The Ill-posed Nature of the Satellite Temperature Retrieval Problem and the Limits of Retrievability

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ABSTRACT

The inverse problem of satellite temperature profile retrieval is well known to be ill-posed. This means that not only is a vertical temperature profile solution not unique, but that two solutions can be very different from each other. A set of atmosphere-like, and true atmospheric examples of significantly dissimilar inverse solutions, were sought and found, using an 11-channel simulated HIRS sounding radiometer. Using the Riemann–Lebesgue Lemma for guidance, it is shown that simultaneous, numerical solutions of an atmospheric character may differ by as much as 10 K between 10–1000 mb. However, an empirical search for dissimilar solutions in the natural atmosphere reveals an extremely low probability of finding two significantly different RAOBs which produce radiance measurements whose differences cannot be resolved by the satellite radiometer. The empirical results are used to derive a first estimate of the limits of retrievability, analogous to the limits of predictability derivable from the ill-posed nature of the numerical weather prediction problem.

1. Introduction

One of the important aspects of appreciating satellite temperature and moisture sounding techniques is to firmly grasp the implications of the ill-posed nature of the problem. Briefly stated, the ill-posed character of this problem means that two very different temperature profiles may, in fact, produce indistinguishable sets of satellite measured spectral radiances. There is an analogy here to ill-posedness of the numerical weather forecast problem where indistinguishable differences in initial conditions may lead to large differences in the forecast, if care is not taken to regularize the problem. [See for example, Lorenz (1969a,b).] There are two ways of looking at ill-posedness of the satellite retrieval problem: directly, and inversely. The more common way is to consider the inverse problem and to refer to some mathematical properties of the governing integral equations, or to refer to the near singularity of the linear matrix approximation of those integral equations. For example, in their classic paper on temperature sounding with the SIRS instrument, Wark and Fleming (1966) show the so-called minimum information solution algorithm as the regularizing parameter γ becomes zero, which yields a solution algorithm involving the inverse of a near-singular (ill-conditioned) matrix. Twomey (1977) illustrates the ill-posedness by an analysis of the spectrum of eigenvalues of the matrix equation representing radiative transfer, a demonstration that one or more eigenvalues may be undesirably small, and also shows that without some additional constraint on the problem, many very different solutions may be found that agree with a finite number of radiance measurements to within some specified noise levels.

From a practical point of view, one faces up to the ill-posedness of the satellite sounding problem by using a retrieval algorithm which has been regularized or conditioned, with some additional information not depending on the current radiance measurements. This may take the form of a purely mathematical smoothness constraint (for example, see Twomey, 1977, Chapter 6), some constraint involving past atmospheric behavior, (for example, Strand and Westwater, 1968, or Smith et al., 1970), or a constraint involving a forecast or guess of the ambient field [see Smith et al. (1972)]. One supposes, then, that because the inverse algorithm has been stabilized, the fundamental ill-posedness has been removed from the problem. But this supposition is incorrect. Each regularized inverse solution algorithm produces a generally different thermal profile which reproduces the satellite measurements within a given envelope of error. The validity of any solution, in a given case, depends fundamentally on the validity of the particular additional constraints imposed for that case.

Recently, Thompson et al. (1985) and Chedin et al. (1985) have demonstrated pattern recognition retrieval algorithms that search for historical analogs of given temperature profiles using spectral radiances as the search variable. Such an analog approach to the tem-

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perature sounding problem raises the question of how likely it is that one could find an historical case that is a good analog to a current case in radiance space, but a very poor analog in temperature space. Such a question embodies the problem of ill-posedness of the original satellite temperature retrieval problem in the *forward* sense, rather than ill-conditioning of a particular *inverse* matrix solution algorithm.

Thus, in this paper we set out to find some "atmosphere-like" and real atmospheric versions of the basic ill-posed character of the temperature retrieval problem operating in the forward sense of the radiative transfer equation. As a result of this study, we believe that we may have a way to define limits of retrievability for satellite temperature sounders and algorithms.

