A New Analysis for the Retrieval of Three-Dimensional Mesoscale Wind Fields from Multiple Doppler Radar

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ABSTRACT

The present study is devoted to a new analysis of the Doppler information from multiple Doppler radar scanings aimed at studying the three-dimensional wind field at mesoscale. This analysis referred to as MANDOP (for Multiple Analytical Doppler) is mathematically described. An application of the method to simulated and to real data is also presented in the paper, including a comparison with VAD and DVAD analyses and with observations from radiosoundings and sodar. The method consists in expanding the three wind components in terms of orthonormal functions. Physical constraints, such as the boundary condition at ground level and the continuity equation, are included in the analysis as variational constraints in order to improve the vertical component retrieval. Solving the resulting linear system provides the coefficients for the analytical expansion of the wind. Thus, the wind, as well as its derivatives and associated physical parameters (e.g., vorticity, pressure, and temperature perturbations) may be expressed with an analytical form.

The data of any number of radars (at least two) may be analyzed simultaneously without reformulating the problem. When only one radar is available, the coefficients retrieved without ambiguity are provided by the method.

Application of MANDOP analysis to real data requires that the time-induced advection problem is solved. This aspect is also addressed in the paper. The solution of this problem benefits from the analytical formulation of the wind.

1. Introduction

The rapid developments, in the last ten years, of numerical models, theoretical research, and new experimental tools (Doppler radar, etc.) have permitted an identification of the mesoscale organization (~100 km) of precipitation in convective systems. The crucial role played by the associated mesoscale circulation on the evolution of larger scale structures through energy transfer and on convective motions through moisture supply has been suggested in recent studies. For example, tropical studies have shown that trailing regions of stratiform cloud associated with squall lines play an important part not only by contributing to the global water budget of the squall line (Camache and Houze 1983; Chong et al. 1987; and others), but also because they are implied in the feedback mechanism responsible for the squall line regeneration (Zipser 1977; Smull and Houze 1985). Similarly, studies on kinematical and thermodynamical structures at the rear of fronts in stratiform precipitation have shown the important role of mesoscale circulations in rainfall formation (Harrell 1973; Testud et al. 1980), by showing for example the possible role of specific mechanisms such as the conditional symmetric instability (Bennetts and Hoskins 1979; Seltzer et al. 1985; Lemaître and Testud 1988; Lemaître et al. 1989). However, the study of the dynamical processes leading to the mesoscale organization of precipitation has been difficult, in particular, because of the limited number of experimental tools allowing a detailed description of the associated mesoscale circulation.

In the present paper, particular emphasis is given to mesoscale studies (a few hundreds of kilometers), and in particular to mesoscale frontal rainband studies, which are still very scarce (Browning et al. 1973; Heymsfield 1979; Houze et al. 1976, 1981; Wang et al. 1983), by proposing a new analysis of data from several Doppler radars in order to retrieve the three-dimensional mesoscale wind field. Nevertheless, this analysis may be applied to convective scale studies, as shown later. Matejka and Srivastava (1982, hereafter referred to as MS82) proposed a promising technique based on a functional development of the horizontal components of the wind, thus allowing an analytical representation of them, of their space derivatives, and other related parameters. In the present paper, we propose a new analysis based on a variational concept. This analysis, which includes physical constraints on the wind field, generalizes MS82 technique to the vertical component of the wind in an attempt to describe the three-dimensional dynamical characteristics of
stratiform as well as of convective precipitations. The present paper describes this paper and is referred to as MANDOP in the following (for Multiple ANalytical DOPpler). Moreover, the paper also illustrates an application to simulated and real data in order to perform an experimental test of the analysis accuracy.

In section 2, we give the mathematical formulation of the MANDOP analysis with special emphasis on its analytical and variational aspects. The analysis of real data calls for the solution of advection effects induced by nonsimultaneous measurements. This problem is also addressed in section 2. An application to simulated and real data given in section 3 shows the adequacy of the present analytical variational approach to retrieve mesoscale circulations. Also included in section 3 is the main outline of the analysis, which shows, in particular, that data of any number of radars may be retained in the analysis in spite of some restrictions. Section 4 lists possible extensions resulting from the analytical form of the wind field to the retrieval of air parcel and particle trajectories and to the thermodynamic perturbations fields.

