Detection and Estimation of Radar Reflectivity from Weak Echo of Precipitation in Dual-Polarized Weather Radars

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ABSTRACT

In operational weather radar, precipitation echoes are often weak when compared to the underlying noise. Coherence properties of dual polarization can be used for enhancing the detection and for the improved estimation of weak echoes of precipitation. The enhanced detectability results from utilizing coherent averages of precipitation signals, while the uncorrelated noise vanishes asymptotically, explicit in the off-diagonal element $R_{hv}$ of the echo covariance matrix. In finite sums, the noise terms as well as the uncertainties associated with them are suppressed. A signal can be detected in weaker echo by an analytically derived censoring policy. The coherent sums are readily available as the cross-correlation function of the antenna voltages $H$ and $V$, which estimates $R_{hv}$ in the mode of simultaneous transmission and reception. The magnitude of $R_{hv}$ is a consistent estimate of the copolar echo power, leading to the copolar radar reflectivity of precipitation, which refers to the geometric mean of the reflectivities in H and V polarizations. Because of the intrinsic noise suppression, estimates of the copolar reflectivity are, in relative terms, more precise and more accurate than the corresponding estimates of reflectivity in specific channels, for weak signals of precipitation. These aspects are discussed quantitatively with validation of the key features in real conditions. The advances suggest for dedicated dual-polarization surveillance scans of weak echo of precipitation.

1. Introduction

Echo power is a basic observable of weather radar, from which the equivalent radar reflectivity is derived. Echo power can be defined as the variance of the received voltages, and in Doppler radar it can be obtained as the zero-lag element of the autocorrelation function of the complex voltages. The radar range equation of volume precipitation associates the power of the precipitation echo (signal) at a specific range to be proportional to the radar reflectivity $\eta$, which is the volumetric backscatter cross section of hydrometeors. Subsequently, this is used to estimate the equivalent reflectivity factor of precipitation $Z_{e}$, which in Rayleigh approximation is the sixth moment of the drop size distribution in the measurement volume.

Often, precipitation echo is weak, and the presence of noise makes the observation difficult. A weak signal power is usually estimated by subtracting the mean noise power, known at a finite accuracy, from the total measured power. The limit of detection is set by the overall uncertainty in the noise power. In weather radar literature, it is customary to normalize the signal powers to a power of the bandwidth limited noise [signal-to-noise ratio (SNR) = 1 or 0 dB], called the “minimum detectable signal,” because SNR = 1 is a good first estimate of the signal detection limit. In details, detection is the process of identifying the echo as signal in the presence of noise, or simply as noise (no signal). The procedure consists of the echo estimator and censoring (thresholds), constructed on the models of noise and signal. Operationally, precipitation signals are rendered as fields of radar reflectivity, which is a function of echo power. This

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requires that all the detected signals are associated with estimates of their respective values of power.

Dual-polarization Doppler weather radar provides further information as samples of the complex voltages are acquired in two orthogonal polarizations, horizontal (H) and vertical (V). Signal correlation in sample time (Doppler coherence) allows for estimating the mean radial velocities and the spectral widths (Doviak and Zrnić 1993), while the power ratios and relative phases of the correlated signals in the H and V polarizations relate to the microphysics of precipitation (Bringi and Chandrasekar 2001). These additional types of coherence in range and in time differentiate the precipitation signal from other components in the received voltages, and the features have been used in several contexts in literature and in operations. They enable identification of the precipitation signal and improved estimation of its parameters in presence of other components in the data (Chandrasekar et al. 2008, 2013). The splitting of the transmitted power in two polarization channels in the mode of simultaneous transmission and reception (STAR) has been viewed as a specific trade-off. This “3-dB loss” can be mitigated by modifying the legacy censoring on SNR in the H channel to consider the auto- and cross-correlations in the H and V channels, gate by gate, while taking the estimates of power and reflectivity unchanged (Ivić 2009). The various combinations of censoring have been analyzed in Ivić et al. (2014).

The impacts of noise in estimating dual-polarization parameters and the uses of sample time coherence for improved accuracy in the parameter estimation have been topics of continued study (Seminario et al. 2001; Melnikov and Zrnić 2007; Lei et al. 2012). A complementary approach is to improve the accuracy of noise through momentary estimates (Ivić et al. 2013), followed by steps of noise subtraction or corrections. The limit of precision in momentary estimates is set by the number of noise samples available, while the limit of accuracy may relate with the purity of noise in the samples. We note that the measurement bias due subtraction can be mitigated by the improved noise estimates, while both the variances and the biases are mitigated in approach of intrinsic noise suppression.

The objective of this paper is to consider the copolar correlations of dual polarization in range and in time for detection of the precipitation signal, and for estimation of the weak signal power, improving from the methodology of single polarization radars (Keränen and Chandrasekar 2011; Keränen et al. 2012). The echo from precipitation volumes is often weak (SNR < 3 dB) when received from far distances or when the reflectivity factor is small. We recognize the operational values of weather radar spanning from detailed, precise, and versatile information of Doppler and dual polarization obtained at moderate ranges, up to coarse observations at maximal ranges, where the primary objective is to determine the existence of precipitation, followed by the first estimates of echo power with relatively large uncertainties, with no reliable estimates of other moments. Nevertheless, the observations of reflectivity alone are sufficient inputs to key products of operational weather radar. This paper presents a novel method to extend the operational range of detecting and estimating the echo power of weak signal and expressing it as an estimator of reflectivity, by utilizing the dual-polarization properties of precipitation. The consistency of the estimator and the legacy power estimates builds on the microphysical properties of precipitation.

Our paper is organized as follows. Dual-polarization echoes are modeled and estimators of copolar echo power are described in section 2. Section 3 presents the methodology of weak echo detection and its application to the coherent copolar echo. In section 4, the estimates of the radar reflectivity of weak echoes of precipitation are characterized quantitatively in terms of variances (precision) and of biases (accuracy) with emphasis on low signal-to-noise ratios. Mutual consistency of the various reflectivity estimates is considered. Examples of operational applications are presented in section 5. The key results of the paper are summarized in section 6.

2. Dual-polarized radar echoes of precipitation

Dual-polarization weather radar considers the echo from scatterers in the resolution volume illuminated by the transmitted pulses. The received radar echoes are acquired as series of $M$ complex antenna voltages $H$ and $V$ sampled in the orthogonal horizontal and vertical polarization states, respectively. Generally, the information contained in these time series can be expressed as the covariance matrix $\mathbf{K}$ of the expectation values of all the combinations of products of $H$ and $V$, including those lagged in time. The characteristics of the precipitation signal can be expressed as the covariance matrix $\mathbf{C}$ of the feature vector of the backscattering properties, modified by the effects of signal propagation in the atmosphere (Bringi and Chandrasekar 2001). The relation between the matrices $\mathbf{K}$ and $\mathbf{C}$ can be understood as a generalization of the scalar relation between the received power $P$ and radar reflectivity $\eta$.

