrieval of small or large vortices because the increase in resolution diminishes quickly with decreasing azimuthal sampling interval. The results using a $\frac{1}{8}$ beamwidth, especially with overlapping, do yield better retrievals for smaller vortex sizes including $R_v = 0.05$ km. However, using a $1^\circ$ beamwidth for the CASA radars would be inconsistent with the goal of developing small, inexpensive radars because a larger antenna would be required.

When comparing results of retrievals using conventional CASA parameters to those of the WSR-88D at close ranges, it is not surprising that the latter, with its narrower beamwidth, produces smaller errors. However, when a CASA radar uses overlapping, the results are nearly equal to those for a WSR-88D. This again confirms the benefit of overlapping to CASA.

### Table 3. Percent error in $V_x$ for a vortex of radius $R_v = 0.05$ km, beamwidth (bw) of $2^\circ$, and an azimuthal sampling interval (daz) of $2^\circ$ (center column) and $1^\circ$ (right column) for various ranges; $-999$ indicates that no value of $V_x$ could be retrieved for that range.

<table>
<thead>
<tr>
<th>Range (km)</th>
<th>PE $V_x$, bw = $2^\circ$, daz = $2^\circ$</th>
<th>PE $V_x$, bw = $2^\circ$, daz = $1^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>$-999$</td>
<td>$-999$</td>
</tr>
<tr>
<td>5</td>
<td>$-999$</td>
<td>$-20.68$</td>
</tr>
<tr>
<td>7.5</td>
<td>$-23.51$</td>
<td>$-999$</td>
</tr>
<tr>
<td>10</td>
<td>$-61.46$</td>
<td>$-33.98$</td>
</tr>
<tr>
<td>12.5</td>
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<td>$-42.29$</td>
</tr>
<tr>
<td>15</td>
<td>$-54.18$</td>
<td>$-999$</td>
</tr>
<tr>
<td>17.5</td>
<td>$-999$</td>
<td>$-999$</td>
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<tr>
<td>20</td>
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<td>$-61.31$</td>
</tr>
<tr>
<td>27.5</td>
<td>$-999$</td>
<td>$-999$</td>
</tr>
<tr>
<td>30</td>
<td>$-82.54$</td>
<td>$-999$</td>
</tr>
</tbody>
</table>

### Table 4. Percent error in $V_x$ for a vortex of radius $R_v = 0.05$ km, beamwidth (bw) of $1^\circ$, and an azimuthal sampling interval (daz) of $1^\circ$, $0.5^\circ$, and $0.25^\circ$ for various ranges; $-999$ indicates that no value of $V_x$ could be retrieved for that range.

<table>
<thead>
<tr>
<th>Range (km)</th>
<th>PE $V_x$, bw = $1^\circ$, daz = $1^\circ$</th>
<th>PE $V_x$, bw = $1^\circ$, daz = $0.5^\circ$</th>
<th>PE $V_x$, bw = $1^\circ$, daz = $0.25^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
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<td>$-999$</td>
<td>$-999$</td>
</tr>
<tr>
<td>5</td>
<td>$-999$</td>
<td>$-999$</td>
<td>$-999$</td>
</tr>
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<td>$-45.27$</td>
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<td>$-999$</td>
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<td>$-23.56$</td>
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<td>$-29.77$</td>
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<td>$-38.50$</td>
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</tr>
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<td>$-999$</td>
</tr>
<tr>
<td>30</td>
<td>$-74.24$</td>
<td>$-999$</td>
<td>$-999$</td>
</tr>
</tbody>
</table>

### Fig. 10. Plot of percent error in $V_x$ vs range for $R_v = 0.1$ km. Three sampling strategies are tested using the same beamwidth (bw) of $2^\circ$ with various factors of overlapping (see Fig. 3). Increased overlapping results in decreased effective beamwidths (ebw).