2. The MANDOP analysis

a. Principle

As pointed out in the Introduction, the MANDOP analysis is a three-dimensional extension of the MS82 analysis. It consists in expressing each of the three wind components as a product of three expansions in terms of orthonormal functions series (each of these expansions depending upon each spatial coordinate). The radial wind is also expressed in this analytical form. In order to retrieve the coefficients of the development, the analytical form of radial winds is variationally adjusted to the observed ones including the anelastic continuity equation and a lower kinematic boundary condition simultaneously satisfied by the analytical form of the three wind components. The MANDOP analysis allows the three components of the wind to be analytically obtained, under constraints. By comparison, in the MS82 analysis, the vertical component, first assumed to be negligible in the horizontal components' processing, is then obtained by integration of the continuity equation. A first advantage of the present method, compared to MS82, is that there is no limitation (except the one associated with the cutoff wavelength of the method) to stratiform precipitation conditions since no assumption has to be made on the vertical component of the wind. Moreover, as shown later, numerical errors linked to the classical integration of the continuity equation are minimized by the present approach. Note that integration from some top altitude would also minimize these errors but would necessitate accurate determination of the vertical velocity at this top altitude. As a consequence, a second advantage is that horizontal components are obtained without contamination by the vertical one. Two remarks are appropriate at this stage. First, it must be noted that both methods require an estimate of the terminal fall velocity. Second, in both methods, physical parameters related to the wind components such as vorticity, pressure, and temperature perturbations may also be expressed analytically. Therefore a better accuracy of the retrieved winds also improves the accuracy of the related retrieved parameters (a third advantage of the MANDOP analysis).

In the following subsections, we describe the mathematical formulation of this analysis, by successively examining (i) the role of the different terms of this variational problem, namely the wind adjustment (subsection 2b), and the additional constraints (2c and 2d) (i.e., the continuity equation of the wind and the boundary condition at ground level), or any other additional constraint (2e), and (ii) the correction for advection (2f).

b. Analytical representation of the wind

1) ANALYTICAL FORMULATION OF THE RADIAL WIND

All notations used in the following are defined in appendix A. The basic hypothesis of this MANDOP analysis is that the variations of the wind components, \( U_1, U_2 \) and \( U_3 \), with respect to each coordinate, may be written as a product of three functions of each coordinate (or reduced coordinate). For example, the analytical form \( V_i \) of the component \( U_i \) may be expressed as

\[
V_i = \prod_{i=1}^{3} f_{ij}(x_i)
\]

with

\[
f_{ij}(x_i) = \sum_{k=1}^{n_i} a_{ijk} F_{lj}(x_l)
\]

where \( a_{ijk} \) are the development coefficients. The base of the orthonormal functions \( F_{lj} \) and the order of expansion \( n_i \) of the component \( i \) on the corresponding \( X_i \) axis are chosen in order to best represent the observations. Combining and rearranging Eqs. (1) and (2) lead to the following analytical form of the three wind components \( V_i \):

\[
V_i = \sum_{k=1}^{N_i} b_{kj} g_{ij}(x_1, x_2, x_3)
\]

with

\[
N_i = n_1 n_2 n_3
\]

and \( b_{kj} \) is a product of three coefficients: \( a_{ijk} \), \( a_{klm} \), \( a_{ijk} \). Coefficients \( k', k, k \) and \( k'' \) range from 1 to \( n_1 \), \( n_2 \) and \( n_3 \), respectively, and \( K_i \) is given by the equation:

\[
K_i - 1 = (k' - 1)n_2n_3 + (k'' - 1)n_3 + k'' - 1;
\]

(5)
$g_{K_i}$ is the product of the three corresponding orthonormal functions:

$$g_{K_i}(x_1, x_2, x_3) = F_{1iK_i}(x_1)F_{2iK_2}(x_2)F_{3iK_3}(x_3).$$

(6)

Assuming an estimate of the fall velocity is independently derived, the analytical expression of the wind components $U_1-U_3$ allow an analytical expression of the observed radial velocity $u_j$, since

$$u_j = L_{ij}U_1 + L_{2j}U_2 + L_{3j}(U_3 + V_1).$$

(7)

The $V_1$ estimate may be obtained independently (i) from radar reflectivity $Z$ through $Z-V_1$ empirical relationships (Atlas et al. 1973; Joss and Waldvogel 1970; Hauser and Amayenc 1986), or (ii), in the case of stratiform precipitation, from conical scanning by linear regression on the horizontal velocity divergence term at each altitude obtained by VAD analysis (Srivastava et al. 1986; Scialam and Testud 1986). It must be noted that, in the first case, the main difficulty is to identify the type of hydrometeor involved in the process leading to the observed radar signal (rain, snow, hail, etc.).