The copolar power $P_{co}^H = \langle |H|^2 \rangle$ is the lag-0 autocorrelation of the sampled voltage $H$ (and analogously for
$P_{\text{co}}^H$; $P_{\text{co}}^V$ can be associated with the mean squares of the backscattering amplitudes of the droplets in the resolution volume, or the radar reflectivities $\eta_{h,v}$ in the corresponding polarization $H$ or $V$. The complex copolar correlation at lag-0 $R_{\text{co}} = (H^*V)$ refers to the cross-correlation of voltages $H$ and $V$. Note that $P_{\text{co}}^H$ and $P_{\text{co}}^V$ are directly observed by transmitting separately in the respective polarization $H$ or $V$. The cross-correlation $R_{\text{co}}$ can be inferred from nearly coincident $H$ and $V$ samples in pulses transmitted that alternate in $H$ and $V$—subject to corrections due to variable Doppler correlations. Alternatively, one can estimate $R_{\text{co}}$ from $H$ and $V$ voltages acquired in coincidence in the STAR mode—subject to impacts of cross-polarization terms in backscattering and in propagation, not omitting the potential instrumental effects primarily in the antenna, which can be managed to a low level in state-of-the-art operational radars. We consider here the STAR mode (Brunkow et al. 2000; Doviak et al. 2000), as it is used in many operational radars. The STAR radar observes the hybrid version of the covariance matrix $\mathbf{R}$ (Bringi and Chandrasekar 2001),

$$\mathbf{R} = \begin{bmatrix} R_{hh} & R_{hv}^* \\ R_{hv} & R_{vv}^* \end{bmatrix},$$

in which $R_{hv} = [R_{hh}R_{vv}]^{1/2} \rho_{\text{co}}$ and $\rho_{\text{co}} = |\rho_{\text{co}}| e^{j \Phi_{\text{co}}}$, for $H$ and $V$ voltages acquired in coincidence. The elements of $\mathbf{R}$ associate with those of the coherency signal matrix reduced from the full covariance matrix $\mathbf{C}$. The matrix elements $R_{hh}$ and $R_{vv}$ can be used to estimate the copolar reflectivities $\eta_{h,v}$. The observed differential reflectivity $\varepsilon_{\text{DR}}$ and the copolar correlation coefficient with the magnitude $|\rho_{\text{co}}|$ and with the differential phase $\Phi_{\text{co}}$ relate with the corresponding microphysical properties of hydrometeors, too. The approximations made in interpreting the $\mathbf{R}$ elements as “copolar” have been quantified (Wang et al. 2006).

The concept of the “coherent copolar echo” relates to the complex element $R_{hv}$. The coherence builds on two persistent dual-polarization features of precipitation, characteristics in the $\mathbf{C}$ matrix and transferred to the properties of $\mathbf{R}$: 1) the distribution of the relative phase of the backscattering amplitudes is narrow for scatterers in each measurement volume ($|\rho_{\text{co}}| \approx 1$), and 2) the real parts of the effective wavenumbers of forward scattering are nearly equal in $H$ and $V$; that is, the mean differential phase $\Phi_{\text{co}}$ evolves slowly in range, in particular in weak and modest precipitation and at lower radar frequencies such as S and C bands. Both features 1 and 2 allow summing the products of $H^*V$ as a phase-coherent copolar echo, where the summation may span substantial intervals in sample time and in range time. The principal upper limit of summing in the sample times is set by the requirement of stationary precipitation. Operationally, a limit is the total measurement time that can be afforded. In range, the differential phase can be taken locally constant $\Phi_{\text{DP}}$ through intervals of many kilometers—excluding the most intense cores of rain at higher frequencies. The resultant of the summed $H^*V$ has the magnitude

$$P_{\text{co}} = |R_{hv}| = |R_{hh}R_{vv}|^{1/2} |\rho_{\text{co}}|,$$

where $|\rho_{\text{co}}| \approx 1$ accounts for the copolar correlation in each gate as well as for any slow evolution of $\Phi_{\text{DP}}$. Essentially, $P_{\text{co}}$ is equal to the geometric mean $[R_{hh}R_{vv}]^{1/2} = [P_hP_v]^{1/2}$ of the copolar echo powers $P_h$ and $P_v$ in locally homogenous precipitation.

Features 1 and 2 are specific to precipitation echo, which thus differentiates from the uncorrelated noises $n_h$ and $n_v$ in the horizontal and vertical channels, respectively. In the presence of precipitation signal (3) and additive noise $(n_h, n_v)$, $\mathbf{R}$ can be expressed as

$$\mathbf{R} = \begin{bmatrix} H^S + n_h \\ V^S + n_v \end{bmatrix}^*[H^S + n_h \\ V^S + n_v] = \begin{bmatrix} R_{hh}^S + \overline{N}_h \\ R_{hv}^S + \overline{N}_v \end{bmatrix},$$

where $\overline{N}_h$ and $\overline{N}_v$ in the diagonal terms refer to the mean noise powers in the respective $H$ and $V$ channels, while the off-diagonal element $R_{hv}$ is composed of signal only. We notice that noise terms are absent in the off-diagonal element $R_{hv}$.

**Estimators of coherent copolar echo from finite sets of voltage samples**

Using voltage vectors $\mathbf{H}$ and $\mathbf{V}$ of length $M$, acquired in the STAR mode, the conventional echo power estimates are given as $P_h = M^{-1} \sum |H_i|^2$ and $P_v = M^{-1} \sum |V_i|^2$. Similarly, the copolar echo can be directly estimated as the magnitude of the copolar covariance:

$$\hat{P}_{\text{co}} = M^{-1} \sum_i H_i^*V_i.$$

The summation is carried out through intervals of sample times and range times. An alternative way to
estimate magnitude of $\langle H^* V \rangle$ can be constructed as $P_{co} = M^{-1} \text{Re} \left[ \sum \text{H}^* V \right] \exp (\phi \mathcal{F}_\text{DP})$ in which the locally constant differential phase $\mathcal{F}_\text{DP}$ is used as additional information (constraint) in order to project the magnitude in the real component. For brevity, this paper discusses the properties of the estimate in Eq. (4), while the methodology is generally applicable to the constrained estimate in which the expected value of noise vanishes, too. The additional information of $\mathcal{F}_\text{DP}$ allows constructing mixed estimators of copolar autocorrelation and cross-correlation functions (Melnikov et al. 2011), in which thermal noise $\nbar_h$ and $\nbar_v$ are present.

Using the radar range equation, the radar effective reflectivities $Z_{h,v} = C_{h,v}^{-1} P_{h,v} - \nbar_{h,v}$ are estimated at each range $r$, in the specific polarizations H and V; $C_{h,v}$ are the constants of system calibration in H and V polarizations. Equation (2) leads us to an analogous estimate of copolar reflectivity $Z_{co}$, with a simple relation with $Z_{h}$ and $Z_{v}$ for precipitation echo, as follows:

$$Z_{co} = C_{co}^{-1} r^2 P_{co} \approx \left[ Z_{h} Z_{v} \right]^{1/2}. \tag{5}$$

The copolar reflectivity is the geometric mean of the reflectivities in the horizontal and vertical polarizations. The calibration constant $C_{co}$ can be obtained from the calibration constants $C_{h}$ and $C_{v}$ simply as follows:

$$C_{co} = (C_{h} C_{v})^{1/2}. \tag{6}$$

### 3. Detecting precipitation in the presence of noise

Detection and estimation of a weak signal of precipitation depends on the statistics of the echo components and of their measurement uncertainties. We first characterize the detection of copolar echo by deriving analytical expressions and by carrying out numerical evaluations with experimental validation of the key features, as detailed in the appendix. The numerical methods are applied in section 4, where we consider the copolar echo as an estimate of reflectivity of precipitation.

