Statistical Quality of Spectral Polarimetric Variables for Weather Radar

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ABSTRACT

Spectral polarimetry for weather radar capitalizes on both Doppler and polarimetric measurements to reveal polarimetric variables as a function of radial velocity through spectral analysis. For example, spectral differential reflectivity at a velocity represents the differential reflectivity from all the scatterers that have the same radial velocity of interest within the radar resolution volume. Spectral polarimetry has been applied to suppress both ground and biological clutter, retrieve individual drop size distributions from a mixture of different types of hydrometeors, and estimate turbulence intensity, for example. Although spectral polarimetry has gained increasing attention, statistical quality of the estimation of spectral polarimetric variables has not been investigated. In this work, the bias and standard deviation (SD) of spectral differential reflectivity and spectral copolar correlation coefficient estimated from averaged spectra were derived using perturbation method. The results show that the bias and SD of the two estimators depend on the spectral signal-to-noise ratio, spectral copolar correlation coefficient, the number of spectrum average, and spectral differential reflectivity. A simulation to generate time series signals for spectral polarimetry was developed and used to verify the theoretical bias and SD of the two estimators.

1. Introduction

In addition to Doppler measurements, application of polarimetry to meteorological observations is one of the most important advancements in weather radar. Polarimetric weather radar has proven its capability in improving rainfall rate estimation, hail detection, data quality, hydrometer classification, etc. The applications and principles of polarimetric weather radar have been well summarized and discussed in a number of articles over the years (e.g., Bringi and Chandrasekar 2001; Doviak and Zrnić 1993; Meischner 2003; Zrnić and Ryzhkov 1999). The benefits of polarimetric weather radar to operational forecast and warning are demonstrated in Scharfenberg et al. (2005). Moreover, polarimetric capability is one of the requirements and challenges for the future multimission phased array radar (Zrnić et al. 2007).

Typical polarimetric measurements include differential reflectivity, copolar correlation coefficient, differential phase, and linear depolarization ratio, which contain information integrated over the radar resolution volume. However, the size of radar resolution volume can be too large. Spectral polarimetry is to combine Doppler and polarimetric measurements so that the distribution of polarimetric variables as a function of radial velocity within the radar resolution volume can be obtained (e.g., Unal et al. 2001; Yanovsky 2002; Russchenberg et al. 2008; Yanovsky 2011). The relationship between the

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commonly used polarimetric variables and spectral polarimetric variables is similar to the one between average power and Doppler spectrum. The Doppler spectrum is the power spectrum of a complex signal expressed as a function of the Doppler frequency or velocity and can be interpreted as the reflectivity weighted distribution of radial velocities of scatterers in the resolution volume (e.g., Yanovsky 2011). In other words, average power is a single value resulted from the integration of Doppler spectrum. Specifically, for spectral polarimetric application, spectral differential reflectivity, spectral copolar correlation coefficient, and spectral differential phases can be estimated, which are represented by the corresponding polarimetric variables as a function of radial velocity. When it is available, spectral linear depolarization ratio can be obtained (Unal 2009).

Spectral polarimetry has been used for improving data quality, especially in clutter identification and suppression. For example, Bachmann and Zrnić (2007) applied spectral polarimetric variables to filter out the contamination from biological clutter and, consequently, better wind estimates from clear air are obtained. The spectral copolar correlation coefficient and the texture of spectral differential reflectivity and spectral differential phase were used to discriminate different types of clutter using a fuzzy logic inference system for improved estimation of Doppler and polarimetric moments (Moisseev and Chandrasekar 2009; Moisseev et al. 2010). Unal (2009) reported good suppression of ground clutter based on spectral linear depolarization ratio. In addition to clutter mitigation, a technique of velocity dealiasing for alternative transmission was developed by using spectral differential phase (Unal and Moisseev 2004). Moreover, spectral polarimetry has been used to obtain microphysical information of precipitation as well as environmental parameters such as background wind and turbulence within the radar resolution volume. For example, Yanovsky et al. (2005) showed that the turbulence intensity in rain can be retrieved from spectral differential reflectivity. Moisseev and Chandrasekar (2007a) estimated drop size distribution (DSD) from the Doppler spectrum after turbulence broadening is deconvolved, where the broadening kernel is obtained from spectral differential reflectivity. Moreover, the individual DSDs of different ice particles mixed in the radar resolution volume are retrieved using Doppler spectrum and spectral differential reflectivity (Spek et al. 2008; Dufournet 2010). The hydrometeor classification is carried out in the spectral domain using all the three spectral polarimetric variables (Moisseev and Chandrasekar 2007b; Moisseev et al. 2008). Although spectral polarimetry has shown promising results, the quality of these spectral polarimetric estimators has not been studied comprehensively. In this work, the statistical error of spectral differential reflectivity and spectral copolar correlation coefficient were derived and verified.

The quality of Doppler and polarimetric estimates has been well studied and reported in several articles, where the perturbation method is often used. For example, Zrnić (1977) applied the perturbation analysis to derive the variance of mean radial velocity and spectrum width estimated by autocovariance method. Liu et al. (1994) extended the perturbation method to derive the bias and standard deviation (SD) of the copolar correlation coefficient estimated from alternating transmission mode. A comprehensive derivation of the SD of differential reflectivity, copolar correlation coefficient, differential phase in both alternating and simultaneous modes is provided in Bringi and Chandrasekar (2001). Melnikov and Zrnić (2007) presented the SD of differential reflectivity and copolar correlation coefficient estimated using the noise-free correlation function at the first lag.

In this work, the perturbation method was also used to theoretically derive the bias and SD of the spectral differential reflectivity and spectral copolar correlation coefficient. This paper is organized as follows. A brief introduction of the spectral polarimetric variables is provided, and their relationship with the commonly used polarimetric variables is derived in section 2. In addition, the estimation of spectral differential reflectivity and spectral copolar correlation coefficient is defined. In section 3, the bias and SD of these two estimators are derived and discussed. The derived statistical errors are subsequently verified using simulation in section 4. Finally, a summary and conclusions are presented in section 5.

