NOTES AND CORRESPONDENCE

Selected Comments on the Use of the Divergence Equation to Obtain Temperature and Geopotentials from an Observed Wind

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ABSTRACT

Recent suggestions to use the full divergence equation and nearly continuous (in time) wind observations to retrieve the geopotential and temperature of large-scale and mesoscale (but still hydrostatic) phenomena are examined. By extrapolating from results obtained in small-scale meteorology the following is suggested: 1) boundary conditions for the normal derivative of temperature (or the geopotential) can be obtained from the wind data itself and 2) objective criteria can be devised to indirectly test the quality of the results and also to decide whether Neumann (i.e., a case in which the normal derivative is specified) or Dirichlet (i.e., the function itself is specified) type boundary conditions should be enforced.

1. Introduction

A few years ago Gal-Chen (1978) and Hane and Scott (1978) noted that if sufficiently accurate measurements of the wind and its time history could be obtained from Doppler radars then this information would, in principle, also define the thermodynamic structure. In essence, this is done by requiring that the data also satisfy the momentum equations in the least-squares sense. This has reduced the problem to a classical calculus of variation problem (Courant and Hilbert, 1953). The form of the momentum equations assumed in these studies is quite general and is in principle applicable to small-, meso- and large-scale atmospheric motions. While not immediately obvious, when the approximations appropriate to large-scale atmospheric flows are employed, the above variational formulation is reduced to solving a classical balance equation (Haltiner and Williams, 1980) for obtaining the geopotential from the wind.

Gal-Chen and Kropfli (1984), Roux et al. (1984) and Hane and Ray (1985) have tested the practical utility of the aforementioned variational formulation on a variety of observed small scale phenomena: Planetary Boundary Layer (PBL) convection in the Gal-Chen and Kropfli (1984) case; severe storms for the Hane and Ray (1985) case and a tropical squall line for the Roux et al. (1984) case. In all three case studies, temperature and pressure are deduced from observed Doppler radars wind. Satisfactory agreement with in situ thermodynamic observations was reported in all three cases.

As Gage and Balsley (1978) pointed out, sensitive Doppler radars can be used to obtain mesoscale wind profiles under all weather conditions. The vertical resolution is up to 100 m. The time resolution is about 1 h and the horizontal resolution is determined by the average distance between the profilers. Comparable resolution is not obtainable from radiometric measurements of the atmosphere either from the ground or from satellites.

This has led several researchers to investigate whether the divergence equation could also be used in this case to obtain the temperature from the wind (Bleck et al., 1984; Kuo and Anthes, 1985). In both of these papers, the solution was sensitive to the accuracy of the imposed geopotential (or Montgomery stream-function) on the boundaries. Furthermore, in both cases the accuracy is judged against model output data used as a control run. Both the control run and the retrieval procedure use the same model. As has been pointed out by Atlas et al. (1985) this procedure may lead to the “identical twin syndrome,” an unrealistically optimistic assessment of errors due to the model’s affinity for its own products. The purpose of this note is: 1) to show that the divergence equation is a least-squares approximation of the horizontal momentum equations; 2) to show that boundary conditions for the solution of the divergence equation can be obtained from the observed wind and that this choice is optimal in some sense; and 3) to point out that when using real data, objective indirect verification criteria may be used in addition to the desirable but difficult direct verification.

2. Modeling assumptions and motivations

To simplify the discussions, a midlatitude primitive equation β-plane approximation in pressure coordinates (Haltiner and Williams, 1980) has been chosen. The reader should bear in mind, however, that the
Conclusions and methodologies are applicable to any other geometry or coordinates. The only modifications needed are the forms of the dynamical equations applicable to the particular geometry. The horizontal momentum equations and the continuity equation are

**Continuity equation:**

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0 \tag{1}
\]

**Horizontal u momentum equation:**

\[
Du/Dr = -\phi/\partial x + F_1 +fv \tag{2}
\]

**Horizontal v momentum equation:**

\[
Dv/Dr = -\phi/\partial y + F_2 -fu. \tag{3}
\]

Here \(D/Dr\) is a symbol for total derivative:

\[
D/Dr = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial z} \tag{4}
\]