#### a. Signal detection from radar echo in presence of noise

Formally, signal detection (Skolnik 1990; McDonough and Whalen 1995) is a procedure that considers observations with a method of decision (“censoring policy” or “thresholds”). The observations are labeled either as signal (in the presence of noise) or no signal. The performance is characterized by detectability, which is the weakest signal detected at a prescribed probability of detection (POD) in the presence of noise. A censoring policy defines thresholds, set to an acceptable false alarm rate (FAR) in which signal is reported in the absence of signal.

In single polarization radar, the power $\hat{P}_h$ is observed and the precipitation signal is estimated as $\hat{P}_h = \rho_h - \nbar_h$, obtained from $M$ samples. The noise power $\nbar_h$ is most easily estimated from independently observed samples. The threshold $T_h$ can be applied on the power ratio $P_h/\nbar_h$. The procedure is chosen for simple consideration of its uncertainty. Statistically, FAR$_h$ is the integral of the signal tail of the probability density function (pdf) of observations from noise. Similarly, POD$_h$ is the integral of the coexisting signal and noise declared as signal. Ideally, the noise power $\nbar_h$ is an accurate estimate of the true value $\nbar_h$. The threshold setting of $T_h$ can be solved from the integral of the pdf of power computed from $M$ samples of noise:

$$\text{FAR}_h = \int_{T_h}^{\infty} \text{pdf}[P_h | \text{noise}(\nbar_h)] dP_h. \tag{7}$$

Similarly, the minimum detectable signal $R_{hh}^{S_{\text{min}}}$ can be solved from the equation of pdfs for $M$ samples of signal and noise corresponding to the prescribed value of POD$_h$ resulting from the just obtained $T_h$:

$$\text{POD}_h = \int_{T_h}^{\infty} \text{pdf}[P_h | \text{precipitation}(R_{hh}^{S_{\text{min}}}) \text{ and noise}(\nbar_h)] dP_h. \tag{8}$$

Combining Eqs. (7) and (8), we have the detection process for an accurately known mean noise power, as illustrated in Fig. 1a. Ideally, very small signals can be detected as small excesses on top of the noise at the limit of very large number of samples, where the sampling variance $\text{Var}(P_h) = M^{-1} \hat{P}_h^2$ becomes vanishingly small.
In reality, the noise power $N_h$ is estimated with an uncertainty $\delta N_h$. The pdfs in Eqs. (7) and (8) are modified by the uncertainty $\delta N_h$, which dominates the widths of pdfs instead of the sampling variances. In practice, good data quality is maintained by setting the threshold $T_h$ conservatively high enough to maintain the desired FAR$_h$ even in the cases when the actual noise power is temporarily higher than estimated. This procedure of the realistic threshold $T_h$ with a safety margin can be modeled by resolving Eq. (7), with the noise scaled up by its uncertainty:

$$\text{FAR}_h = \int_{T_h}^{\infty} \text{pdf}[P_h | \text{noise}(\overline{N}_h + \delta N_h)] dP_h.$$  \hspace{1cm} (9)

The realistic minimal detectable signal $R_{hh}^\text{min}$ is obtained by considering the threshold with a margin and resolving Eq. (9). The value of $R_{hh}^\text{min}$ becomes higher than the ideal, in particular at large number of samples. This is illustrated in Fig. 1b.

**b. Detection with coherent copolar echo power**

Detection with copolar echo can be carried out analogously to the case of $P_h$ echo. Moreover, the censoring policies can be matched for equal image quality (i.e., for the same rate of false echo due noise) for ease of use. We consider the observation of $\check{P}_c$ from which the signal is estimated as the copolar element $|R_{hh}^\text{cop}|$. Note that $\check{P}_c$ is censored by a threshold $T_c$ at FAR$_c$ that is required to be equal to FAR$_h$ resulting from $T_h$. Ideally, $T_c$ can be solved from

$$\text{FAR}_c = \int_{T_c}^{\infty} (P_c | \text{noise}(\overline{N}_h, \overline{N}_v)) dP_c = \text{FAR}_h.$$  \hspace{1cm} (10)

In the realistic description of detection, we account for the uncertainties $\delta N_h$ and $\delta N_v$, and we solve Eq. (10) for $T_c$ with the scaled noises, analogous to Eq. (9). For a given common probability of detection POD$_c$ = POD$_h$, the minimum detectable signals of $|R_{hh}^\text{min}|$ can be solved analogously to Eq. (8),

$$\text{POD}_c = \int_{T_c}^{\infty} \text{pdf}(P_c | \text{precipitation} R_{hh}^\text{min}$ and noises $\overline{N}_h, \overline{N}_v) dP_c.$$  \hspace{1cm} (11)

for the ideal threshold, and for the threshold with a margin that accounts for the realistic description of noises.

The detection processes of $\check{P}_c$ with the ideal and the realistic noise models are illustrated in Figs. 2a and 2b, respectively. Given uncorrelated nature of noise, pdf($\check{P}_c$) is distributed at low values of $\check{P}_c$ allowing low settings of $T_c$. Most importantly, the impact of noise uncertainty is suppressed (i.e., the setting of $T_c$ does not change too much when a margin of uncertainty is allowed in the noise powers). It will be quantified next how the performance of the coherent copolar echo estimator $\check{P}_c$ differs substantially from that of $P_h$ in detecting signal in presence of noise. The detection performances of $P_h$ and $\check{P}_c$ can be evaluated in a comparable basis.
The appendix describes the statistics of the received voltages acquired in dual polarization. The pdf of the residual noise in copolar echo can be expressed analytically in terms of the statistical moments. The expectation value and the variance are derived in particular, for practical uses. These key outcomes are validated at actual radar available for operational uses. Using the same principles, a model of dual-polarization signal describes the essential features of precipitation.

c. Minimum detectable precipitation signal in the coherent copolar echo

The characterizations of the noise and the precipitation signal, described in the appendix, allow evaluating the minimal detectable signals of precipitation in the copolar coherent estimator $P_{co}$ and in the estimator $P_{hs}$, as follows. We evaluate $|R_{hs}^{S\min}|$ in $P_{co}$ as a function of the number of samples $M$ in a large span, at POD = 50% for the prescribed FARs in the ranges from $10^{-3}$ up to $10^{-5}$. The lower value corresponds to negligible rates of raw speckle in radar sweeps at a constant elevation [plan position indicator (PPI)] at 1° resolution in azimuth and 1000 gates in each radial—typical in operational radars. The higher value is applicable to processing with additional speckle filtering, such as in Vaisala (2013). We have repeated the computations with $P_{hs}$ as a cross-check with the reference evaluation (Skolnik 1990).