2. Spectral analysis of polarimetric variables

a. Definition of spectral polarimetric variables

The spectral polarimetric variables are defined from auto- and/or cross spectra of signals from horizontal (H) and vertical (V) channels and therefore are a function of Doppler frequency, whereas the commonly used polarimetric variables are only a single value. Spectra and correlation function are a Fourier transform pair for a stationary process, as shown in the following equation (Papoulis and Pillai 2002):

\[ R_i(\tau) = \int_{-\infty}^{\infty} S_i(f)e^{i2\pi f \tau} df, \quad i = H, V, \text{or} \ X, \] (1)

where \( \tau \) is temporal lag, and \( f \) denotes the Doppler frequency. For \( i = H \) or \( V \), \( R_i(\tau) \) and \( S_i(f) \) are the auto-correlation function and autospectrum (i.e., Doppler spectrum) of signals from the \( i \) channel. Additionally,
$R_X(\tau)$ and $S_X(f)$ are the cross-correlation function and cross spectrum of signals from H and V channels. Note that autospectrum is always real and positive, whereas cross spectrum is generally complex valued.

The spectral differential reflectivity in linear scale is defined in the following equation:

$$sZ_{dr}(f) = \frac{S_H(f)}{S_V(f)}.$$  \hspace{1cm} (2)

The spectral differential reflectivity in decibel (dB) scale is defined by $sZ_{DR}(f) = 10 \log_{10}sZ_{dr}(f)$. Note that all spectral polarimetric variables have a character of s in the notation to differentiate them from the commonly used polarimetric variables. In addition, spectral polarimetric variables can be represented as a function of radial velocity using $v = -\lambda f/2$, where $\lambda$ is the radar wavelength. Differential reflectivity (linear scale) is related to the spectral differential reflectivity in the following manner:

$$Z_{dr} = \frac{P_{s,H}}{P_{s,V}} = \int_{-\infty}^{\infty} sZ_{dr}(f)S_{n,V}(f) \, df,$$  \hspace{1cm} (3)

where $P_{s,i} = R_i(0)$ is the average power for H or V channels, and $S_{n,V}(f) = S_V(f)/P_{s,V}$ is the V-channel Doppler spectrum normalized by its average power.

The spectral copolar correlation coefficient is defined by the modulus of the coherence function $C_X(f)$ in the following equation:

$$sp(f) = |C_X(f)| = \frac{|S_X(f)|}{[S_H(f)S_V(f)]^{1/2}}.$$  \hspace{1cm} (4)

The commonly used copolar correlation coefficient ($\rho_{co}$), which is defined by the modulus of normalized cross-correlation function, is related to the spectral correlation coefficient through the coherence function in the following form:

$$\rho_{co} = \frac{|R_V(0)|}{\sqrt{R_H(0)R_V(0)}} = \left\{ \int_{-\infty}^{\infty} C_X(f) \sqrt{S_{n,H}(f)S_{n,V}(f)} \, df \right\}^2,$$  \hspace{1cm} (5)

where $S_{n,H}(f)$ is the normalized Doppler spectrum of the H channel.

It is interesting to point out that differential reflectivity is equivalent to the integration of spectral differential reflectivity weighted by normalized Doppler spectrum from V channel, as shown in (3). For copolar correlation coefficient as shown in (5), it can be observed that the normalized cross correlation is the integration of the coherence function weighted by the square root of the product of normalized spectra from H and V channels. It is evident from (3) and (5) that the spectral polarimetric variables provide an additional dimension of velocity to differential reflectivity and copolar correlation coefficient through spectral processing. For example, for polarimetric measurements made with sufficiently high elevation angle, different size of hydrometeors can be sorted by the radial components of their size-dependent terminal velocities. As a result, differential reflectivity as a function of drop sizes can be obtained using spectral polarimetry (e.g., Moisseev and Chandrasekar 2007a; Spek et al. 2008). If spectral polarimetric variables are uniform in Doppler velocity, then the commonly used polarimetric variables (differential reflectivity and copolar correlation coefficient) would have the same value as the spectral polarimetric variables. This can be verified by substituting a constant for $sZ_{dr}(f)$ and $C_X(f)$ into (3) and (5), respectively. It is interesting to point that the uniform spectral differential reflectivity could be an indication of the presence of strong turbulence within the resolution volume (Yanovsky et al. 2005; Yanovsky 2011).

b. Estimation of spectral polarimetric variables

Now let us examine how to estimate spectral differential reflectivity and spectral copolar correlation coefficient from a finite number of complex radar samples. The first step is to perform the discrete Fourier transform (DFT) of radar signals from both channels using the following equation:

$$Z_j(f) = \sum_{m=0}^{M-1} d(m)V_j(m) e^{-j2\pi mf}, \quad j = H \text{ or } V,$$  \hspace{1cm} (6)

where $d(m)$ is the data window for $V_j(m)$. The second step is to estimate the auto- and cross spectra using the following equations:

$$\hat{S}_j(f) = \frac{|Z_j(f)|^2}{M} - N_j, \quad j = H \text{ or } V, \quad \text{and}$$  \hspace{1cm} (7)

$$\hat{S}_X(f) = \frac{Z_H(f)Z_V^*(f)}{M},$$  \hspace{1cm} (8)

where $\hat{S}_j(f)$ is the estimated Doppler spectrum from either H or V channel, $\hat{S}_X(f)$ is the estimated cross spectrum, and $N_j$ is the constant noise level in H or V channel spectrum and was assumed to be known. It was further assumed that the noise from H and V are white and uncorrelated. To reduce the variance in the spectrum estimator (Doviak and Zrnić 1993), K spectra were averaged in step 3:
where $\hat{S}_{ik}(f)$ is the auto- or cross spectra estimated using (7) or (8). It is assumed that the velocities in spectra are not aliased, otherwise unfolding both Doppler and cross spectra is needed to obtain unambiguous velocity information. The last step is to estimate spectral differential reflectivity and spectral correlation coefficient from averaged spectra using the following equations:

$$s\hat{Z}_{dr}(f) = \frac{\bar{S}_H(f)}{\bar{S}_V(f)}, \text{ and}$$

$$s\hat{\theta}(f) = \frac{|\bar{S}_X(f)|}{\sqrt{\bar{S}_H(f)\bar{S}_V(f)}}.$$  

Note that one can choose to estimate the spectral differential reflectivity and spectral correlation coefficient from each spectrum and, consequently, average the spectral polarimetric variables. Nevertheless, in this work we only consider the case where the spectrum averaging is performed first.

### 3. Statistical analysis of spectral polarimetric variables

#### a. Derivation

The perturbation method was used in this work, where the estimators of interest were modeled by a deterministic true value and a random perturbation term. For example, the spectrum estimator has the form of $\hat{S}_i(f) = S_i(f) + \delta S_i(f)$, where $S_i(f)$ is the true spectrum and $\delta S_i(f)$ is the perturbation. Note that hereafter the function form of the variables in spectrum domain is omitted for simplicity, that is, $\hat{S}_i = \hat{S}_i(f)$, for example.