\(x\) is the west-east direction, and \(y\) is the north-south direction. The pressure \(p\) is the vertical coordinate, \(\phi\) is the geopotential, \(\nu\) and \(v\) are horizontal velocities, \(\omega\) is \(Dp/Dr\) and \(f\) is the Coriolis parameter \((=2\Omega \sin \Theta)\) with \(\Theta\) the latitude). Furthermore, \(F_1\) and \(F_2\) are symbols for turbulent stress derivative (of dimension Newton/kg) representing scales of motions which cannot be explicitly resolved by the model. In this study we assume that they can be either measured directly by a Doppler radar (Kropfli, 1986) or parameterized from resolvable wind observations, e.g., as in Kuo and Anthes (1985). By differentiating (2) with respect to \(x\) and (3) with respect to \(y\) and adding the results, a divergence equation can be obtained (Haltiner and Williams, 1980, pp. 26–27). For the purpose of this discussion, we write this equation symbolically as

\[-\nabla^2 \phi = H(u, v, \omega, F, \partial u/\partial t, \partial v/\partial t, \cdots) \tag{5}
\]

where the rhs of (5) is a known function of observed variables (wind, vertical velocity, etc.). It is tacitly assumed, as in Kuo and Anthes (1985) that the vertical velocity may be computed diagnostically from the continuity equation (1). Alternatively, one may adopt the approach of Bleck et al. (1984) and simply neglect certain terms in the divergence equation. Among the terms neglected are those dependent on the vertical motion. The net result is the so-called balance equation (Haltiner and Williams, 1980, p. 68). It is apparent from (5) that, at least in principle, knowledge of the wind and its tendency could yield [via solution of (5)] an estimate of \(\phi\) and therefore of the temperature (via the hydrostatic relation), provided that appropriate lateral boundary conditions could be specified.

Relation (5) is the one used by Kuo and Anthes (1985) as the starting point in their discussions of how to retrieve the temperature from the wind. Bleck et al. (1984), on the other hand, add the additional constraint that the calculated geopotential (actually, the Montgomery streamfunction) be as close as possible to the geopotential retrieved using remotely sensed radio-

metric data. In order to satisfy this latter constraint, (5) cannot be enforced to each grid point. Instead it is used as an additional weak constraint, i.e., the calculated geopotential (or Montgomery streamfunction) must satisfy the divergence equations as close as possible. The resulting calculus of variations problem is a fourth-order elliptic differential equation for \(\phi\), as compared to the second-order elliptic equation (5).

Both Bleck et al. (1984) and Kuo and Anthes (1985) found it necessary to specify the temperature at the lateral boundaries, recognizing that this specification might cause some errors. As is mentioned in the introduction, the purpose of this note is to point out that the lateral boundary conditions can actually be specified from the observed wind and the utilization of the appropriate dynamical equations. For convenience, we show (in the next section) how to derive such boundary conditions for the Kuo and Anthes (1985) formulation. Nevertheless, one should bear in mind that a similar exercise could be applied for the Bleck et al. (1984) case.

3. Boundary conditions for the divergence equation; the divergence equation as a least-squares approximation of the horizontal momentum equations

Assuming that the wind and its time tendency is known, the dynamical equation (2) and (3) can be written symbolically as

\[
\frac{\partial \phi}{\partial x} = F \tag{7}
\]

\[
\frac{\partial \phi}{\partial y} = G, \tag{8}
\]

where \(F, G\) are known (specified) functions of position and time. As discussed before, the horizontal divergence of (7) and (8) is the divergence equation (5). At the same time, it is also self-evident that a solution \(\phi\) of the divergence equation (5) does not necessarily satisfy the horizontal momentum equations (7) and (8). A necessary condition for the equivalence of (7) and (8) to (5) is that the wind data also satisfy the compatibility conditions,

\[
\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}. \tag{9}
\]

If the wind measurements were exact and available continuously in time and if, in addition, mesoscale models of the type discussed by Kuo and Anthes (1985) were able to perfectly simulate the motions which are explicitly resolved, then the compatibility condition (9) must be satisfied. In the more practical case (9) is not satisfied, thus (7) and (8) do not have a solution in the usual sense. To overcome these difficulties, Gal-Chen (1978) and Hane and Scott (1978) proposed a least-squares solution of (7) and (8) by solving the variational problem

\[
\int \int (\frac{\partial \phi}{\partial x} - F)^2 + (\frac{\partial \phi}{\partial y} - G)^2 = \text{min}. \tag{10}
\]

The solution of the above variational problem is a
Poisson equation for $\phi$ identical to the divergence equation (5), i.e.,
\[ \nabla^2 \phi = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}. \] (11)

The calculus of variation approach suggests that two types of boundary condition may be employed to solve (11), namely:

1) Dirichlet type boundary conditions, which require specifying $\phi$ on the boundary. This reduces the problem to the one treated by Kuo and Anthes (1985) and makes it similar to that of Bleck et al. (1984).