We compute the thresholds $T_{co}$ and $T_{hs}$ corresponding to the specified FAR$_{co} = $ FAR$_{hs}$ from pdfs parameterized by the noise levels $N_{hs}$ and $N_{hs}^\prime$ known within uncertainties of $\delta N_{hs}$. We compute the ideal thresholds as well as the thresholds with margins due to the uncertainties $\delta N_{hs}$ of 2 dB that roughly characterize the observed contemporary variations in noise (Seminario et al. 2001; Melnikov and Zrnić 2007). We then compute $|R_{hs}^{S\min}|$ and $h_{hs}^{S\min}$ from pdfs of dual-polarization complex voltages composed of precipitation signal and noise.
The minimum detectable signals \(|R_{hh}^h|_{\text{min}}\) and \(|R_{hh}^y|_{\text{min}}\) are visualized for \(\text{FAR} = 10^{-3}\) in Fig. 3a for the ideal case of accurately known noises as well as with the quoted margins of noise uncertainty. Figure 3b represents the analogous outcomes for \(\text{FAR} = 10^{-3}\). The detectabilities of \(P_h\) level off at values close to 0 dB because of noise uncertainty, while the detectability curves of the copolar echo \(P_{co}\) preserve the downward trends of \(-M^{-1/2}\) even when noises are known with an uncertainty. At the noise uncertainty of 2 dB, the relative advantage of \(P_{co}\) with respect to \(P_h\) reaches and exceeds 10 dB for thousands of samples and beyond. If the mean noise powers were known accurately, detectabilities of \(P_h\) would be better than those of \(P_{co}\) by about 2 dB. These outcomes are consistent with the independent evaluations in Ivić et al. (2014).

The curves of \(|R_{hh}^h|_{\text{min}}\) and \(|R_{hh}^y|_{\text{min}}\) are normalized to \(N_h\) and \(N_y\), respectively. These normalizations allow for interpreting the cases where \(N_h\) differs from \(N_y\), in which \(|R_{hh}^h|_{\text{min}}\) readings are to be scaled by \((N_y/N_h)^{1/2}\) for comparison to \(|R_{hh}^y|_{\text{min}}\). The curves are computed for \(z_{DR}^h = 1\), while the minimal copolar signals corresponding to \(R_{hh}^y_{\text{min}}\) at an arbitrary \(z_{DR}^y\) can be obtained by rescaling the readings of \(|R_{hh}^y|_{\text{min}}\) by \((z_{DR}^y)^{-1/2}\). The factors of \(|\rho_{co}^h| < 1\) and of nonvanishing \(K_{DP}^h\) within the measurement volume may have impacts on detectability, too. The impacts on detectability are generally modest. For fair PODs around 50%, they are approximately equal to the biases in signal estimation, quantified to be at the level of 1 dB in specific cases, typically less than 1 dB (see section 4). The impacts of finite Doppler spectral widths of the signal are not significant in detectability, which, in general terms, is a convolution signal from low PODs to high PODs where the effects of spectral width tend to cancel—our choices of POD = 50% and the limit of large spectral limit are representative, in this sense. Furthermore, detectability may be affected by the effects of propagation induced by precipitation in the signal path prior to the measurement volume. Attenuation may grow large (tens of decibels) at higher radar frequencies, while it is present at S band, too. It is typically stronger in the H polarization plane, and the conditions may favor signal being detectable in \(P_{co}\) echo when differential attenuation is significant. There are other significant factors in far distance precipitation, such as beam overshooting, which are common to all radar-based estimates. Beam overshooting is a specific case of partial beam filling, which is a potential cause of reduced correlation in copolar echo. In high-quality dual-polarization antennas, the main lobe patterns can be arranged to match well (Moisseev et al. 2010), which maintains the high copolar correlations in partially filled measurement volumes.

4. Copolar echo as measure of radar reflectivity of precipitation

When the copolar echo \(P_{co}\) is taken as the estimate of the precipitation signal, we are concerned about the variances due to finite number of samples \(M\) (relating with precision) as well as about potential systematic errors, or biases (relating with accuracy), in varied conditions of precipitation. In particular, the performance at and below the legacy detection limit of \(R_{hh}^y\) (SNR = 1) is of interest (see Fig. 3). Given the linear relation between the signal power estimates and the corresponding estimates of reflectivities \(\eta_X\), the estimates of relative variances and of systematic errors in signal directly translate into the properties of \(\eta_X\), where \(X\) refers to \(h\) or \(co\). At the level of reflectivity estimates, we are interested about the relation between the copolar reflectivity \(\eta_{co}\) obtained from \(P_{co}\) and the reflectivities \(\eta_{h}\) and \(\eta_{v}\) obtained from \(P_{h}\) and \(P_{v}\) (relating with consistency). The basic aspects of precision, accuracy, and consistency are discussed briefly in the following subsections.

At all SNRs, the calibration constant \(C_X\) is a known technical source of systematic uncertainty in estimates of reflectivity. From Eq. (6) we find out how \(\mathcal{C}_{co}\) can be obtained from estimates of \(\mathcal{C}_h\) and \(\mathcal{C}_v\). In practice, a more simple approach may be to combine \(\mathcal{C}_h\) with the relative gain \(p_{DRK}\) obtained at a more accuracy in a procedure for the estimates of differential reflectivity. We get \(\mathcal{C}_{co} = \mathcal{C}_h^{1/2}p_{DRK}\), and the uncertainties in the calibration of \(\eta_{co}\) effectively reduce close to those of \(\eta_h\). Alternatively, one can apply Eq. (6) and consider the uncertainties in \(\mathcal{C}_h\) and \(\mathcal{C}_v\) to result from arbitrarily correlated random factors. In case both channels are calibrated at the same general quality (same uncertainty) and the uncertainty in \(\mathcal{C}_{co}\) is that of \(\mathcal{C}_h\) or better.

a. Precision of the copolar echo power estimates in presence of noise

We evaluate the variances of the copolar precipitation echo estimates obtained from a finite number of \(M\) samples and at finite SNR (and equivalently of the copolar reflectivity), using the Monte Carlo description of the precipitation signal in presence of noise, analogously to the evaluation of detectability. We express the results as relative standard deviations \(1 + \text{STD}(|R_{hh}^y|)/|R_{hh}^y|\) of the signal \(|R_{hh}^y|\).
estimated as the \( \bar{P}_c \) echo and compare them with the relative standard deviations of the signal \( R_{\text{hh,min}}^S \) in the \( \bar{P}_h \) echo as a function of \( \text{SNR}_{\text{hh}} \), in the ranges where signals are detectable at a significant POD. The normalizations are made with respect to the true signal, which we consider objective. We notice that \( \text{STD}(R^S_{\text{HH,min}}) / R^S_{\text{HH}} \) are independent of \( z^S_{DR} \); that is, outcomes obtained at \( z^S_{DR} = 1 \) apply to any \( z^S_{DR} \) at a given \( (R^S_{\text{HH,min}}, R^S_{\text{HH}})^{1/2} \).

The outcomes are displayed in Fig. 4, where we find that the sampling variations of the \( R^S_{\text{HH,min}} \) signal in copolar echo improve from that of the signal in the legacy echo in the region of low SNR down to the limit of detection. Generally, noise dominates the variances of weak echo. This is illustrated by the curves computed with noise powers increased by 2 dB. The results are consistent with the analytical expressions of the variance of \( \bar{P}_c \) for noise only [see Eq. (A9) in the appendix]. For larger numbers of samples, the “new” signals detected at \( \text{SNR}_h < 0 \text{ dB} \) are measured at precision of 1 dB, which is a typical operational target.