The bias and variance of spectra estimated using (7) or (8) have been provided in the literature (e.g., Bringi and Chandrasekar 2001; Doviak and Zrnić 1993). In (B1) the bias of auto- and cross spectra can be reduced by a larger number of samples. In this work, a sufficient number of samples are used so that the bias of spectrum estimators is negligible. The number of samples to achieve such conditions will be discussed in more detail in section 3b. Under this condition, the average spectra ($\overline{S}_i = S_i + \delta S_i$) are also assumed to be unbiased. Specifically, the following assumption was used for the unbiased spectrum estimator:

$$\langle \delta S_i \rangle = \frac{\delta S_i}{S_i} = 0, \quad i = H, V, \text{ and } X,$$

where $\langle \cdot \rangle$ is the expected value.

Moreover, we assumed the perturbation of average spectrum estimator is much smaller than the true spectrum as $\delta S_i \ll S_i$. It is known that the SD of periodogram spectrum estimator is comparable to its expected value (Doviak and Zrnić 1993). After $K$ averaging of independent spectra, the SD is reduced by $\sqrt{K}$. In this work, a minimum of $K = 5$ was considered, which results in $[\text{SD}(\delta S_i)/S_i] \approx 0.45$.

The estimator of spectral differential reflectivity is represented in (13) by using the perturbation method and can be approximated by (14):

$$s\hat{Z}_{dr} = sZ_{dr} + \delta s\hat{Z}_{dr}$$

$$\approx S_H + \frac{S_H}{S_V}\left(\frac{\delta S_H}{S_H} - \frac{\delta S_V}{S_V} + \frac{\delta S_H^2}{S_H} + \frac{\delta S_V^2}{S_V}\right).$$

where (14) was obtained by substituting $\overline{S}_i = S_i(1 + \delta S_i/S_i)$ into (10), taking out the term of $S_H/S_V$, and performing binomial expansion of the denominator. Note that the terms with an order higher than the second order were neglected because of $\delta S_i/S_i \ll 1$. Subsequently, it can be shown that the bias of $s\hat{Z}_{dr}$ is $b(s\hat{Z}_{dr}) = (\delta s\hat{Z}_{dr})$, and the variance of $s\hat{Z}_{dr}$ is $\text{var}(s\hat{Z}_{dr}) = (\delta s\hat{Z}_{dr})^2$. The normalized bias and variance of spectral differential reflectivity can then be derived in the following forms after applying (12) and considering up to the second-order terms:

$$b(s\hat{Z}_{dr}) = -\frac{\langle \delta S_H^2 \rangle}{S_H^2} + \frac{\langle \delta S_V^2 \rangle}{S_V^2},$$

and

$$\text{var}(s\hat{Z}_{dr}) = \frac{\langle \delta S_H^2 \rangle}{S_H^2} + \frac{\langle \delta S_V^2 \rangle}{S_V^2} - 2\frac{\langle \delta S_H \delta S_V \rangle}{S_H S_V}.$$
where \( sp \) is defined in (4), and \( s\text{SNR}_j = S/N_j \) is the spectral signal-to-noise ratio (SNR) in linear scale for the \( j \) channel, \( j = H, V \).

Similar procedure was applied to spectral copolar correlation coefficient estimator and the following equations were obtained:

\[
\hat{s}p = sp + \delta \hat{s}p,
\]

\[
\approx \frac{|S_X|}{\sqrt{S_H S_V}} + \frac{|S_X|}{\sqrt{S_H S_V}} \left\{ \frac{1}{2} \frac{\delta H}{S_H} - \frac{1}{4} \frac{\delta V}{S_V} + \frac{3}{8} \frac{\delta H^2}{S_H} \right. \\
+ \frac{3}{8} \frac{\delta V^2}{S_V} + \frac{3}{4} \frac{\delta H \delta V}{S_H S_V} + \frac{1}{2} \frac{\text{Re} \left( \frac{\delta \delta H}{S_X} \right)}{S_X} - \frac{1}{2} \frac{\text{Re} \left( \frac{\delta \delta V}{S_X} \right)}{S_X} \left\}.
\]

(19)

where \( \text{Re}(\cdot) \) denotes the real part of a complex variable. The normalized bias and variance of spectral copolar correlation coefficient can be approximated using the following equations:

\[
b(s\hat{p}) = \frac{3}{4} \frac{\delta H^2}{S_H} + \frac{3}{4} \frac{\delta V^2}{S_V} + \frac{1}{2} \frac{\text{Re} \left( \frac{\delta \delta H}{S_X} \right)}{S_X} - \frac{1}{2} \frac{\text{Re} \left( \frac{\delta \delta V}{S_X} \right)}{S_X},
\]

(20)

\[
\text{var}(s\hat{p}) \approx \frac{1}{4} \frac{\delta H^2}{S_H} + \frac{1}{4} \frac{\delta V^2}{S_V} + \frac{1}{2} \left( \frac{\text{Re} \left( \frac{\delta \delta H}{S_X} \right)}{S_X} \right)^2 \left( \frac{\text{Re} \left( \frac{\delta \delta V}{S_X} \right)}{S_X} \right)^2 + \frac{1}{2} \frac{\text{Re} \left( \frac{\delta \delta H}{S_X} \right)}{S_X} \frac{\text{Re} \left( \frac{\delta \delta V}{S_X} \right)}{S_X},
\]

(21)

Note that \( (\delta \hat{s}p)^2 = 0 \) was used implicitly in the derivation of (21). By using (A6) to (A10), the following bias and variance of spectral correlation coefficient can be obtained:

\[
b(s\hat{p}) = \frac{1}{K} \left\{ \frac{(1 - sp)^2}{4sp^2} + \frac{2s\text{SNR}_H + 3}{8s\text{SNR}_H^2} + \frac{2s\text{SNR}_V + 3}{8s\text{SNR}_V^2} \\
+ \frac{s\text{SNR}_H + s\text{SNR}_V + 1}{4sp^2s\text{SNR}_H^2s\text{SNR}_V} \right\},
\]

(22)

\[
\frac{\text{var}(s\hat{p})}{sp^2} = \frac{1}{K} \left\{ \frac{(1 - sp)^2}{2sp^2} + \frac{1 - 2s\text{SNR}_H}{4s\text{SNR}_H^2} + \frac{1 - 2s\text{SNR}_V}{4s\text{SNR}_V^2} + \frac{s\text{SNR}_H + s\text{SNR}_V + 1}{2sp^2s\text{SNR}_H^2s\text{SNR}_V} \right\}. 
\]