2. Ill-posedness of the retrieval problem

Although the basic radiative transfer equations involve the nonlinear Planck function of frequency and temperature, the ill-posed nature of the inverse problem can just as well be demonstrated in a simplified system. Thus, we begin with the retrieval problem reduced and represented in the form of a set of Fredholm integrals [see, for example, Conrath (1972)]:

$$g(\nu_i) = \int_{x_0}^{x_T} K(\nu_i, x) t(x) dx \qquad (1)$$

where x is a vertical coordinate ranging over the effective atmosphere $[x_0, x_T]$; the kernal function is taken to be

$$K(\nu_i, x) = \left.\frac{\partial B(\nu_i, T(x))}{\partial T(x)}\right|_{T^*(x)} \left.\frac{\partial \tau(\nu_i, x)}{\partial x}\right|_{T^*(x)},$$

where $B(\nu_i, T(x))$ is the Planck function at frequency ν_i and for temperature profile $T(x)$; $\tau(\nu_i, x)$ is the atmospheric spectral transmittance; the indicial function $t(x) = T(x) - T^*(x)$; the difference between some object profile $T(x)$ and a known reference profile $T^*(x)$; and the $g(\nu_i)$ represent the difference between upwelling spectral radiance emanating from an atmosphere with temperature $T(x)$ and that emanating for $T^*(x)$. In practice, the true values of $g(\nu_i)$ are not known but are measured as values $\tilde{g}(\nu_i)$, subject to uncertainty $E(\nu_i)$.

The inverse retrieval problem is to determine $t(x)$ [or $T(x)$] from some discrete set of measurements $\tilde{g}(\nu_i)$. The ill-posedness of this inverse problem is expressed by the Riemann–Lebesgue Lemma concerning the convergence of coefficients of a Fourier series [see Titchmarsh (1948)]. This lemma holds that if $t_1(x)$ is a solution of (1) in the sense that its corresponding numbers $g_1(\nu_i)$ satisfy

$$g_1(\nu_i) - \tilde{g}(\nu_i) \leq E(\nu_i),$$

then a second, *arbitrarily different* solution $t_2(x) = t_1(x) + C\cos[n\pi x/(x_T - x_0)]$ also exists, since

$$\lim_{n\to\infty} \int_{x_0}^{x_T} K(\nu_i, x) C \cos[n\pi x/(x_T - x_0)] dx = 0$$

for arbitrarily large C (provided $K(\nu_i, x)$ is absolutely integrable on $[x_0, x_T]$, etc.). Thus, given a set of satellite radiometer measurements, $\tilde{g}(\nu_i)$, not only is an inverse solution not unique, but two very different solutions may be found even for infinitesmally small measuring uncertainties, $E(\nu_i)$. It is to be noted that this ill-posedness relates to the general forward problem and not to a particular inverse algorithm.

3. Some very different atmosphere-like solutions

The arbitrarily different nature of the solutions $t_1(x)$ and $t_2(x)$ given previously, while mathematically clear from the lemma, is not so clear atmospherically. We do not expect to find real atmospheric profiles that look like $t_2(x)$ with infinite n and arbitrarily large C. But, then, neither do we expect to make observations $\tilde{g}(\nu_i)$ with infinitesmally small uncertainty, $E(\nu_i)$. Our first problem, therefore, was to determine the range of C and n for some realistic set of instrument functions, $K(\nu_i, x)$, and uncertainties, $E(\nu_i)$.

As a hypothetical sounding instrument, we chose an 11-channel sounder comprised of 4.3 and 15 μ HIRS-like weighting functions. Table 1 shows spectral wavenumbers and two sets of instrument noise values for this sounder.