**b. Accuracy of the copolar echo power estimates in presence of noise**

We evaluate the systematic errors (biases) in the Monte Carlo approach applied earlier in the study of detectability. We express the results as the relative bias \( \text{BIAS}(\bar{R}) = R^S / \bar{R}^S \) where \( \bar{R}^S \) is the estimate of the true value \( \bar{R} \), which allows easy comparisons of different power estimators.

A scale of the accuracy of weak power estimates can be inferred from the properties of signals \( R^S_{\text{HH,min}} \) estimated in weak \( \bar{P}_h \) echo. As known, the \( \bar{R}^S_{\text{HH}} \) estimates tend to get biased in the step of noise subtraction as soon as the actual mean noise power deviates from the power estimate \( N_h \). The subtraction bias becomes relevant and grows fast for the weakest signals, while the trend is truncated by the limits of detection, as illustrated in Fig. 5. Using \( M = 64 \) samples as an example, variations in the actual noise power levels by 2 dB introduce relative biases in \( \bar{R}^S_{\text{HH}} \) up to 1.5 dB, at the limit of detecting signals at POD\(_h = 50\% \). Biases of 1 dB are typical for operational numbers of samples, while they would be significantly bigger if larger \( M \) were used.

Apart from the intrinsic estimator biases, censoring modifies the statistics of the detected weak signals in any estimation method by removing the fraction of pdf\((\bar{P}_X|\text{signal and noise})\) that falls below the threshold. Thus, weak signals detected at a small POD tend to be biased, even if the power estimator is intrinsically accurate. The censoring bias is about 1 dB (2 dB) for signals of \( R^S_{\text{HH,min}} \) detected at POD\(_h = 50\% \) (POD\(_h = 25\% \)) in the \( \bar{P}_h \) echo in a large range of number of samples. The \( \bar{P}_c \) echo behaves similarly to...
a function of POD$_\text{co}$, which is higher than POD$_h$ for a given SNR$_h$, however.

We notice the direct estimate of the copolar echo $\hat{P}_\text{co}$ [Eq. (4)] is positive definite, also for the input of noise. This suggests that a residual component of noise power is present in a very weak signal, when the signal estimate is chosen as $|R_{\text{co}}|^2 = P_\text{co}$ (i.e., no subtraction applied). In Fig. 5 we see that the residual term is below 0.5 dB in the domain of “new” signals detected in the $\hat{P}_\text{co}$ echo from $M = 64$ samples and remains low with high numbers of samples such as $M = 4096$, down to the detection limit of $|R_{\text{co}}|^\text{min}$. The bias is about 0.6 dB at the detection limit of 0 dB for $M = 16$, which is distinctly lower than the bias in $R_{\text{hh}}$ for the same signal strength. For simplicity, we choose not to apply any correction on top of the intrinsic noise suppression.

We can summarize the evaluations of variances and of systematic errors, as follows. The copolar echo estimator $\hat{P}_\text{co}$ is a reliable measure of the precipitation echo in the significant domain of SNR$_h < 0$ dB, undetectable in the legacy echo $\hat{P}_h$. In addition, the non-negligible relative biases and variances of the estimator $R_{\text{hh}}$ at SNR $\approx 0$ dB are reduced in the $\hat{P}_\text{co}$ estimator, which thus may improve estimates of those signals of precipitation.

c. Consistency of the copolar echo as an estimator of radar reflectivity

We are further interested about the relation between the copolar reflectivity estimates $Z_{\text{co}}$ obtained from the $\hat{P}_\text{co}$ estimator and the reflectivities of $Z_{h,v}$ obtained from $\hat{P}_{h,v}$ (relating with consistency). As indicated in Eq. (2), the three quantities $Z_{\text{co}}$, $\hat{Z}_h$, and $\hat{Z}_v$ are closely related. Moreover, in the conditions of light rain (characterized by $|\rho_{\text{co}}^S| \approx 1$, $\hat{Z}_{\text{DR}} = 1$, and $k_{\text{DP}} = 0$) $Z_{\text{co}}$, $\hat{Z}_h$, and $\hat{Z}_v$ become equivalent for each sample. In the general case of precipitation, their close relations are well understood and can be quantified through a study of variations of SNR, $|\rho_{\text{co}}^S|$, $\hat{Z}_{\text{DR}}$, and $k_{\text{DP}}$ in their natural ranges. In the following subsections, we characterize the key dependencies on SNR and $|\rho_{\text{co}}^S| < 1$ while the full evaluation of the space spanned by the parameters $|\rho_{\text{co}}^S|$, $\hat{Z}_{\text{DR}}$, and $k_{\text{DP}}$ is beyond the scope of this paper.

As a guideline, we consider the impacts of $|\rho_{\text{co}}^S| < 1$ in a few relevant nontrivial types of precipitation. The copolar echo power $\hat{P}_\text{co}$ is proportional to $|\rho_{\text{co}}^S|$, and hence the relative bias is obtained by converting the linear values of $|\rho_{\text{co}}^S|$ into units of decibels. In intense rain, the values of $|\rho_{\text{co}}^S|$ are known to keep above 0.95, which translates into less than $-0.2$-dB bias in $\hat{P}_\text{co}$. In mixed phases (melting layer, melted large drops, bright band), $|\rho_{\text{co}}^S|$ is in the range 0.7–0.9, which translates into a bias of $-1.5$ to $-0.5$ dB. In dry hail, $|\rho_{\text{co}}^S|$ is above 0.85 (i.e., a bias of $-0.7$ dB). It may grow more significant for wet hail, while wet hail is a known source of positive bias in $Z_h$ if interpreted as an estimate of rainfall intensity. These characteristics are weakly dependent on radar frequency. In general, we find modest biases due to $|\rho_{\text{co}}^S| < 1$ compared to the known uncertainties in interpretations of reflectivity, such as the $R(Z)$ rainfall relations.

Generally, the relation $Z_{\text{co}} = Z_h|\rho_{\text{co}}^S|Z_{\text{dr}}^{1/2}$ is the comprehensive guide to interpretations and operational uses of $Z_{\text{co}}$. It defines a simple mapping between $Z_h$ and $Z_{\text{co}}$, in the mean sense as well as in gate-by-gate estimates at sufficient SNR, where $|\rho_{\text{co}}^S|$, $\hat{Z}_{\text{DR}}$, and $k_{\text{DP}}$ are observed at sufficient accuracy.

5. Experimental evaluations

We have evaluated the coherent copolar echo power estimator in varied conditions for meteorological applications. Several applications of the enhanced detection can be envisaged, such as shallow winter precipitation that is often weak in echo power. In warm seasons, early detection of significant weather systems at far distances is one of the primary tasks of weather radar. In events of intense rain, additional margins of detectability may be desired for maintaining the coverage of observations subject to large rain-induced attenuation, especially for higher-frequency radars.

The estimates of the radar reflectivity $\eta_h$ and of the copolar reflectivity $\eta_{\text{co}}$ are available as standard features of IRIS/RDA signal processing (Vaisala 2013) in the Vaisala WRM200 dual-polarization weather radar. Continuous data streams through varied seasons are available from the development site in Kerava, Finland, at high latitudes. Similarly, data streams are acquired at an operational site of WRM200 near Belo Horizonte, Brazil, including events of intense rain in subtropical climate (Keränen et al. 2012).