(23)

b. Discussions

Before investigating the dependence of the statistical errors on different parameters, we need to determine the condition when the assumption of unbiased spectrum estimators in (12) is approximately valid. Theoretically, the spectrum estimators using (7) or (8) are asymptotically unbiased (Papoulis and Pillai 2002). In this work, it was assumed that the spectrum bias can be neglected if \( \delta \hat{s}_j/S_j \leq 0.2 \) from spectrum peak to 30 dB down from the peak. A simulation was carried out to determine the minimum number of samples to achieve the requirement. As shown in (B1), the expected value of the estimated spectrum was obtained by the convolution of a model spectrum and window function. The window function was obtained by the Fourier transform of the lag window. In other words, no statistical fluctuations were considered for this ideal case. The bias is the difference between the model spectrum and the expected value of estimated spectra. In other words, the minimum number of samples to achieve the requirement can be determined by increasing the number of samples until the requirements are met for a given type of window. Figure 1 presents the minimum number of samples as a function of normalized spectrum width for rectangular, Chebychev, and von Hann data windows. It is interesting to point out that the rectangular data window demands the largest number of samples for a given spectrum width. This is because the window function for rectangular data window has the highest sidelobes, which biases the spectrum most. On the other hand, Chebychev and von Hann data windows produce similar results, while von Hann data window is slightly better. With 64 samples, the requirements for unbiased assumption is fulfilled for normalized spectrum width larger than 0.05. Hereafter, von Hann data window with 64 or more samples is considered. Note that the threshold of \( \delta \hat{s}_j/S_j <= 0.2 \) and 30 dB are somewhat arbitrary. A larger number of samples is needed if a more stringent condition is used.

The bias and SD of the two spectral polarimetric estimators depend on the spectral SNRs, true spectral copolar correlation coefficient, and the number of spectrum average. In this work, we assumed the noise level from the two channels is equal and known. Thus, the spectral SNR for H channel can be represented by
sSNR$_{HI} = sSNR_V sZ_{dr}$. According to (17), (18), (22), and (23), the error statistics of spectral differential reflectivity estimator and spectral correlation coefficient estimator as a function of $sSNR_V$ are shown in Figs. 2–4 by varying one variable at a time. In Fig. 2, the error statistics as a function of $sSNR_V$ is shown for three true spectral copolar correlation coefficients given the true spectral differential reflectivity of 3 dB and 20 spectra averaging. It is evident that all the errors (bias and SD) decrease with increasing $sSNR_V$. For high spectral SNR (larger than approximately 20 dB), both the bias and SD are mainly determined by $sp$ and $K$ and only vary slightly with $sSNR_V$. Additionally, all the errors decrease as the model spectral copolar correlation coefficient increases.

In Fig. 3, $sZ_{DR}$ = 3 dB and $sp$ = 0.9 were used and error statistics for three different values of $K$ are presented. The dependence of error statistics on $sSNR_V$ for this case is similar to the previous one. It is evident that $K$ has larger impact on the bias of $sZ_{DR}$ and $sp$ for smaller $sSNR_V$. Moreover, the error statistics as a function of $sSNR_V$ for three different values of $sZ_{DR}$ for $sp = 0.9$ and $K = 20$ are shown in Fig. 4. As expected, the bias of $sZ_{DR}$ does not depend on the true $sZ_{DR}$. It is evident that the impact of spectral differential reflectivity on error statistics is relatively small compared to $sp$ and $K$ in Fig. 2 and Fig. 3, respectively.

Practically, if specific bias and SD of spectral polarimetric variables are desirable, it is of interest to estimate the number of spectrum averages to achieve these requirements. Furthermore, after spectral polarimetric variables are estimated, it is helpful to know which portion of the estimates, associated with sufficiently high spectral SNR, meets the required statistical performance. These can be achieved by using the error statistics in (17), (18), (22), and (23). First, a minimum requirement of the bias and SD of two spectral polarimetric variables was set using (C1). Subsequently, each minimum error requirement can be postulated by an inequality of quadrature form as $Ax^2 + Bx + C = 0$, where $x = sSNR_V$ as shown in (C2)–(C5). The coefficients of $A$, $B$, and $C$ were derived as a function of $sp$, $K$, and/or $sZ_{DR}$ and are different for each minimum error requirement. To ensure the inequality to have a real-valued solution, the conditions of $A > 0$ and $B^2 - 4AC > 0$ are required. From (C2) to (C5) the condition of $A > 0$ demands a minimum number of spectrum average $K$ to meet the requirement of error statistics. As a result, the minimum $K$ to achieve each error requirement can be derived and is shown in the following equation:

$$K_1 = \frac{(1 - sp^2)}{gR_1}, \quad K_2 = \frac{2(1 - sp^2)}{g^2R_2^2},$$
$$K_3 = \frac{(1 - sp^2)^2}{4sp^2R_3^2}, \quad K_4 = \frac{(1 - sp^2)^2}{2sp^2R_4^2},$$

where $g = (\ln10/10)$ is a constant. Thus, the minimum number of $K$ to meet all the four requirements can be obtained by the maximum of $K_1$–$K_4$. The results are depicted in Fig. 5 for three values of $sp$. For example, if one requires the bias of spectral differential reflectivity and spectral copolar correlation coefficient better than 0.04 dB and 0.05%, and the SD of the two estimators better than 0.6 dB and 3%, the minimum number of spectrum averages is approximately 20 for $sp = 0.9$. For designing an experiment for spectral polarimetric observations, these equations provide a guidance to determine the number of spectrum averages to meet the minimum error requirement given a targeted value of spectral copolar correlation coefficient. In this work, the statistical quality for both spectral polarimetric variables was derived under the assumption that the $K$ spectra are independent in appendix A. For example, $K$ independent spectra can be obtained from nonoverlapping consecutive blocks of $M$ samples in time (e.g., Moisseev et al. 2006; Spek et al. 2008) or in range. Note that the running average of frequency bins used in Bachmann and Zrnić (2007) is similar to the Welch method (Stoica and Moses 1997), where data blocks are overlapped in time. Although the number of spectrum averages increases for the same data sequence, the $K$ spectra are no longer independent. Therefore, modifications to the statistics of spectral polarimetric variables are needed. However, this is out of the scope for this work. Furthermore, it should be...
cautioned that increasing $K$ can degrade the time or range resolutions of the measurement, depending on the domain the averaging is performed.

After the data were collected and processed after $K$ averaging, the minimum $s$SNR$_V$ to achieve required accuracy and precision can be obtained from $s$SNR$_V$ $\geq \frac{(-B + \sqrt{B^2 - 4AC})}{2A}$ for each error requirement. A more detailed description is provided in appendix C. This procedure will be demonstrated in section 4.