We chose $T^*(x)$ to be a standard atmosphere, and we specified the uncertainties $E(\nu_i)$ to be approximately 1% of band-averaged upwelling spectral radiances corresponding to $T^*(x)$ (noise set 1 in Table 1). In examining the practical quantitative implications of the Riemann–Lebesgue Lemma to the satellite sounding problem, we are most interested in values for *low* wavenumbers n (longer vertical wavelengths). Table 2 shows, for each channel, the maximum value of C, for each $n = 0, 1, 2, \cdots, 10$, such that

$$\int_{x_0}^{x_T} K(\nu_i, x) C \cos[n\pi x/(x_T - x_0)] dx \leq E(\nu_i) \qquad (2)$$

TABLE 1. Central wavenumber (cm^{-1}) and noise values [$mW/(m^2\ sr\ cm^{-1})$] for 11 HIRS-like sounder channels.

Wavenumber	Noise set 1	Noise set 2
668.60	0.6300	0.8200
679.05	0.6300	0.1500
689.70	0.6300	0.1100
703.80	0.6300	0.0800
716.70	0.6300	0.0500
731.85	0.6300	0.0600
2192.50	0.0044	0.0011
2211.65	0.0044	0.0012
2237.35	0.0044	0.0009
2271.20	0.0044	0.0007
2506.60	0.0044	0.0005

TABLE 2. Maximum absolute value of C which satisfies inequality (2) for vertical wavenumber n ranging from 0 (bias) to 10, and using noise set 1 of Table 1. Also tabulated are results using $C \sin[n\pi x/(x_T - x_0)]$ in inequality (2).

		Sounding channel wavenumber (cm^{-1})										
n	Function	668.6	679.05	689.70	703.80	716.70	731.85	2192.50	2221.65	2237.35	2271.20	2506.60
0	cos	0.57	0.46	0.44	0.48	0.58	0.73	0.53	0.30	0.18	0.11	4.03
	sin	—	—	—	—	—	—	—	—	—	—	—
1	cos	8.00	−1.03	−0.85	−1.54	104.04	2.97	0.81	0.52	0.49	−4.02	5.03
	sin	0.78	0.87	2.62	−1.17	−1.13	−1.29	−1.01	−0.54	−0.29	−0.25	−9.52
2	cos	−2.40	−15.69	51.50	−2.18	−2.52	−6.88	2.54	3.00	−1.15	−0.30	8.73
	sin	3.21	−2.50	14.25	3.87	−5.45	−2.61	−1.03	−0.62	−0.53	1.32	−7.51
3	cos	−3.57	−69.60	10.61	5.76	−144.26	−27.83	8.32	−99.07	−1.90	−29.84	16.15
	sin	11.60	−52.10	−9.03	14.16	−48.70	−5.94	−1.53	−1.05	−2.04	0.90	−8.59
4	cos	−5.18	−86.45	−12.70	−26.61	−30.91	−27.98	18.13	−29.23	−4.56	−5.31	27.24
	sin	−15.49	35.18	81.63	−26.14	−17.21	−7.89	−2.07	−1.52	−2.63	−10.65	−10.43
5	cos	−14.73	150.20	30.89	77.06	−165.58	−74.38	29.47	−47.83	−8.01	−5.28	41.41
	sin	−8.22	−29.86	−82.68	208.48	−26.79	−10.86	−2.61	−1.97	−2.92	−6.56	−12.51
6	cos	28.84	−82.56	−40.64	−58.10	−63.80	−70.00	42.06	−65.87	12.62	−5.80	59.32
	sin	−12.02	204.61	−147.17	−97.43	−26.30	−12.76	−3.14	−2.37	−3.21	−4.90	−14.67
7	cos	15.50	73.53	43.15	89.09	−1559.43	−208.91	51.69	−195.92	−31.65	203.08	79.93
	sin	−124.11	−152.88	−861.91	−562.31	−33.11	−15.29	−3.68	−2.80	−3.76	−6.06	−16.91
8	cos	35.24	−207.53	−86.75	−85.27	−97.89	−123.48	68.65	−299.92	−49.84	51.63	104.42
	sin	26.38	240.61	798.28	−146.61	−33.20	−16.99	−4.23	−3.23	−4.43	−8.20	−19.14
9	cos	−265.02	131.34	76.36	129.59	2962.00	−380.32	82.17	2823.83	−200.56	11.52	132.28
	sin	28.02	1044.49	419.34	−1250.68	−42.18	−19.85	−4.80	−3.69	−5.34	−18.84	−21.43
10	cos	−49.86	−199.24	−99.76	−134.12	−160.09	−198.03	110.69	−655.37	−92.59	28.31	163.44
	sin	56.02	193.45	267.80	−389.85	−45.83	−21.95	−5.34	−4.12	−6.13	−79.48	−23.67