We have selected a typical event of large-scale winter precipitation observed by the Kerava radar. Fields of reflectivities from the copolar echo $P_{\text{co}}$ and the legacy $P_h$ echo are computed in a volume scan covering ranges exceptionally up to 400 km. At the lowest elevation of 0.5°, the moments are summed from $256 \times 16 = 4096$ samples in time and in range spanning 4 km in depth (250-m gate spacing, pulse width of 2 $\mu$s), followed by four sweeps at higher elevations using gradually lower numbers of samples at finer radial resolutions, down to 150 m ($M = 64$, pulse width of 1 $\mu$s). The total volume time is less than 4 min, meeting typical operational needs. The low-elevation echoes of $P_h$ are censored at FAR $= 2 \times 10^{-3}$ with
a margin of uncertainty of 1 dB in the noise power estimates. The modest margin is sufficient to maintain FAR in cool season (small variations in thermal due to precipitation), and in this type of radar system, throughout the seasonal observing period. Using Eq. (9), the threshold of \( T_h/N_h = 1.2 \text{ dB} \) is obtained. This threshold is applied to \( P_h \) and it corresponds to censoring SNR\(_h\) at \(-5.0 \text{ dB}\). The threshold for the \( P_{co} \) echo is computed from Eq. (10) with the same FAR and at the same safety margin as in censoring \( P_h \). The \( P_{co} \) echoes are censored by the threshold of \( T_{co}/(N_h/N_v)^{1/2} = -13.9 \text{ dB} \). Both \( N_h \) and \( N_v \) are estimated from samples of sky noise at high elevations, obtained in the annual maintenance procedures. The reference value of FAR = \( 2 \times 10^{-3} \) accounts for the speckle filtering applied equivalently for both types of echo, leading to FAR = \( 3 \times 10^{-5} \) in reported data fields. The thresholds would be marginally higher if the FAR level was realized without speckle filters, equivalently for both types of echo. With these settings the detectability of \( P_{co} \) is enhanced by 9 dB with respect to that of \( P_h \).

In Fig. 6, the data at ranges less than 64 km are projected to the constant altitude of 900 m (CAPPI) by using the high SNR data at higher elevations computed at a high resolution in range. The data at distances larger than 64 km are displayed as the direct sweep data (PPI) computed from 4096 samples at a resolution that is comparable in azimuth and in range. The coherent copolar echo visibly improves the detection of weak signals at distances of 200 km and beyond, in comparison to the legacy echo. The enhanced detections are validated by the observations from seven regional weather radars covering the area, displayed as the concurrent hourly composite of reflectivities as a public service of the Finnish Meteorological Institute.

In Fig. 7, we display a different type of case, observed by the Kerava radar in the warm season. A system of convective precipitation developed in the south in the period of 6 h, sampled with the enhanced
Fig. 7. An example of the enhanced capability of coherent copolar echo $\hat{P}_{co}$ to detect a convective weather system at far distances. The red rings locate the region of the weather system. Shown are (left) the reflectivity fields, estimated from the $\hat{P}_{co}$ echo, and (right) the reflectivity fields from the $P_{h}$ echo using the same antenna voltages as input: (top) the first persistent $\hat{P}_{co}$ signal observed; (middle) $\hat{P}_{co}$ signal, no $P_{h}$ signal; and (bottom) the first persistent signal in $P_{h}$ seen about 5 h later. The data are displayed at the elevation of $0.5^\circ$. The processing settings are described in text.
surveillance scans every 15 min. The processing settings and the censoring specifications are the same as in the cool season case, except that a variability of 1.8 dB is accounted for in noise powers throughout the warm season. The echoes of \( P_h \) and of \( P_{co} \) are censored at \( T_h/N_h = 2.0 \text{ dB} \) (corresponding to an SNR\(_h\) threshold of \( -2.3 \text{ dB} \)) and \( T_{co}/(N_{co}N_{co})^{1/2} = -12.8 \text{ dB} \), respectively. The data from the lowest elevation of 0.5\(^{\circ}\) are displayed (PPI). Precipitation is unambiguously detected as the copolar echo \( P_{co} \) at 0115 UTC at the maximum range of 400 km, followed by continuous observations of the system moving northward. When the same antenna voltages are used to compute the legacy \( P_h \) echo, the system is detected 5 h later at 0615 UTC at the approximate range of 300 km. Such advances in lead time and in range of detection are significant in events of major weather systems.

6. Conclusions

We have developed a coherent copolar estimate of precipitation echo in dual polarization and characterized it with the objective of improved detection (section 3) and of estimation of weak signal of precipitation (section 4). Given the microphysical properties of precipitation (section 2), the coherent copolar echoes can be summed in range and in time for detecting and estimating precipitation at low signal-to-noise ratios that were previously not observed (Fig. 3). Thousands of samples can be obtained in operationally affordable scan times in dedicated dual-polarization surveillance scans by considering radar observation volumes that become more symmetric in width and in depth at far ranges.

Using such inputs, the dual-polarization weather radar is able to detect precipitation at SNR\(_h\) below \(-10 \text{ dB} \), which compensates and significantly exceeds the intrinsic loss of 3 dB of the STAR mode radar, when compared to radar transmitting in a single polarization. For numbers of samples such as 16 to 64, typically used for obtaining observations at a high temporal and spatial resolution (in timely sweep times and no summing in range), the coherent copolar echo restores the intrinsic loss of 3 dB in the precipitation echo, when common uncertainties in noise are accounted for.

The precision (i.e., the variances due to the finite number of samples of these weak echo powers) meets typical operational needs (Fig. 4). The biases are modest and quantifiable (Fig. 5), including the step of interpreting \( Z_{co} \) as an estimate of the radar reflectivity factor. We conclude that the copolar echo estimator \( P_{co} \) is a reliable measure of the precipitation echo in the significant domain of SNR < 0 dB, undetectable in the legacy echo \( P_h \).

Operational realizations of the coherent copolar echo are straightforward, given the real-time censoring at the equivalent image quality of the copolar and conventional echo, and given the calibration of copolar echo using the existing calibration methodology of dual-polarization radars (section 5). The copolar coherent echo estimates can be used as input to copolar radar reflectivity, leading to improved detectability, precision (variances at finite \( M \)), and accuracy (residual biases at low SNR) of estimates of weak signals of precipitation. This kind of enhanced potential introduces uses of dual polarization in the weather surveillance scans with significant operational impacts. In our field evaluations, the lead times of detecting far distance weather systems improved by hours.

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APPENDIX

Statistics of Dual-Polarization Echoes

The statistical properties of dual-polarization radar antenna voltage vectors \( \mathbf{U} \) can be described within the model of complex Gaussian, which leads to the multivariate complex Gaussian probability density function (Bringi and Chandrasekar 2001, and references therein)

\[
\text{pdf}(\mathbf{U}) = \frac{1}{\pi^{|\mathbf{K}|}} \exp[-\mathbf{U}^* \mathbf{K}^{-1} \mathbf{U}], \tag{A1}
\]

where \( \mathbf{U} \) refers to the vector of complex return signals and \( q \) is the dimension of \( \mathbf{U} \); also \( \mathbf{K} \) is the covariance matrix of observations, and \( |\mathbf{K}| \) is its determinant. In the STAR mode, it suffices to consider the subspace of the matrix \( \mathbf{K} \) consisting of the correlations between all the coincident and collocated pairs of \( \{U_{ij}, U_{ij}\} \), \( U_{ij} = H_{ij} \) or \( V_{ij} \) taken at sample times \( T_i \) and at range times \( \tau_j \).