4. Verification of statistical errors using simulations

a. Description of simulation method

To further verify the statistical error of spectral polarimetric variables derived in section 3, a simulation of time series signals from dual-polarimetric weather radar was developed based on Zrnić (1975). The idea was to first generate $K$ V-channel spectra independently and each spectrum has $M$ velocity bins. Subsequently, H-channel spectrum at each velocity bin over the $K$ spectra was produced to meet the given spectral differential reflectivity and spectral copolar correlation coefficient. This process was repeated for $M$ velocity bins. In simulation, a model spectrum for V-channel $S_V(m)$, spectral differential reflectivity $sZ_{dr}(m)$, and spectral copolar correlation coefficient $sp(m)$ were given for $M$ velocity bins. The outputs of the simulation are the time series of complex signals from both H and V channels. The simulation was performed in the following three steps. Note that a more complex and realistic spectrum model based on scattering properties and turbulence intensity (e.g., Yanovsky et al. 2001; Spek et al. 2008) can be used for the model spectra in the simulation.

In step 1, the Fourier transform of V-channel time series signal ($Z_V$) was generated for $M$ samples based on Eq. (7) in Zrnić (1975) for adding statistical fluctuations to a model spectrum and, subsequently, this process was repeated $K$ times for spectrum averaging as shown in the following:

$$Z_V(k,m) = [-S_V(m) \lnu(k,m)]^{1/2} e^{i\theta(k,m)},$$

$$k = 1, 2, \ldots, K,$$

where $u(k, m), k = 1, 2, \ldots, K$, and $m = 1, 2, \ldots, M$ are independent, identically distributed (iid) random variables with uniform distribution between 0 and 1 and, similarly,
\( \theta(k, m) \) are also iid random variables but with uniform distribution between \(-\pi\) and \(\pi\). Note that \( Z_V(k, m) \) is the DFT of the time series samples for \( K \) independent spectra as defined in (6) for a given \( k \).

In step 2, the Fourier transform of H-channel time series signals, \( Z_H(k, m) \), was generated to produce model spectral differential reflectivity and spectral correlation coefficient at each velocity bin. This was done by

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**FIG. 3.** As in Fig. 2, but for \( K = 10, 20, \) and \( 30 \) given \( Z_{DR} = 3 \) dB and \( \rho = 0.9 \).

**FIG. 4.** As in Fig. 2, but for \( Z_{DR} = 0, 1.5, \) and \( 3 \) dB given \( \rho = 0.9 \) and \( K = 20 \).
considering $Z_V(k, m), k = 1, 2, \ldots, K$ at a velocity bin $m$ as an random sequence of $K$ samples. The following equation was used to generate the random sequence of $Z_H(1: K, m)$, which has a mean amplitude of $Z_V(1: K, m) \sqrt{s Z_{dr}(m)}$ and correlation coefficient of $s r(m)$ with $Z_V(1: K, m)$ and it was repeated $M$ times for all the velocity bins:

$$Z_H(k, m) = \sqrt{s Z_{dr}(m)} s r(m) Z_V(k, m) + \sqrt{1 - s p^2(m)} R(k, m), \quad m = 1, 2, \ldots, M,$$

(26)

where $R(k, m)$ was generated using (25) with the same model spectrum $S_V(m)$, but $u(k, m)$ and $\theta(k, m)$ were generated independently. In other words, $Z_V(1: K, m)$ and $R(1: K, m)$ are $K$ realizations from two independent random variables with identical distribution.

In step 3, the time series signals for H and V channels for each $k$ were generated by the inverse Fourier transform of (26) and (25), respectively. Finally, independent noise sequences were added to the two time series signals individually based on the desirable SNR.

### b. Simulation results

In this work, the spectral polarimetric variables and the derived statistical errors were demonstrated and verified using three cases. For all three cases, the V-channel Doppler spectrum was modeled by a Gaussian spectrum with mean velocity of $-5$ m s$^{-1}$ and spectrum width of 4 m s$^{-1}$ for a maximum unambiguity velocity of 25 m s$^{-1}$. The spectral copolar correlation coefficient used in the model was uniform over all the velocities and has values of 0.8, 0.9, and 0.95 for cases I, II, and III, respectively. The modeled spectral differential reflectivity has a constant of 0 dB for case I, has an exponential variation for case II, and is a linear function for case III. For all the three cases, the number of samples is 64 and the number of spectrum averages is 20. Moreover, the theoretical SDs for spectral differential reflectivity and spectral copolar correlation coefficient can be calculated using (18) and (23), respectively. The modeled spectral polarimetric variables and their associated theoretical SDs are denoted by dotted, dashed, and solid lines for the three cases, respectively, in Fig. 6. For each case, 100 realizations of time series signals for both channels were generated using a constant SNR of 30 dB for V channel. Before investigating the statistical performance of the spectral polarimetric estimators, let us first verify whether the simulated time series signals can produce the desirable conventional polarimetric variables. Using (3), the differential reflectivity of 0, 1.112, and 3.016 dB are expected for cases I, II, and III, respectively. Based on (5), the expected copolar correlation coefficient is equal to its constant spectral correlation coefficient. At the same time, the differential reflectivity and correlation...
coefficient can be estimated using simulated time series signals (Bringi and Chandrasekar 2001). The mean of estimated differential reflectivities obtained for the three cases is $-0.007$, $1.119$, and $3.026$ dB, and the mean of the estimated copolar correlation coefficients is $0.799$, $0.900$, and $0.949$. The good agreement between the expected and estimated polarimetric variables verifies the relationship and suggests the feasibility of proposed simulation.

For each realization, the spectral polarimetric variables were estimated using (10) and (11) with von Hann data window. Note that the spectral polarimetric variables are not defined for a spectral SNR less than 0 dB. The mean of estimated spectral differential reflectivity and spectral correlation coefficient are presented on the top-left and -right panels, respectively. Additionally, the SD of the two estimators is shown on the lower two panels. It is evident that the mean of both spectral differential reflectivity and spectral correlation coefficient agree well with the model. The bias of spectral correlation coefficient becomes identifiable only toward both tails of the spectrum, where spectral SNR is evidently low. The bias of spectral differential reflectivity is also present at low spectral SNR, but it is difficult to observe because of its relatively small value for these cases. Moreover, the theoretical SDs for the two spectral polarimetric estimators are verified using simulation as shown in the bottom two panels. As discussed earlier, the SD of both estimators is not affected much by the modeled spectral differential reflectivity. This is manifested by relatively constant values of SDs between approximately $-10$ and $8$ m s$^{-1}$ for cases II and III, despite the model $sZ_{DR}$ varies within that region. Moreover, the effect of modeled spectral correlation coefficient on SD of both spectral polarimetric variables is evident for a given $K$ at large spectral SNR.