where x ranges over 100 equal increments of $p^{2/7}$ from 1000 to 0.01 mb. Also shown are corresponding C values using a sine function in (2). Surprisingly, for these channels and uncertainty levels, the permissible C values can be of meteorologically significant size for even the large vertical wavelengths corresponding to small n. For example, for $n = 1$, a temperature signal of $C \cos[(1)\pi x/(x_T - x_0)]$ would be undetectable in the 716.70 cm channel for C values upward of 104 K! (Such an anomaly would clearly be detectable in the other channels since 104 K exceeds the other critical values in this row of Table 2.) However, according to Table 2, an anomaly of this vertical structure would be undetectable by *any* of the 11 channels if its amplitude were less than 0.49 K, which is of meteorologically significant magnitude. To further illustrate, we picked a set of several particular (C, n) pairs, and generated the profiles $T^*(x)$, a standard atmosphere and $T^*(x) + \sum_n C_n \cos[n\pi x/(x_T - x_0)]$, both of which produce identical sounder radiances in each channel to within the noise values in set 1 of Table 1. The profiles are illustrated in Fig. 1 and have a 10–1000 mb rms difference of 9.51 K. These two solutions are not only mathematically distinct, but meteorologically distinct as well. Unlike the unstable inverse of Wark and Fleming (1966), which could be discarded due to its clearly nonatmospheric character, the two solutions in Fig. 1 have temperature values well within the normal range of atmospheric temperature and neither may be re-

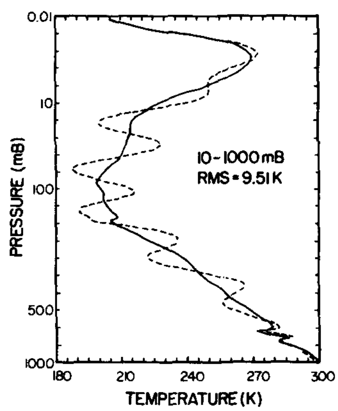

FIG. 1. Two vertical profiles of temperature which produce indistinguishable radiances in 11 CO_2 sounding channels of HIRS operating with about 99% measuring accuracy. The solid curve is a standard atmosphere $T^*(x)$, and the dashed curve is the function $T^*(x) + \sum_n C(n) \cos[n\pi x/(x_T - x_0)]$ for $C(5) = 2$, $C(7) = -8$, $C(8) = 7$.

jected casually, although the wavelike character of the perturbed profile could be considered unlikely.

4. Some very different true atmospheric solutions

The examples in Fig. 1, while atmosphere-like, are not true observed temperature profiles. As a second problem, we searched through a large, heterogeneous library of actual RAOB temperature profiles to find pairs of RAOBs which produce "indistinguishable" radiances. The procedure is considered to be a perfect analog retrieval algorithm here within the representational limits of the RAOB sample, the radiometer noise values and solution criterion used. That is, two different RAOBs producing indistinguishable satellite radiances may be defined as perfect analog retrieval estimates of each other. Note that this definition of perfect retrieval relates to the original, ill-posed sounding problem, free of additional constraints (except that retrievals are thermal profiles that have occurred in nature) and not to a well-posed approximation to the original problem. Note also that this definition of perfect retrieval solution relates to the solution domain of satellite measured *radiances.* Given the Riemann–Lebesgue Lemma, this definition does *not,* by any means, guarantee that such perfect retrievals are even acceptably similar to each other in the atmospheric temperature profile domain.