Let us show how to practically proceed with the radial velocity computation. In fact, the observed velocity to be considered is the radial air wind given by

$$u_j = (u_j' - L_{3j}V_1) = L_{ij}U_1 + L_{2j}U_2 + L_{3j}U_3.$$  

(8)

The analytical expression of this radial air wind is

$$v_j = (v_j' - L_{3j}V_1) = L_{ij}V_1 + L_{2j}V_2 + L_{3j}V_3.$$  

(9)

With $N_i$ coefficients necessary to define each component $V_i$, Eq. (9) implies that $N (= N_1 + N_2 + N_3)$ coefficients are necessary to define analytically the radial air wind or $v_j$ observed with radar $j$. Thus, the velocity $v_j$ may be expressed by

$$v_j = \sum_{i=1}^{3} L_{ij}V_i = \sum_{i=1}^{3} L_{ij} \sum_{K=1}^{N_i} b_{K} g_{K_i}(x_1, x_2, x_3)$$

$$= \sum_{K=1}^{N} b_{K} h_{K}(x_1, x_2, x_3)$$

(10)

with the following correspondence:

$$K = K_i \quad \text{for} \quad i = 1$$

(11a)

$$K = N_i + K_i \quad \text{for} \quad i = 2$$

(11b)

$$K = N_i + N_2 + K_i \quad \text{for} \quad i = 3$$

(11c)

$$h_{K}(x_1, x_2, x_3) = L_{ij}g_{K_i}(x_1, x_2, x_3).$$

(12)

2) RADIAL WIND ADJUSTMENT

The $N$ coefficients $b_K$ are obtained by minimizing, in the least-squares sense, for all the experimental points obtained from all the radars $j$ ($j = 1, 2, \text{etc.}$) in the retrieval domain; the expression $P$ is defined as

$$P = \sum_{j} \sum_{e} (v_j - u_j)^2$$

with respect to the $N$ coefficients $b_K$. We thus solve the following linear system of $N$ equations with $N$ unknowns $b_K$, for $K = 1, 2, \cdots, N$:

$$\frac{\partial P}{\partial b_K} = \sum_{j} \sum_{e} [\sum_{K'=1}^{N} M_{jk} b_{K'} g_{K'}(x_1, x_2, x_3)$$

$$\times M_{jk} g_{K}(x_1, x_2, x_3)] - \sum_{j} \sum_{e} u_j M_{jk} g_{K}(x_1, x_2, x_3) = 0$$

(14)

with the notations

$$M_{jk} = M_{jk'} = L_{ij}(x_1, x_2, x_3) \quad \text{and} \quad g_K = g_{K_i}$$

for $K, K' \leq N_1$ (15a)

$$= L_{2j}(x_1, x_2, x_3) \quad \text{and} \quad g_K = g_{K_2}$$

for $N_1 + 1 \leq K, K' \leq N_1 + N_2$ (15b)

$$= L_{3j}(x_1, x_2, x_3) \quad \text{and} \quad g_K = g_{K_3}$$

for $N_1 + N_2 + 1 \leq K, K' \leq N$. (15c)

System (14) is equivalent to the matrix equation

$$CB = A$$

(16)

in which

- $B$ is the $N$ dimension vector of the unknowns $b_{K'}$;
- $C$ is an $N \times N$ dimension symmetric matrix, the elements of which, $C_{KK'}$, consist of analytical information [orthonormal functions through their products $g_K(x_1, x_2, x_3)$, and direction cosines],

$$C_{KK'} = \sum_{j} \sum_{e} M_{jk} g_{K}(x_1, x_2, x_3)M_{jk'} g_{K'}(x_1, x_2, x_3);$$

(17)

- $A$ is an $N$ dimension vector, the elements of which $A_K$ contain experimental informational data $u_j$

$$A_K = \sum_{j} \sum_{e} u_j M_{jk} g_{K}(x_1, x_2, x_3).$$

(18)

Then, the unknowns $b_{K'}$ are determined through inversion of Eq. (16) and a direct determination of the three wind components can be obtained.

c. Additional constraints

Direct estimate of the three wind components is possible (without taking account of the continuity equation) with three radars at upper levels. On the contrary, at low levels, if horizontal components are derived with sufficient accuracy, the vertical one suffers a great uncertainty [see section 3a(2)]. This is due to the low elevation radar scannings, which induce a poor contribution of the vertical velocity to the observed
radial one or equivalently lead to an ill-conditioned matrix $\mathbf{C}$ (see appendix B). However, the estimate of the three wind components may be improved by constraining the direct previous estimate to satisfy the continuity equation and a boundary condition at ground.