An example of estimated spectra and spectral polarimetric variables from one realization is shown in Fig. 7. The estimated average spectra for both H and V channels for the three cases are shown on the top three panels from left to right, respectively. The noise level is at approximately $-50$ dB. The estimated spectral differential reflectivity and spectral correlation coefficient are denoted by dashed lines in the middle and bottom panels, respectively. The model values are denoted by solid lines. Note that all the estimates associated with $\text{SNR} < 0$ are denoted by gray dashed lines, which generally follow the model values with increasing fluctuations as $\text{SNR}_F$ decreases. Thus, it is important to obtain some idea about which portion of the estimates meets the error.
As discussed in section 3 and appendix C, the minimum spectral SNR that meets the desirable bias and SD can be determined given $K$, $sZ_{DR}$, and $s$. Here, $K = 20$ has been set. As discussed previously, the impact of spectral differential reflectivity on statistical error is limited. Thus, the spectral differential reflectivity of 0 dB is used for all the three cases. Additionally, the estimated spectral correlation coefficient averaged over the region where it starts from the peak of the spectrum to 20 dB below the peak was used to approximate the true spectral copolar correlation coefficient. As an example, we requested the maximum bias and SD of spectral differential reflectivity to be 0.08 and 0.85 dB, respectively. Additionally, the maximum bias of 0.3% and maximum SD of 8% were requested for the normalized spectral correlation coefficient. The resulted spectral SNR thresholds for the three cases are denoted by horizontal dotted lines on the top panels. The estimated spectral differential reflectivity and spectral correlation coefficient that meet the requirements are highlighted by black dashed lines. The adaptive threshold for spectral SNR can help to identify the region of spectral polarimetric variables with desirable data accuracy and precision.

5. Summary and conclusions

In this work, the relationship between the commonly used polarimetric variables and the spectral polarimetric variables was established. Both the differential reflectivity and copolar correlation coefficient can be represented as the weighted sum of their spectral components. Furthermore, the estimators for spectral differential reflectivity and spectral cross-correlation coefficient were defined based on averaged auto- and cross spectra. The bias and SD of the two spectral polarimetric variables were derived using perturbation method. These statistical errors have similar forms to those from polarimetric variables and decrease as the increasing spectral SNR. Moreover, the number of spectrum averages and spectral copolar correlation coefficient play a significant role in these errors. These derived statistics can also be used to determine the minimum number of spectrum averages to achieve desirable requirements of the statistical
errors. Consequently, a threshold on spectral signal-to-
noise ratio can be determined for a given number of
spectrum averages to obtain the region where spectral
polarimetric estimates have achieved the expected qual-
ity. One of the limiting factors for the application of
spectral polarimetry to operational observations could
be the requirement of a relatively long time sequence
\((M \times K)\) to achieve reasonable frequency resolution
and statistical quality. In this work, we assumed that \(M\)
is sufficiently large so spectrum estimators are unbiased.
To effectively increase the number of spectrum averages,
a combination of time and range averaging could be used.
Another possibility is to apply overlapped data windows
so that the frequency resolution is maintained, while the
data quality is compromised.

These derived statistical errors were further verified
using simulations, where the time series signals for both
H and V channels were generated based on modeled
spectral polarimetric variables and regular SNR. Three
cases with different values of spectral copolar correlation
coefficient and three variations of spectral differential
reflectivities were simulated. The results demonstrate
that not only the model spectral differential reflectivity
and spectral copolar correlation coefficient can be recon-
bstructed, but also the bias and SD obtained from simula-
tions are consistent with the theoretical derivations.

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APPENDIX A

Statistics of Perturbation Terms

In this work, the averaged auto- and cross spectra
defined in (9) were used to estimate polarimetric vari-
ables as shown in (10) and (11). The bias and SD of
spectral polarimetric variables depend on the correlation
of the perturbation terms of averaged spectra in
(15), (16), (20), and (21). In this appendix, the correla-
tion function and second moment of these perturbations
are first derived to be as a function of nonaveraged spectra.
Subsequently, the statistics of these nonaveraged spec-
tra obtained in appendix B are used.

a. Effect of spectrum averaging

It is assumed that the \(K\) spectra for averaging, \(\hat{S}_{ik}\) in (9),
\(k = 1, 2, \ldots, K\), are independent, identically distributed
(iid) random variables, where \(i = H, V,\) or \(X\). In other
words, the variance and covariance of averaged spectra can
be reduced by a factor of \(K\), \(\text{var}(\hat{S}_{i}/S_{i}) = (1/K)\text{var}(S_{i}/S_{i})\),
and \(\text{cov}(\hat{S}_{i}/S_{i}, \hat{S}_{j}/S_{j}) = (1/K)\text{cov}(S_{i}/S_{i}, S_{j}/S_{j})\) for \(i \neq j\)
(Papoulis and Pillai 2002). Note that all spectrum es-
timators are normalized by their true spectra (\(S_{i}, i = H, V,\) and \(X\)), which are deterministic variables. In
this work, it was further assumed that sufficient number of
samples are used so that the spectrum estimators are
unbiased, that is, Eq. (12) and, consequently, \(\langle \hat{S}_{i} \rangle =
\langle S_{i}/S_{i} \rangle = 1\) is established. Thus, the second moment of
the perturbation of normalized autospectra can be derived in the following manner:

\[
\langle \delta S_{i}^{2} \rangle = \text{var}(\hat{S}_{i}) = \frac{1}{K} \text{var}(S_{i}) = \frac{1}{K} \frac{1}{S_{i}^{2}} \left( \frac{S_{i}^{2}}{S_{j}^{2}} - 1 \right), \quad j = H, V. \quad (A1)
\]

Similarly, the following two relationships can be obtained:

\[
\left\langle \delta S_{X}^{2} \right\rangle = \frac{1}{K} \left\langle \delta S_{H}^{2} \delta S_{V} \right\rangle, \quad (A2)
\]