To obtain results with some practical value, we abandoned the linearization that leads to Eq. (1) and, instead, sought pairs of RAOBs $\{T_k(x), T_l(x)\}$ having the property that the difference in their corresponding sounder radiances $R(\nu_i)$, calculated from the full radiative transfer equation with temperature corrected transmittances, is within instrument noise level. The two sets of noise values in Table 1 were used for independent searches. Noise set 1 is approximately 1% of the band-averaged radiance emanating from a standard atmosphere, while noise set 2 is channel specific values provided by NOAA/NESDIS as representing HIRS instrument capabilities. Noise set 2 represents an upper bound on HIRS accuracy since it does not include contributions due to scene noise, clouds, surface term corrections, etc. The criterion "within instrument noise level" was applied in two different senses. The most restrictive test requires radiance agreement using an infinity norm,

$$\max\left|\frac{R_k(\nu_i)-R_l(\nu_i)}{E(\nu_i)}\right| \leqslant 1 \qquad (3)$$

that is within instrument noise level for each and all sounding frequencies. A less restrictive test uses a unity norm requiring only that

$$(1/11)\sum_{i=1}^{11}\left|\frac{R_k(\nu_i)-R_l(\nu_i)}{E(\nu_i)}\right| \leqslant 1. \qquad (4)$$

The RAOB library used consists of a fairly heterogeneous batch of 1600 RAOBs between 30°S and 60°N, during summer or winter, over ocean or land, assembled by N. Phillips (NOAA/NMC), and interpolated to 65 fixed levels between 0.1 mb and 1000 mb. The search over this library involves 1 279 200 distinct pair-wise tests for each experiment.

Distinct RAOB profiles satisfying one of the solution criteria are each perfect analog retrieval estimates of the others. In our pair-wise tests, two such RAOBs we call a *dissimilar pair* of solutions. Using the infinity-norm criterion [Eq. (3)] with noise set 1, we found only 83 dissimilar pairs out of the total of 1 279 200 distinct pairings. Of these, the most dissimilar pair had a 10–1000 mb rms deviation of 4.82 K, while the least dissimilar pair had 0.77 K deviation. Calculated over the 83 dissimilar pairs, the mean of the 10–1000 mb rms deviation was 1.99 K. Figure 2 shows, on a shifted abscissa, the five most dissimilar pairs for this experiment. The differences are somewhat wavelike, particularly above 700 mb, suggesting the same type of wavelike compensation in the integrations as in the Riemann–Lebesgue Lemma. The discrepancies are generally larger at higher altitudes where there is less information overlap in the HIRS weighting functions used in this study and lower vertical resolving power.

When the solution criterion was relaxed to the unity-norm criterion [Eq. (4)], but using the same noise set 1 values, we found 2369 dissimilar pairs of RAOBs. The most dissimilar pair had 10–1000 mb rms deviation of 7.46 K, and the ensemble mean of 10–1000 mb rms deviations was 2.72 K. Figure 3 shows, on a shifted abscissa, the five most dissimilar pairs for this experiment. The first and third most dissimilar pairs have a profile in common (which has an apparent error near 20 mb), yet they do not form a dissimilar trio.

Figure 4 shows profiles of rms deviations between dissimilar pairs for the ensembles of 83 and 2369 retrievals discussed above. These traces are similar in shape to most retrieval error profiles: larger errors near

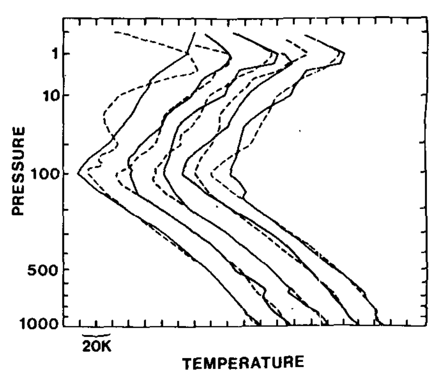

FIG. 2. The five most dissimilar pairs of RAOBs which produce 11-channel HIRS radiances that are indistinguishable in the infinity-norm sense, assuming about 99% measurement accuracy. The 10–1000 mb rms differences between dissimilar pairs are, from left to right, 4.82 K, 4.51 K, 3.68 K, 3.37 K, 3.27 K.