### Fig. 11. Comparison of percent error of a WSR-88D radar to a CASA radar for ranges less than 30 km for $R_v = 0.1$ km.
Several limitations exist for the present study and could be examined in future work. One is the TPVM vortex model, which is simple and may not capture the intricacies of true tornadoes and mesocyclones, including asymmetry and nonvertical orientation. The use of more complicated analytical models would overcome this limitation. Another limitation of the present study is the assumption that the CASA radars have a constant rotation rate regardless of their sampling strategy. This is not true during operation as the radars decrease or increase their rotation rate based on the phenomenon being scanned and the sampling strategy employed. A third limitation is that the scanning radar always has one azimuthal sampling volume that coincides with the center of the vortex, which does not occur very often during actual data collection. Also, this study uses a single radar to sample a single vortex. In reality, CASA is a network of radars that work together to sample multiple atmospheric phenomena. Another limitation is the use of Newton’s method in the retrieval, which can be extremely sensitive to the initial guess field. Finally, we did not take into account the effects of attenuation, which is an issue of great significance to CASA given its operation at X band.

Many extensions of this work could be undertaken to better understand how the CASA radars might sample tornadoes and mesocyclones in an optimal manner. For example, the idealized vortices used to create pseudo-observations could be replaced with high-resolution numerical model simulations of vortices, thus providing more realistic multidimensional wind profiles. To do so, a two-dimensional horizontal cross section of wind ($u$ and $v$) centered on the vortex at low elevations would be needed and could be used in the present code quite easily. Also, it would be interesting to use simulated data at various times so that different stages of the tornado’s life cycle (i.e., while its size is changing) could be studied.

Another obvious extension is the use of real CASA data in the retrieval program. Again, this would be more realistic than the idealized vortex model used and could involve dealing with multiple phenomena simultaneously. The code would most likely have to be altered in order to accommodate the latter, though of course no control solution would be available for comparison.

Finally, this study could be extended by conducting experiments with more than one CASA radar. This would be a relatively straightforward extension by simply adding more terms to the cost function. Understanding how multiple radars work together collaboratively and adaptively—in order to extract maximum information while minimizing the use of radar resources—is a fundamental challenge for CASA radars and must be studied further.

**Acknowledgments.** This research was supported in part by the Engineering Research Centers Program of the National Science Foundation under NSF Cooperative Agreement EEC-0313747. The authors thank Professor Luther White of the University of Oklahoma for his input in the mathematical development of the three-parameter vortex model.
APPENDIX

Retrieval Technique

The retrieval technique used in this study is based on a variational method similar to that used by Wood (1997) and estimates the vortex core radius \( R_x \) and maximum tangential velocity \( V_x \) from the range-degraded Doppler velocity signature of the axisymmetric vortex. Briefly, the technique involves first developing initial guesses of \( V_x \) and \( R_x \) that will be used to solve a set of nonlinear equations. Their solution yields a set of linear equations (also discussed below) from which the retrieved values of \( V_x \) and \( R_x \) are obtained.

Several steps are used to determine the initial guesses of \( V_x \) and \( R_x \); these guesses must be sufficiently close to the true value in order to achieve convergence in the retrieval with “close” depending upon a number of factors as described later. The initial value for \( V_x \) \( (V_x') \) is calculated from the Doppler rotational velocity \( (V_{rot}) \), given by

\[
V_x' = V_{rot} = \frac{1}{2}(V_P - V_N).
\]  \( \text{(A1)} \)

In (A1), \( V_P \) is the extreme positive Doppler velocity value away from the radar and \( V_N \) is the extreme negative Doppler velocity value toward the radar. The initial guess for \( R_x \) \( (R_x') \) is given by

\[
R_x' = \frac{D_E}{2},
\]  \( \text{(A2)} \)

where \( D_E \) is the estimated core diameter and is expressed as (Wood and Brown 1992)

\[
D_E = F D_A.
\]  \( \text{(A3)} \)