1) CONTINUITY EQUATION OF THE WIND

A physical wind field must obey the continuity equation (under anelastic conditions) that may be written on the first order as

$$\text{div}(\rho_0 \mathbf{U}) = 0$$  \hspace{1cm} (19)

where $\rho_0(z)$ is the air density of the basic state under hydrostatic conditions. The air density may then be written as

$$\rho_0(z) = \rho_0(z_r) \exp[-(z - z_r)/H]$$  \hspace{1cm} (20)

where $z_r$ is a reference altitude and $H$ is the scale height for density variations.

Substituting Eq. (20) in Eq. (19) yields

$$\rho_0(z_r) \exp[-(z - z_r)/H] \left[ \text{div} \mathbf{U} - (U_3/H) \right] = 0.$$  \hspace{1cm} (21)

Condition (21) has to be satisfied by the analytical form $\mathbf{V}(V_1, V_2, V_3)$ of the wind $\mathbf{V}(U_1, U_2, U_3)$ for all experimental points of the domain, and thus may also be expressed as a constraint statistically verified in the least-squares sense:

$$P' = \sum_e \left| \text{div} \mathbf{V} - (V_3/H) \right|^2 \exp[-2(z - z_r)/H]$$

minimum.  \hspace{1cm} (22)

This condition is equivalent to the matrix equation

$$\mathbf{C}' \mathbf{B} = 0$$  \hspace{1cm} (23)

in which $\mathbf{C}'$ is an $N \times N$ dimension matrix. Its elements, $C_{KK' \ell}$, contain “analytical” information through $\text{div} \mathbf{V}$ and $V_3$ (see appendix C).

2) BOUNDARY CONDITION AT GROUND LEVEL

This subsection presents the analytical formulation of the boundary condition at ground level. We consider consecutively the general case in which the detailed orography in the retrieval zone is taken into account and the simplified case in which the orography is neglected.

If orography is taken into account, the ground level condition is established as follows: with $G$ a local point at ground (Fig. 1), let the local ground-normal unit vector $\mathbf{n}_g$ be at an angle $\alpha$ to the vertical and the local ground wind be defined by $\mathbf{U}(U_1, U_2, U_3)$. The most general form for the ground level condition is

$$\mathbf{U} \mathbf{n}_g = 0.$$  \hspace{1cm} (24)

Equation (24) expresses that $\mathbf{V}$ belongs to the plane tangent to the ground at point $G$. If the horizontal unit vector $\mathbf{n}_h$ is at an angle $\xi$ to the $OX_1$ axis, this condition becomes

$$U_3 = -(U_1 \cos \xi + U_2 \sin \xi) \tan \alpha;$$  \hspace{1cm} (25)

$\xi$ and $\alpha$ are orographic data about the retrieval domain.

Condition (25) must be verified by the analytical form of the wind for all experimental points at ground level. It may be expressed as a constraint statistically verified in the least-squares sense:

$$P'' = \sum_g \left[ V_3 + (V_1 \cos \xi + V_2 \sin \xi) \tan \alpha \right]^2$$

minimum.  \hspace{1cm} (26)

Condition (26) is equivalent to the matrix equation

$$\mathbf{C}'' \mathbf{B} = 0$$  \hspace{1cm} (27)

in which $\mathbf{C}''$ is an $N \times N$ dimension matrix. Its elements $C''_{KK' \ell}$ contain “analytical” information through $V_1$, $V_2$, and $V_3$ (see appendix D).

If orography is neglected—i.e., if one assumes flat ground of altitude $z_g$ over the whole domain—the ground-level boundary condition implies a zero vertical wind at ground level, so that condition (26) reduces to

$$P'' = \sum_g V_3^2$$

minimum.  \hspace{1cm} (28)

The nonzero terms of matrix $\mathbf{C}''$ now occupy the $N_3$ last lines and columns of it. Its elements $C''_{KK' \ell}$ contain analytical information concerning the third component:

$$C''_{KK' \ell} = \sum_g g_k(x_1, x_2, x_3) g'_k(x_1, x_2, x_3)$$  \hspace{1cm} (29)

for $N_1 + N_2 + 1 \leq K, K' \leq N_1 + N_2 + N_3$.