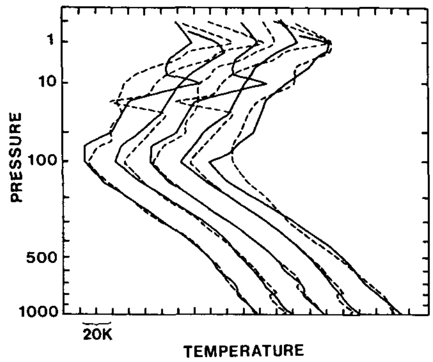

FIG. 3. As in Fig. 2 except from left to right, 7.46 K, 6.49 K, 6.37 K, 6.34 K, 6.11 K.

surface, tropopause height and high levels with intervening local minima.

To relate these experimental results to the Riemann–Lebesgue Lemma, the difference between the most dissimilar pair in the unity-norm experiment (the leftmost pair of curves in Fig. 3) was decomposed into its Fourier representation over x. Table 3 shows the resulting Fourier expansion coefficients for $n = 0, 1, \cdots, 9$ along with three other quantities. The quantity $R(n)$ is the residual 10–1000 mb rms deviation between the solid curve and an approximate Fourier reconstruction of it using the dashed curve plus Fourier components up through and including the value of n shown. The quantity $N_1(n)$ is the corresponding unity-norm of the differences between radiance upwelling from these partial Fourier reconstructions and that upwelling from the solid curve. The quantity $N_2(n)$ is the unity-norm of the radiance difference between the solid curve, and the dashed curve added to *only the single* Fourier component for that value of n. Where these $N_2(n)$ values are greater than 1 (i.e., for $n = 0, 1, 3$), that particular Fourier component would be resolved by the satellite radiometer *if that component existed alone.* However, when Fourier components are simultaneously superimposed, there is a compensation effect within the radiative transfer integral such that the unity norm of the partial Fourier sum [i.e., $N_1(n)$] steadily decreases towards unity as more components are added. A full Fourier expansion exactly reconstructs the solid curve that has a unity-norm radiance difference of 0.99 from the dashed curve. This result means that the Riemann–Lebesgue Lemma, applied channel-by-channel, harmonic-by-harmonic, is not the crucial test of ill-posedness in the satellite sounder problem. Waves with amplitudes which individually exceed Riemann–Lebesgue critical values can be superimposed to provide a complex temperature signal that does not violate an overall solution criterion.

Using noise set 2, we found only eight dissimilar pairs with the unity-norm criterion [Eq. (4)] and only three dissimilar pairs with the infinity-norm criterion [Eq. (3)] out of the total 1 279 200 pair-wise comparisons. We find this result remarkable for it suggests that with state-of-the-art radiometer precision, there may be virtually no *true atmospheric* versions of dissimilar solutions to the satellite sounding problem. A sounder capable of estimating clear column radiances with this very high accuracy appears to be capable of distinguishing virtually all of the RAOBs used in this study.

5. The limits of retrievability

It is interesting to attempt to derive a first approximation to the limits of retrievability from the results of this work. This derivation is somewhat analogous to that for the more familiar numerical weather prediction problem, in which one may establish limits of predictability traceable to the ill-posedness of that problem. Such an exercise has conceptual value in trying to define a performance window for evaluating operational sounding schemes, and should provoke thought about how accurately one can hope to retrieve atmospheric thermal structure by satellite.

Satellite temperature retrieval algorithms might produce arbitrary amounts of retrieval errors, as demonstrated by Table 2, if not properly regularized. Further, such schemes produce numerically generated solutions that are only approximations to what has occurred, or may occur, in nature. We propose to define lower and upper bounds on the accuracy of practical retrieval schemes by assessing the performance of a *zero-order* retrieval method as follows.