In (A3), \( D_A \) is the apparent diameter shown in Fig. A1 between the location of \( V_P \) and \( V_N \) (Wood and Brown 1992) and is given by

\[
D_A = (R_N^2 + R_P^2 - 2R_N R_P \cos \Delta \phi)^{1/2},
\]  \( \text{(A4)} \)

where \( R_N \) and \( R_P \) are the ranges of the extreme negative and positive Doppler velocity values, respectively; and \( \Delta \phi \) is the difference between the azimuths of the extreme positive and negative Doppler velocity values given by

\[
\Delta \phi = \phi_P - \phi_N.
\]  \( \text{(A5)} \)

In (A3), \( F \) is the correction factor (Wood and Brown 1992) expressed as

...
which represent \( V_1, V_2 \) and \( V_{12} \), respectively. In (A9)–(A11), \( i \) is the azimuthal data subpoint, \( j \) is the range data subpoint for a pseudo-observation, \( \Phi_1 = V_1/V_x \), and \( \Phi_2 = V_2/V_x \) [see Eq. (3), p. 7]. The parameters in (A9)–(A11) have been defined in subsection 2b. The azimuth and range must be computed before going on to the next subpoint value within a beamwidth area.

To determine the optimal estimate, the cost function \( J \) is minimized via a first derivative test, yielding the following necessary conditions:

\[
\frac{\partial J}{\partial \mathbf{m}} = 0 = 2 \sum_{i} [\tilde{V}_d(\mathbf{m}) - \tilde{V}_d] \frac{\partial \tilde{V}_d(\mathbf{m})}{\partial \mathbf{m}}. \quad (A12)
\]

We generate a sequence of models \( \mathbf{m}_0 \) and \( \mathbf{m}_1 \), with the hope that \( J(\mathbf{m}) \rightarrow \min J(\mathbf{m}) \) as the number of iterations approaches infinity. In (A12), we expand in scalar form [reading \( J(\mathbf{m}) \) as \( J(m^1, m^2) \)] as

\[
\frac{\partial J(m^1)}{\partial V_x} = 0 = 2 \sum_{i} [\tilde{V}_d(m^1) - \tilde{V}_d] \frac{\partial \tilde{V}_d(m^1)}{\partial V_x}, \quad \text{and} \quad (A13)
\]

\[
\frac{\partial J(m^2)}{\partial R_x} = 0 = 2 \sum_{i} [\tilde{V}_d(m^2) - \tilde{V}_d] \frac{\partial \tilde{V}_d(m^2)}{\partial R_x}, \quad (A14)
\]

where

\[
\frac{\partial \tilde{V}_d(m^1)}{\partial V_x} = \sum_{i} \sum_{j} \frac{1}{2} [\Phi_1(R_x, r) + \Phi_2(R_x, r)] \cos \gamma G(\phi) W(R_d)^2 \frac{\partial \tilde{V}_d(m^1)}{\partial V_x}, \quad (A15)
\]
\[
\frac{\partial V_\phi(m^2)}{\partial R_s} = \sum_{i} \sum_{j} \left[ \frac{1}{2} \sum_{i} \sum_{j} \frac{\partial \Phi_1(R_s, r)}{\partial R_s} + \frac{\partial \Phi_2(R_s, r)}{\partial R_s} \right] \cos \gamma G(\phi) |W(R_s)|^2
\]

where \( I \) and \( J \) are the upper limits of the number of azimuthal and range data subpoints within the beamwidth area, respectively.

Therefore, using \( n = 1 \) and \( n = 2 \) in (3) and differentiating (3) with respect to \( R_s \) yields

\[
\frac{\partial \Phi_1(R_s, r)}{\partial R_s} = \frac{2r^3 - 2R_s^2 r}{(R_s^2 + r^2)^2}
\]

(A17)

and

\[
\frac{\partial \Phi_2(R_s, r)}{\partial R_s} = \frac{(12R_s^2 r^5 - 12R_s^6 r)}{(3R_s^4 + r^2)^2}
\]

(A18)

To solve (A12), Newton’s method is applied. We begin by writing these equations in another form (Gerald and Wheatley 1984, 133–158)

\[
F(m^*) = \frac{\partial f(m)}{\partial m} = 0,
\]