In this subspace, precipitation signals in each polarization channel are known to be correlated to a variable degree in sample times (Doppler spectral widths), while the additive noise can be modeled to be uncorrelated in \( T_i \) and \( T_j \) (\( i \neq j \)). For the sake of robustness, this method does not utilize the Doppler coherence of the signal—the censoring is based on the persistent dual-polarization coherences, only. We notice that the noise terms dominate the variances of
auto- and cross-correlations, at low SNR (Melnikov and Zrnić 2007; see also Eq. (A4)). The detectability of very weak signals is then generally immune to variable Doppler widths of the precipitation signal. Separately, the voltages acquired in range times, \( \tau_i \) and \( \tau_j \) \((i \neq j)\) become correlated when the sampled range time differences are short compared to the transmitted pulse length or to the inverse of the receiver matched filter bandwidth. These correlations can be accounted for in range time by applying the standard text book descriptions (Doviak and Zrnić 1993). Given the fact that the range–time correlations are the same for the signal and for the additive noise in each channel, the range–time parameterizations apply to all SNRs.

\( \text{a. Residual noise in copolar echo} \)

For noise, the covariance matrix \( \mathbf{K} \) is diagonal, and the density function \( \text{pdf}(\mathbf{U}) \) simplifies to a product of the pdfs of individual voltages that are uniformly distributed in phase and Rayleigh distributed in amplitude

\[
[\hat{P}_{\text{co}}(\text{noise})]_k = \frac{4(N_h N_v)^{k/2}}{\Gamma(M)M^k 2^M k + 1} \sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(M - m + 1/2) \Gamma(M + k - m + 1/2)}{\Gamma(m + 1) \Gamma(M - m + 1/2)} 2^{-m}.
\]

Equation (A3) is numerically unstable and has been found to be not computable for large \( M \) (appendix D in Ivić 2009). For arbitrarily large \( M \), we find it convenient to express \( [\hat{P}_{\text{co}}(\text{noise})]_k \) \((k = 1, 2)\) as Taylor expansions of the moments of \( H \) and \( V \), as outlined in the following.

\( \text{b. Expansions of the expected values and variances of residual copolar noise} \)

We calculate the expectation values and variances of the echo estimator \( \hat{P}_{\text{co}} \) in Eq. (4) for \( M \) samples of noise by considering it as a random variable that is a function of amplitudes and phases of the complex voltages \( U_i \), \( h_i \) or \( V_j \), \( v_j \), \( i = 1 \ldots M \), which are assumed to represent a homogenous set, parameterized by the mean value of noise powers \( N_h \) and \( N_v \), which are constants. The power can be expressed as

\[
\hat{P}_{\text{co}} = M^{-1} \left[ \sum_{i=1}^{M} h_i^2 v_i^2 + 2 \sum_{i=1}^{M} \sum_{j<i} h_i h_j v_i v_j \cos(\Delta \Phi_{\text{dp}}^{ij}) \right],
\]

where the voltage amplitudes \( (h_i, v_j) \) are Rayleigh distributed, and the phases \( (\phi_i, \vartheta_i) \) are uniformly distributed and \( \Delta \Phi_{\text{dp}}^{ij} = \vartheta_i - \vartheta_j + \phi_i \). The expectation value \( \langle \hat{P}_{\text{co}} \rangle \) is

\[
\langle \hat{P}_{\text{co}} \rangle = M^{-1} \prod_{\phi_i} \prod_{\vartheta_i} \prod_{\phi_M} \prod_{\vartheta_M} \hat{P}_{\text{co}}(M)
\times \text{pdf}(h_1, \ldots, h_M, v_1, \ldots, v_M, \phi_1, \ldots, \phi_M, \vartheta_1, \ldots, \vartheta_M)
\times dh_1 \cdots dh_M dv_1 \cdots dv_M d\phi_1 \cdots d\phi_M d\vartheta_1 \cdots d\vartheta_M,
\]

where \( \text{pdf}(h_1, \ldots, h_M, v_1, \ldots, v_M, \phi_1, \ldots, \phi_M, \vartheta_1, \ldots, \vartheta_M) \) is the probability density function of noise, Eq. (A4). We follow a standard recipe in statistics and expand Eq. (A5) as Taylor series for statistically independent random variables. The periodicity of the uniformly distributed phases is a challenge in this approach. The complication can be overcome by considering the \( M(M - 1) \) terms of \( \cos(\Delta \Phi_{\text{dp}}^{ij}) \) as the independent random variables corresponding to the phases. The moments need to be known only. They are easily calculable, \( \langle \cos(\Delta \Phi_{\text{dp}}^{ij}) \rangle = 0 \) for example.

It is straightforward although somewhat laborious to calculate the partial derivatives and the moments of

\( \text{pdf}(\mathbf{U} | \text{noise}) \)

\[
= \frac{1}{\pi^{M} \prod_{i=0}^{M-1} h_i \exp \left(-\frac{h_i^2}{2N_h} \right)} \prod_{i=0}^{M-1} v_i \exp \left(-\frac{v_i^2}{2N_v} \right),
\]

\[
\mathbf{U} = [h_0 e^{i \phi_0}, \ldots, h_{M-1} e^{i \phi_{M-1}}, v_0 e^{i \theta_0}, \ldots, v_{M-1} e^{i \theta_{M-1}}],
\]
amplitudes. We obtain for the expectation value \( \langle \hat{P}_{co} \rangle \) the following expression in which the nonvanishing terms up to second order have been included, followed by the general form of the \( n \)th-order terms:

\[
\langle \hat{P}_{co} \rangle = M^{-1} \left\{ M \left( \frac{\pi}{4} \right)^2 N_h N_v \right\}^{1/2} + \frac{1}{2} \left[ M \frac{M-1}{M^{3/2}} \left( \frac{N_u}{N_h} \right)^{1/2} \times \left(1 - \frac{\pi}{4}\right) N_h \right] \langle h - \langle h \rangle \rangle^2 + \left[ M \frac{M-1}{M^{3/2}} \left( \frac{N_h}{N_v} \right)^{1/2} \times \left(1 - \frac{\pi}{4}\right) N_v \right] \langle v - \langle v \rangle \rangle^2 - M(M-1) \frac{1}{4M^{3/2}} \left( \frac{\pi}{4} \right) (N_h N_v)^{1/2} \\
\langle \cos^2(\Delta \Phi_{dp}) \rangle = 1/2
\]

\[
\begin{align*}
\hat{P}_{co} &= M^{-1} \left\{ M \left( \frac{\pi}{4} \right)^2 N_h N_v \right\}^{1/2} + \frac{1}{2} \left[ M \frac{M-1}{M^{3/2}} \left( \frac{N_u}{N_h} \right)^{1/2} \times \left(1 - \frac{\pi}{4}\right) N_h \right] \langle h - \langle h \rangle \rangle^2 + \left[ M \frac{M-1}{M^{3/2}} \left( \frac{N_h}{N_v} \right)^{1/2} \times \left(1 - \frac{\pi}{4}\right) N_v \right] \langle v - \langle v \rangle \rangle^2 - M(M-1) \frac{1}{4M^{3/2}} \left( \frac{\pi}{4} \right) (N_h N_v)^{1/2} \\
&\vdots
\end{align*}
\]