Define a *zero-order* satellite temperature retrieval method as one in which the set of *all* known naturally occurring thermal profiles is searched to find those profiles that produced spectral radiances indistinguishable from radiance measurements at hand, to within some observational error limits. This zero-order method thus seeks solutions to the radiative transfer equations, subject only to the single, external constraint

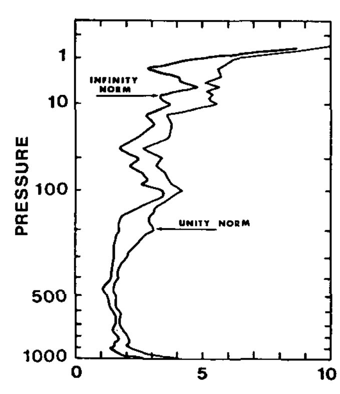

FIG. 4. Profiles of root-mean-square differences, over ensembles of dissimilar pairs, using either infinity-norm agreement or unity-norm agreement with an 11-channel HIRS operating with about 99% measuring accuracy.

that the solutions are known to occur in the atmosphere. This constraint is both the most important and least restrictive constraint that one might impose on a retrieval scheme. When applied to a representative sample of cases, the largest retrieval error produced by the zero-order scheme represents an estimate of the maximum tolerable error for all other useful retrieval schemes, in the sense that this is the *natural atmospheric* upper limit of error attributable to ill-posedness of the original sounding problem. The smallest error obtained by the zero-order method represents an estimate of the minimum tolerable error for all useful retrieval schemes in the sense that any smaller error must relate to approximate solutions, unlike any which have actually occurred in nature.

Calculating limits of retrievability in this manner may not be easy to do since our sample population of 1600 RAOBs, using only 11 HIRS-like channels, produced relatively few zeroth order retrievals. Nevertheless, we use those cases to provide a first estimate of the limits of retrievability. For the 11-channel sounder operating with accuracy represented by noise set 1, a practical retrieval algorithm is suspect if it produces 10–1000 mb rms retrieval error worse than 4.82 K or better than 0.77 K. An acceptable performance level for a retrieval scheme for this instrument would be ensemble averaged errors approaching the ensemble average of the zero-order retrievals, 1.99 K. Using noise set 2, only three zero-order retrievals were obtained for the 1600 cases attempted. This does not permit a rational, quantitative estimate of limits of retrievability for this high precision experiment. It is significant, however, that a sounder which could operationally estimate clear column radiances with this precision could distinguish virtually all the variability in this diverse set of cases, which implies that the limits of retrievability should relate to quite small retrieval error. A reviewer found this result an encouraging rebuttal to criticism of low vertical resolving power of satellite sounders. We agree, but would argue that the ability of a satellite device to discriminate between various thermal fields in a zero-order, forward approach is not the same as the ability of a *particular* retrieval algorithm to resolve vertical structure in the inverse approach. Vertical resolving power involves both the radiometer capabilities and the nature of the retrieval algorithm used to convert radiances into atmospheric properties.

The limitations of these first estimates of retrievability should be emphasized. The results here reflect upon only 11 CO_2 channels of the TOVS system. The variability among zero-order retrievals obtained seems largest at higher altitudes. Omitted TOVS information should be useful at higher altitudes and should lower the limits of retrievability from those given here. It is doubtful that the channels used here could operationally obtain measurements of radiance with the accuracy of noise set 2, owing to uncertainties in the estimation of cloud and moisture corrections, surface temperature and atmospheric transmittances. Furthermore, the set of 1600 RAOBs used here is too small a sample to reliably estimate limits of retrievability for high precision observational systems, since its diversity limits the number of zeroth order retrievals obtained.

6. Conclusions

The ill-posed character of the satellite temperature retrieval problem has been examined in this paper. Whereas ill-posedness in the theoretical sense often involves solutions that can be rejected due to their non atmospheric character, the cases shown here demonstrate ill-posedness in the more practical sense that "very different" temperature profile solutions are sought whose differences are within the range of natural atmospheric variability.