(A19)

where \( m^* = [V^*_x, R^*_s]^T \) are the local minimizers that satisfy (A19). We expand the equations as a Taylor series about the point \( m = [V_x, R_s]^T \) in terms of \( (m^* - m) \), where \( m \) is a point near \( m^* \). The Taylor series expansion of (A19) is given by

\[
F_1(V_x^*, R_s^*) = 0 = F_1(V_x, R_s) + \left. \frac{\partial F_1}{\partial V_x} \right|_{m} (V_x^* - V_x) + \left. \frac{\partial F_1}{\partial R_s} \right|_{m} (R_s^* - R_s) + \text{higher-order terms},
\]

(A20)

and

\[
F_2(V_x^*, R_s^*) = 0 = F_2(V_x, R_s) + \left. \frac{\partial F_2}{\partial V_x} \right|_{m} (V_x^* - V_x) + \left. \frac{\partial F_2}{\partial R_s} \right|_{m} (R_s^* - R_s) + \text{higher-order terms}.
\]

(A21)

In (A20) and (A21), each function is evaluated at the approximate root \( (V_x, R_s) \). By using the Taylor series expansion, we have reduced the problem from a set of two nonlinear equations to a set of two linear equations. The unknown values are the improvements in each estimated variable \( (V_x^* - V_x) \) and \( (R_s^* - R_s) \). To implement (A20) and (A21), the partial derivatives may be approximated by recalculating the functions with a small perturbation \( \delta \) to each of the variables in turn

\[
\frac{\partial F_1}{\partial V_x} \approx \frac{F_1(V_x + \delta, R_s) - F_1(V_x, R_s)}{\delta},
\]

(A22)

\[
\frac{\partial F_2}{\partial V_x} \approx \frac{F_2(V_x + \delta, R_s) - F_2(V_x, R_s)}{\delta},
\]

(A23)

\[
\frac{\partial F_1}{\partial R_s} \approx \frac{F_1(V_x, R_s + \delta) - F_1(V_x, R_s)}{\delta},
\]

(A24)

\[
\frac{\partial F_2}{\partial R_s} \approx \frac{F_2(V_x, R_s + \delta) - F_2(V_x, R_s)}{\delta}.
\]

(A25)

If \( m \) is sufficiently similar to \( m^* \), we can truncate after the first derivative terms in (A20) and (A21) and solve for the unknowns \( (m^* - m) \). Also note that we take the derivatives of \( F_1 \) and \( F_2 \), which is the second derivative of \( J \); therefore, we also are performing a second derivative test of \( J \) to determine whether the extrema are a minimum, which is the desired condition. Thus, by Cramer’s rule

\[
V_{x}^{\ell+1} = V_x^{\ell} + \frac{\begin{vmatrix} -F_1^{\ell} & \frac{\partial F_1}{\partial V_x} \\ -F_2^{\ell} & \frac{\partial F_2}{\partial V_x} \end{vmatrix}}{\det},
\]

(A26)

\[
R_{s}^{\ell+1} = R_s^{\ell} + \frac{\begin{vmatrix} \frac{\partial F_1}{\partial V_x} & -F_1^{\ell} \\ \frac{\partial F_2}{\partial V_x} & -F_2^{\ell} \end{vmatrix}}{\det},
\]

(A27)

where the superscript \( \ell \) is the iteration number and the determinants are given by

\[
\det = \left| \frac{\partial F_1}{\partial V_x} \frac{\partial F_1}{\partial R_s} - \frac{\partial F_2}{\partial V_x} \frac{\partial F_2}{\partial R_s} \right| \neq 0.
\]

(A28)

To achieve convergence, the iterations are terminated when the function values \( F(m^*) \) in (A19) are sufficiently small or the changes in the \( m \) values are sufficiently small.

The maximum number of iterations is set to 50 in this study.
Some judgment is needed to use a subroutine that solves a nonlinear system in which Newton’s method is employed. The initial guesses for values of the **m** variables must be near enough to a solution to give convergence although convergence is not always achieved.

**REFERENCES**