The nonvanishing terms of the expansion of \( \langle \hat{P}_{co} \rangle \) can be arranged as series of rational expressions of \( M \), with a common multiplicative factor \( M^{-1/2}(N_h N_v)^{1/2} \). In the second moments we obtain

\[
\langle \hat{P}_{co} \rangle \approx \left\{ \frac{N_h N_v}{M} \right\}^{1/2} \left\{ \frac{\pi}{4} + \frac{M-1}{M} \left(1 - \frac{9\pi}{8} \right) \right\}. \quad (A8)
\]

The variance of \( \hat{P}_{co} \) is by definition \( \text{Var} \{ \hat{P}_{co} \} = \langle (\hat{P}_{co})^2 \rangle - \langle \hat{P}_{co} \rangle^2 \). The first term equals \( \langle |R_{in}|^2 \rangle = (N_h N_v)/M \), as derived in textbooks. Combining this with the estimate of \( \langle \hat{P}_{co} \rangle \) from Eq. (A8), we obtain

\[
\text{Var} \{ \hat{P}_{co} \} \approx \frac{N_h N_v}{M} \left\{ 1 - \left[ \frac{\pi}{4} + \frac{M-1}{M} \left(1 - \frac{9\pi}{8} \right) \right]^2 \right\}. \quad (A9)
\]

In the lowest orders of the Taylor expansion we obtain the expectation value and the variance to be

\[
\langle P_{co}(\text{noise}) \rangle \approx \left\{ \frac{N_h N_v}{M} \right\}^{1/2} \left[ \frac{\pi}{4} + \frac{M-1}{M} \left(1 - \frac{9\pi}{32} \right) \right] \approx 0.902 \left\{ \frac{N_h N_v}{M} \right\}^{1/2} \quad \text{for large } M,
\]

\[
\text{Var}[P_{co}(\text{noise})] \approx \frac{N_h N_v}{M} \left\{ 1 - \left[ \frac{\pi}{4} + \frac{M-1}{M} \left(1 - \frac{9\pi}{32} \right) \right]^2 \right\} \approx 0.187 \frac{N_h N_v}{M} \quad \text{for large } M. \quad (A10)
\]
which are proportional to $M^{-1/2}$ and $M^{-1}$, respectively, followed by constants and terms proportional to negative power of $M$, which approach zero at large $M$. This pattern is repeated in higher orders, rendering the expansion stable at all $M$.

The expressions show that the residual noise term of $\bar{P}_h$ has a distribution narrower than the corresponding distributions of $\bar{P}_h$ or $\bar{P}_i$. Its moments can be expressed as functions of $M$ and of the geometric mean noise power $N_{hh} = (N_h N_i)^{1/2}$. This is the characteristics of FAR$_{co}$, too. For a fixed FAR$_{co}$, the threshold $T_{co}$ can be thus expressed as a function of $M^{-1}N_{hh}$ where $N_{hh} = (N_h N_i)^{1/2}$, instead of a parameterization in a multidimensional space spanned by $(M, N_h, N_i)$. This outcome simplifies the considerations and the practical uses of the copolar echo.

c. Experimental validation of the copolar noise suppression in radar systems

We validated the theoretical results obtained in the previous section at the Vaisala WRK200 weather radar at University of Helsinki, Finland (Moisseev et al. 2010). Noise samples up to $M = 4096$ were acquired from the sky (i.e., at a high elevation in fair weather conditions) as well as from the sun as a cross-check of an elevated levels of thermal noise. The data were processed through the signal processing software IRIS/RDA (Vaisala 2013) upgraded to report the echo power estimates $\bar{P}_{co}$. Figure A1 displays the distributions of $\bar{P}_{co}$ and $\bar{P}_h$ from a large set of rays of $M = 256$ samples, with markers that indicate the expectations from Eq. (A10). We find good agreements of the observed mean noise and their variance with respect to the analytical estimates.

We repeated the analyses of Fig. A1 for the distributions of noise estimated from samples of $M = 8, 32, 256$, and 4096. The means of the estimates $\bar{P}_{co}$ are displayed in Fig. A2, in which the data results are accompanied with trends of the mean residual noise $\langle \bar{P}_{co} \rangle$ obtained from the analytical Taylor expansion, Eq. (A10) and from the factorial sum in Eq. (A3). The Monte Carlo integrations of Eq. (A2) are overlaid in Fig. A2, too. Analogously, the data results and the theoretical estimates for the variances $\text{Var}(\bar{P}_{co})$ are displayed as standard deviations in Fig. A3. All these results are in good agreement, which demonstrates the validity of the concept in actual radar systems.

d. Model of precipitation signal in copolar echo

For evaluations of POD$_{co}$ and of the minimum detectable signals $|R_{hh}^S|_{\text{min}}$, we continue working on the $H$ and $V$ voltages acquired in the STAR mode. The contributions from precipitation (additive to noise) can be modeled to be highly correlated $|\rho_{hi}^S| \approx 1$. The statistics of Eq. (A1) reduces to the product of the pdfs of the signal in a specific channel, $H$ for example,

$$\text{pdf}(U|\rho_{hi}^S = 1) = \frac{1}{\pi M} \prod_{i=0}^{M-1} \frac{h_i}{N_h} \exp\left(-\frac{h_i^2}{R_{hh}^S}\right),$$

$U = [h_0 e^{i\phi_0}, \ldots, h_{M-1} e^{i\phi_{M-1}}]$,  

$V = (z_{DR}^S)^{-1/2} [h_0 e^{i\phi_0}, \ldots, h_{M-1} e^{i\phi_{M-1}}]$,  

(A11)

in which $z_{DR}^S$ is the differential reflectivity of the precipitation signal.

We explore the properties of POD$_{co}$ and $|R_{hh}^S|_{\text{min}}$ in the limit $|\rho_{hi}^S| = 1$ by varying $z_{DR}^S$ as a parameter in numerical simulation of Eq. (A11). The outcome is analogous to the evaluations of FAR$_{co}$ and $T_{co}$. Both POD$_{co}$ and $|R_{hh}^S|_{\text{min}}$ can be expressed as functions of the geometric mean $(R_{hh}^S)^{1/2}$, very precisely for $M > 4$. Earlier, we found that the threshold $T_{co}$ of the copolar echo can be constructed as a one-dimensional function of the ratio SNR$_{co} = \bar{P}_{co}/(N_h N_i)^{1/2}$. Here, the threshold is found to yield constant $|R_{hh}^S|_{\text{min}}$ for all combinations of $(R_{hh}^S)^{1/2}$, $R_{hi}^S$, $N_h$, and $N_i$, that refer to same $(R_{hh}^S, R_{hi}^S)^{1/2}$, $N_h, N_i$). This outcome simplifies the practical uses of the copolar echo, because the same $T_{co}$ applies for different instances of $z_{DR}^S$. The outcome is consistent with the expectation that the legacy echo of
Ph has a prior detection advantage (disadvantage) factor of \((z_{DR})^{1/2}\) with respect to \(P_{co}\) when \(z_{DR}\) is higher (lower) than unity.

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