Following a similar procedure, the correlation of the
perturbation of the averaged spectra can be derived to
be related to the correlation of spectra without averag-
ing in the following:

\[
\left\langle \frac{\delta S_{H}}{S_{H}} \frac{\delta S_{V}}{S_{V}} \right\rangle = \frac{1}{K} \left\langle \delta S_{H} \delta S_{V} \right\rangle = \frac{1}{K} \left\langle \delta S_{H} \delta S_{V} \right\rangle - 1, \quad (A4)
\]

\[
\left\langle \delta S_{X}^{2} \right\rangle = \frac{1}{K} \left\langle \delta S_{X}^{2} \right\rangle, \quad (A3)
\]

\[
\left\langle \frac{\delta S_{X}^{2}}{S_{X}} \frac{\delta S_{X}^{2}}{S_{X}} \right\rangle = \frac{1}{K} \left\langle \frac{\delta S_{X}^{2}}{S_{X}} \delta S_{X} \right\rangle = \frac{1}{K} \left\langle \frac{\delta S_{X}^{2}}{S_{X}} \delta S_{X} \right\rangle - 1, \quad (A5)
\]

where \(j = H\) or \(V\). The next step is to derive the second
moments and correlations from (A1) to (A5) using the
results from appendix B.

b. Second moments and correlations

By substituting (B9) into (A1), the second moment of
the perturbation of the normalized autospectrum esti-
mator can be derived in the following result:
\[
\frac{\langle \delta S^2_X \rangle \delta S}{S^2_X} = \frac{1}{K} \left( 1 + \frac{2}{s\text{SNR}_j} + \frac{1}{s\text{SNR}_j^2} \right), \quad (A6)
\]

where \(s\text{SNR}_j\) is the SNR measured in the spectrum domain, and \(j = H\) or \(V\).

For the second moment of \(\mathcal{R}(\delta S^2_X/S_X)\), the term of \(\langle \mathcal{R}^2(\delta S^2_X/S_X) \rangle\) in (A2) is rewritten and derived in the following form, using \(\langle \delta S^2_X \rangle = 2\langle S_X^2 \rangle\) which can be obtained with a similar derivation as (B4):

\[
\mathcal{R}\left( \frac{\delta S_X}{S_X} \right) = \left\langle \left( \frac{1}{2} \left( \frac{\dot{S}_X}{S_X} + \frac{\ddot{S}_X}{S_X} \right) \right)^2 \right\rangle = \frac{1}{2} \frac{\langle |\dot{S}_X|^2 \rangle}{\langle S_X \rangle^2}.
\]

By substituting the above result into (A2) and using (B10), the following result is obtained:

\[
\left\langle \mathcal{R}^2(\delta S^2_X/S_X) \right\rangle = \frac{1}{2Ksp^2} \left\{ 1 + sp^2 + \frac{s\text{SNR}_H + s\text{SNR}_V + 1}{s\text{SNR}_H s\text{SNR}_V} \right\}.
\quad (A7)
\]

Similarly, by substituting (B10) into (A3), the second moment of \(\delta S^2_X/S_X\) is derived in the following equation:

\[
\left\langle |\delta S^2_X| \right\rangle = \frac{1}{Ksp^2} \left\{ 1 + \frac{s\text{SNR}_H + s\text{SNR}_V + 1}{s\text{SNR}_H s\text{SNR}_V} \right\}.
\quad (A8)
\]

The correlation between \(\delta S^2_H/S_H\) and \(\delta S^2_V/S_V\) is obtained in the following equation using (A4) and (B11):

\[
\left\langle \frac{\delta S^2_H}{S_H} \frac{\delta S^2_V}{S_V} \right\rangle = \frac{sp^2}{K}.
\quad (A9)
\]

By substituting (B12) and (B13) into (A5) and using \(\mathcal{R}(S^2_X/S_X) = (1/2)(S^2_X/S_X + S^2_H/S_H)\), the following correlation can be derived:

\[
\left\langle S^2_j \right\rangle = \frac{1}{M^2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} \langle V^*_j(m)V_j(n)V^*_q(p)V_q(q) \rangle d^*(m)d(n)d^*(p)d(q)e^{-j2\pi f_0(n-m)}e^{-j2\pi f_0(q-p)},
\quad (B2)
\]

where \(j = H\) or \(V\). The expected value of the four variables in (B2) can be simplified in the following manner (Reed 1962):

\[
\langle V^*_j(m)V_j(n)V^*_q(p)V_q(q) \rangle = \langle V^*_j(m)V_j(n) \rangle \langle V^*_q(p)V_q(q) \rangle + \langle V^*_j(m)V_q(q) \rangle \langle V^*_q(p)V_j(n) \rangle = R_j(l)R_j(l') + R_q(o)R_q(o'),
\quad (B3)
\]

APPENDIX B

Statistics of Auto- and Cross Spectra

In this section, the second-order statistics (i.e., second moment and correlation) of the nonaveraged auto- and cross spectra are derived. Before the derivation, it is of interest to review the expected value of the auto- and cross spectra estimated from \(M\) complex signals using (7) and (8), respectively (e.g., Doviak and Zrnić 1993; Papoulis and Pillai 2002):

\[
\langle S_j \rangle = \sum_{l=-M+1}^M \langle V^*_j(m)V_j(m+l) \rangle e^{-j2\pi f_0 l} = S_j(f) \otimes W(f),
\quad (B1)
\]

where \(\otimes\) denotes the convolution, \(S_j\) is the true spectrum, and its corresponding inverse Fourier transformation is denoted by \(R_j(l)\). Specifically, the autocorrelation is defined by \(R_j(l) = \langle V^*_j(m)V_j(m+l) \rangle, j = H\) or \(V\), and the cross correlation is defined by \(R_{j}(l) = \langle V^*_j(m)V_h(m+l) \rangle\). Moreover, the lag window \(w(l)\) is defined by the autocorrelation function of data window \(d(m)\). The window function \(W(f)\) is the Fourier transform of the lag window. It can be observed from (B1) that the bias of \(\hat{S}_j\) depends on \(W(f)\) and decreases as increasing \(M\). To maintain the generalization of the derivation, the assumption of unbiased spectrum is not applied until the last step. The second-order statistics of those spectrum estimators are first considered for signal and noise together using \(\hat{S}_j = \hat{S}_j + \hat{N}_j\). The second moment of the autospectrum can be written in the following form by substituting (6) into (7):
where \( l = n - m, l' = q - p, o = q - m, \) and \( o' = n - p. \) Substituting (B3) into (B2) and applying (B1), the following result can be derived:

\[
\begin{align*}
\langle \hat{S}_X^2 \rangle & = \sum_{l=-(M-1)}^{M-1} R_l(l) w(l) e^{-j2\pi fl} + \sum_{l'=-\tilde{M}(M-1)}^{\tilde{M}(M-1)} R_{l'}(l') w(l') e^{-j2\pi fl'} \\
& \times \sum_{o=-(M-1)}^{M-1} R_l(o) w(o) e^{-j2\pi fo} = 2\langle \hat{S}_X^2 \rangle.
\end{align*}
\] (B4)