2) Neumann type boundary conditions whereby the normal derivative of the geopotential is obtained from the observed wind data, viz.,
\[ (\frac{\partial \phi}{\partial x})n_x + (\frac{\partial \phi}{\partial y})n_y = F n_x + G n_y. \] (12)

Here $n_x$ and $n_y$ are the cosines of the angles that the normal to the boundary is making with the $x$ and $y$ axes, respectively. For a rectangular domain (12) is reduced to the more familiar condition
\[ \frac{\partial \phi}{\partial x} = F; \quad \text{on} \quad x = \text{const. boundary} \quad (13a) \]
\[ \frac{\partial \phi}{\partial y} = G; \quad \text{on} \quad y = \text{const. boundary}. \quad (13b) \]

The boundary condition (12) reduces the mathematics to the one treated by Gal-Chen and Kropfli (1984), Roux et al. (1984) and Hane and Ray (1985). It must be noted that when Neumann-type boundary conditions are used, the resulting $\phi$ are unique at each horizontal plane only to within an arbitrary constant. This arbitrariness can be removed in two ways; either by subtracting the horizontal average, thus obtaining the geopotential fluctuations, or by obtaining at one point in each horizontal plane an absolute measurement of the temperature (hence, an absolute measurement of the geopotential). This latter procedure would define the geopotential field uniquely.

4. Approaches to verification

In the previous section, two different approaches to specifying the boundary conditions for the divergence equation were discussed. Both approaches are well posed and lead (in some sense) to an optimal approximation of the horizontal equations of motion. The choice of which to use in practice is not an easy one and depends on a variety of factors such as

1) The accuracy of the imposed boundary conditions and the sensitivity of the results in the interior of the domain to these boundary conditions.

2) Modeling uncertainties which would cause the real atmosphere not to satisfy the divergence equation (5).

3) Random and systematic errors in estimating the wind, thus, in estimating the rhs of (11).

While simulation studies can be quite revealing in addressing the above issues, Atlas et al. (1985) have pointed out that the results obtained from such simulations usually underestimate the actual errors quite substantially. This points out to the importance of experiment with real data to find out whether the geopotentials (or temperature) retrieved by the use of the divergence equation are correct. While direct verifications are obviously the most desirable and must be pursued, the interpretation is not easy, largely because the radar-retrieved geopotentials represent some averaging in space and time while the radiosonde measurements of temperature are point measurements. Taking these reservations into account, Gal-Chen and Kropfli devised an indirect method of verification which they have dubbed “momentum checking.” They have applied this technique to check the quality of pressure retrievals from dual Doppler radar observations of winds in the PBL. Roux et al. (1984) and Hane and Ray (1985) applied it to other small-scale phenomena. It is suggested that the same technique may be applied in the mesoscale case. In particular, it could be used as guidance in the decision whether to use Neumann or Dirichlet type boundary conditions.

In essence, the momentum-checking criteria involves calculating a quantity $E_r$, defined by
\[ E_r = \frac{\iint \left[ \left( \frac{\partial \phi}{\partial x} - F \right)^2 + \left( \frac{\partial \phi}{\partial y} - G \right)^2 \right] \iint \left( F^2 + G^2 \right) \]. (14)

As mentioned previously, only under error-free conditions would the deduced geopotentials from (11) exactly balance (7) and (8) and thus yield $E_r = 0$ in (14). On the other hand, Gal-Chen and Kropfli (1984) showed that if the observations were dominated by a random noise, one would get $E_r = 0.5$. Based on some comparisons with in situ measurements, they have suggested that when $E_r > 0.3$ the results are suspect; retrievals with $E_r > 0.5$ are virtually useless. Perhaps the same criteria could be applied in the mesoscale case, but it must be kept in mind that these criteria are only one element of the important issue of verification.

5. Summary and discussions

The full divergence equation has been applied quite extensively to retrieve pressure from Doppler radar observations of a variety of small-scale phenomena. In using that technique researchers have been obliged to use the Neumann-type boundary conditions since no pressure data is generally available on the boundaries. In the mesoscale case, there are two plausible ways to specify the boundary conditions for the geopotential. One is to specify the geopotential, e.g., by utilizing predictions from some larger scale models or from vertically crude first-guess satellite retrievals. The other option is to specify the geopotential normal derivative. This latter boundary condition can be deduced directly
from the wind observations. At any rate, the use of the criteria (14) could help one to decide objectively which boundary conditions to use and also to determine the general quality of the retrieved geopotentials.

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REFERENCES