Following theorems of Riemann and Lebesque, examples of temperature profiles that are very different have been constructed mathematically for a linearized model of radiative transfer. Those examples show that two atmosphere-like temperature profiles differing by nearly 10 K rms between 10–1000 mb can easily be constructed, which produce indistinguishable radiances in each of 11 HIRS-like sounding channels operating at about a 1% error level. Presumably, current numerical retrieval algorithms could also produce individual retrievals this much in error, even though convergence to a satellite measurement has been achieved.

An empirical search of a large, heterogeneous sample of real RAOBs was performed seeking pairs of RAOBs that produce satellite radiometer radiance values indistinguishable within some prescribed envelope of error. Radiances for this search were computed with a nonlinear radiative transfer model. For a sounder capable of measurement at about a 1% error level, two temperature profiles drawn at random from the heterogeneous batch had less than 0.2% chance of being indistinguishable by the radiometer in a unity-norm sense, and less than a 0.007% chance of being indistinguishable in an infinity-norm sense. Of those pairs producing indistinguishable radiances for every channel (infinity-norm), the ensemble mean 10–1000 mb rms deviation was 1.99 K with maximum value of 4.82 K. For a HIRS sounder capable of operating with bench-test accuracy, no more than three pairs of RAOBs, of a total of 1 279 200 candidate pairs, were found to produce indistinguishable radiances for every channel.

The empirical tests conducted on atmospheric ill-posedness indicate that modern, high precision sounders, such as TOVS, are capable of distinguishing between naturally occurring thermal fields in a forward problem sense. This does *not* imply, however, that inverse retrieval algorithms adapted to these sounders are automatically capable of resolving the thermal structural differences. Such algorithms, being solutions

TABLE 3. Characteristics of the Fourier decomposition, and partial reconstruction of the difference between the most dissimilar pair of profiles in Fig. 3. (See text for definition of quantities.)

Vertical wavenumber	Cosine coefficient	Sine coefficient	$R(n)$ (K)	$N_1(n)$	$N_2(n)$
0	0.937	—	—	2.12	2.12
1	−0.463	−1.861	7.18	1.33	2.53
2	−3.469	−1.236	6.61	1.96	0.99
3	1.486	−4.094	5.62	2.39	1.99
4	0.266	−0.049	5.32	2.24	0.34
5	0.967	2.492	4.99	1.56	0.69
6	−1.268	1.856	4.62	1.12	0.45
7	−0.922	−0.555	4.48	1.16	0.07
8	0.210	0.656	4.38	1.09	0.10
9	−0.956	0.372	4.34	1.02	0.07

to well-posed, regularized approximations to the sounding problem, have been conditioned to avoid numerical instabilities in the algorithm. Such conditioning may make the radiance measurement information somewhat subservient to a priori statistics, forecast first-guess fields, or mathematical smoothing constraints such that the single solution produced from the infinity of possible solutions to the radiative transfer equations may be undesirably far from the ambient state of the atmosphere.

When the ill-posed satellite retrieval problem is coupled with the ill-posed numerical weather prediction problem—and both are regularized in some fashion—one discovers less impact of the former on the performance of the latter than one might hope, given the vast amounts of new observational information that are added by the satellite radiometers. One may wonder if the regularizing and initializing constraints are removing useful information which exists in the satellite radiance observations.

Earlier, we cited several studies involving an *analog* retrieval procedure in which one selects one or a few RAOBs from an historical database that produce radiances close to a current satellite radiometer measurement. Our present study demonstrates that one might only rarely find a perfect analog by this technique. Yet, the analog retrieval approach is a method of seeking solutions to the ill-posed problem without any subjectively defined external constraints that might drive the solution toward some preconceived atmospheric state. Using a few near analogs in constructing a well-posed model of the retrieval problem might minimize the negative impacts of unrepresentative a priori data and optimize the positive influence of the satellite measurements on any subsequent analysis or forecast.

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