For the second moment of the modulus of cross-spectrum estimator, \( \langle |\hat{S}_X|^2 \rangle \), it can be obtained by substituting (6) into (8) and performing the absolute value operation:

\[
\langle |\hat{S}_X|^2 \rangle = |\langle \hat{S}_X \rangle|^2 + \langle \hat{S}_V \rangle \langle \hat{S}_H \rangle.
\] (B5)

For the correlation between the two autospectrum estimators, \( \langle \hat{S}_H \hat{S}_V \rangle \), the following result can be obtained by following similar procedure:

\[
\langle \hat{S}_H \hat{S}_V \rangle = \langle \hat{S}_H \rangle \langle \hat{S}_V \rangle + |\langle \hat{S}_X \rangle|^2.
\] (B6)

Similarly, the correlation between the auto- and cross spectra, \( \langle \hat{S}_X \hat{S}_H \rangle \) and \( \langle \hat{S}_X \hat{S}_V \rangle \), can be derived as follows:

\[
\langle \hat{S}_X \hat{S}_H \rangle = 2\langle \hat{S}_X \rangle \langle \hat{S}_H \rangle, \quad \text{and}
\] (B7)

\[
\langle \hat{S}_X \hat{S}_V \rangle = 2\langle \hat{S}_X \rangle \langle \hat{S}_V \rangle.
\] (B8)

Practically, the noise level can be estimated from spectrum (e.g., Hildebrand and Sekhon 1974) or obtained from the system calibration. Therefore, the next step is to substitute \( \hat{S}_f = \hat{S} + N_f \) in (B4)–(B8) to extract the signal component of the autospectrum. Additionally, \( \hat{S}_X \) is unchanged in these equations because the noise between the two orthogonal channels is assumed to be uncorrelated. After simplification, the following results can be obtained:

\[
\langle \hat{S}_f^2 \rangle = 2\langle \hat{S}_f \rangle^2 + 2N_f \langle \hat{S}_f \rangle + N_f^2,
\] (B9)

\[
\langle |\hat{S}_f|^2 \rangle = |\langle \hat{S}_f \rangle|^2 + N_f \langle \hat{S}_f \rangle + N_f^2,
\] (B10)

\[
\langle \hat{S}_H \hat{S}_V \rangle = \langle \hat{S}_H \rangle \langle \hat{S}_V \rangle + |\langle \hat{S}_X \rangle|^2,
\] (B11)

\[
\langle \hat{S}_X \hat{S}_H \rangle = 2\langle \hat{S}_X \rangle \langle \hat{S}_H \rangle + \langle \hat{S}_X \rangle N_H, \quad \text{and}
\] (B12)

\[
\langle \hat{S}_X \hat{S}_V \rangle = 2\langle \hat{S}_X \rangle \langle \hat{S}_V \rangle + \langle \hat{S}_X \rangle N_V.
\] (B13)

Note that if the assumption of unbiased spectrum estimator is applied, all the expected values of spectrum estimators in (B9)–(B13) are approximated by their true values.

**APPENDIX C**

**Derivation of the Thresholds for sSNR_V**

If the required quality of the spectral polarimetric variables are specified in the following manner,

\[
\begin{align*}
\text{b}(s Z_{DR}) & \leq R_1, \\
\text{var}(s Z_{DR}) & \leq R_2, \\
\frac{\text{var}(s \rho)}{s \rho} & \leq R_3,
\end{align*}
\] (C1)

then these requirements can be represented as quadratic inequalities of \( x = s \text{SNR}_V \) by substituting (17), (18), (22), and (23) into the above four inequalities:

\[
\begin{align*}
\text{SNR}_V & - (1 - s \rho^2)] x^2 - 2x - 1 \geq 0, \quad \text{(C2)}
\end{align*}
\]

\[
\begin{align*}
s Z_{DR}^2 [s^2 + 2(1 - s \rho^2)] x^2 - 2(s s_{Z_{DR}} + s_{Z_{DR}}) x - (1 + s_{Z_{DR}}^2) \geq 0, \quad \text{(C3)}
\end{align*}
\]

\[
\begin{align*}
8 s \rho^2 s_{Z_{DR}} s_{Z_{DR}} & \left[ 4 s_{Z_{DR}}^2 - \frac{(1 - s \rho^2)^2}{4 s \rho^2} \right] x^2 - 2 s_{Z_{DR}} (s^2 + s \rho^2 s_{Z_{DR}}) x + s_{Z_{DR}} + 1)x - (3 s \rho^2 + 3 s \rho^2 s_{Z_{DR}}^2 + 2 s_{Z_{DR}}) \geq 0, \quad \text{(C4)}
\end{align*}
\]

\[
\begin{align*}
4 s \rho^2 s_{Z_{DR}} & \left[ 4 s_{Z_{DR}}^2 - \frac{(1 - s \rho^2)^2}{2 s \rho^2} \right] x^2 - 2 s_{Z_{DR}} (-s \rho^2 - s \rho^2 s_{Z_{DR}}^2) + s_{Z_{DR}} + 1)x - (s \rho^2 + s \rho^2 s_{Z_{DR}}^2 + 2 s_{Z_{DR}}) \geq 0, \quad \text{(C5)}
\end{align*}
\]

where \( g = \ln 10/10 \) is a constant. Specifically, each requirement can be written as \( Ax^2 + Bx + C \geq 0 \), where the coefficients \( A, B, \) and \( C \) are different for each case. The conditions to satisfy the inequalities and ensure the real-value solution are that the coefficient \( A \) has to be non-negative and \( B^2 - 4AC > 0 \). Note that the coefficient \( A \) is a function of spectrum average, the error requirement,
and spectral copolar correlation coefficient. In other words, the condition of $A > 0$ can be used to determine the minimum number of average to achieve the required bias and SD given the $sp$ value of interest.

Furthermore, given the value of $sp$ and $sz_{dr}$, the minimum $sSNR_V$ to achieve each required error statistic (i.e., $R_1-R_4$) can be obtained by solving the corresponding inequality. Each inequality in (C2)–(C5) has two solutions of $x \leq x_1$ or $x \geq x_2$, where $x_1 = \left[-B + (B^2 - 4AC)^{1/2}/2A\right]$ and $x_2 = \left[-B + (B^2 - 4AC)^{1/2}/2A\right]$. For this work, the $x_1$ is always negative and therefore only the later solution is selected for each requirement. The threshold of $sSNR_V \geq x_2$ to meet all the requirements is determined by the maximum of the four thresholds.

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