d. Mathematical formulation of the variational problem

The variational problem to be solved consists in finding the coefficients which give the best fit, in the
least squares sense, to the radial wind in the whole domain, under the subsidiary conditions that the continuity equation and the boundary condition at ground level are verified by the analytical form of the wind field. Thus, condition (26) or (28) must be satisfied together with conditions (13) and (22). This may be expressed as

$$\lambda P + \lambda' P' + \lambda'' P'' \quad \text{minimum} \quad (30)$$

with respect to the N coefficients to be determined, so that the matrix equation (16) has now to be replaced by

$$DB = [\lambda C + \lambda' C' + \lambda'' C'']B = A; \quad (31)$$

$$\lambda, \lambda' \text{ and } \lambda'' \text{ are weighting factors whose evaluation is discussed in the following.}$$

The main effect of including the physical constraints in the retrieval process is to improve vertical component accuracy, as shown in appendices B, C, and D. Equation (30) shows that the variational form of the analysis generates a boundary condition best fulfilled in the least-squares sense. This means that the condition $w = 0$ at ground level is statistically verified, i.e., the vertical velocity at ground level fluctuates about 0 with a variance depending in this case on the weighting factor $\lambda''$.

Including the continuity equation, constraint in the variational problem allows retrieval of the third (vertical) component homogeneously throughout the vertical domain at the same time as the horizontal component. This minimizes the numerical errors related to the classical upward integration of the continuity equation.

An estimate of the weighting factors, $\lambda, \lambda'$ and $\lambda''$, may be done following the calculus of variations (Courant and Hilbert 1953). In the present simple case, a standard deviation approach may also be used. It leads to the following estimation for the weighting factors:

$$\lambda = 1 \quad (32a)$$

$$\lambda' = \exp\left[+2(z - z_{\text{ref}})/H\right] \delta z^2 \quad (32b)$$

$$\lambda'' = N_z \quad (32c)$$

where $\delta z$ is the mean vertical resolution of the calculations and $N_z$ is the mean number of measurements in the vertical. Note that the height variation of $\lambda'$ makes divergence contribution as efficient at the top as it is at ground.

e. Other conditions

Other physical conditions may be introduced in the process of the wind retrieval and also expressed as additional matrices. For instance, the vertical wind may be assumed to be zero at some “top” altitude, $z_t$. The corresponding condition may be written as

$$P'' = \sum V_3^2 \quad \text{(minimum).} \quad (33)$$

exp points at top

However, in the present case, no upper-limit condition is imposed, since we consider a direct estimate of the vertical velocity at upper levels to be sufficiently accurate.

Additional information, of an experimental nature, or mathematical constraints on the wind field characteristics, such as those needed for the pressure and temperature retrieval, could also be taken into account under a variational form and expressed in the same matricial manner.

f. Correction for advection

A mathematically exact formulation for correcting advection may be given. The basic hypothesis (as in Gal-Chen 1982), is that there is a reference frame in which the precipitating system air circulation is nearly stationary.

We seek an analytical form of the instantaneous 3D wind field taken at the time $t_0$ (for example, midtime of the sequence). In order to find the coefficients of this analytical form, one must compare this analytical form to the observed wind. The problem is that these observations are not all taken at time $t_0$ and thus, in particular, the advection ($C_x, C_y$) of the observed system must induce time evolution of the observed wind field. If one considers that the time evolution of the wind field observed at ground level between $t_0$ and time $t$ of an observation (at point $x, y$), is only due to the advection of the observed wind field, this comparison can be done only if we compare this observation at $t$ with the analytical form displaced by $-C_x(t - t_0)$ and $-C_y(t - t_0)$. Therefore, the correction for advection simply consists in minimizing the difference between a radial wind observed at time $t$ and the radial component of the analytical form of the wind field taken at the location it occupies at time $t$. Thus, this procedure provides a mathematically exact solution to the problem of correcting for advection, and one obtains an instantaneous analytical three-dimensional wind field.

The continuity equation, which is included in the process under a variational form as a constraint on the analytical form of the wind field, is applied on an instantaneous three-dimensional wind field. Thus, the vertical velocity is not contaminated by the advection if advection is accurately determined.

This advection speed $C(C_x, C_y)$ is determined by one of the classical methods such as a visual correlation of successive radar reflectivity patterns or other methods (Gal-Chen 1982). Concerning internal evolution of the system, it can be neglected if its characteristic time of evolution is larger than the sequence duration. This means that if convective scale motions are studied,
one must use fast scanning. Otherwise, classical conical
scannings with longer duration may be performed.

g. Remarks for the application of the method

In conclusion, in this approach the mathematics is
expressed in matrix notation and with variational for-
mulation. It allows for the use of information of the
vertical component of air motion. It uses the statistical
boundary condition adjustment. Practically, it leads to
a matricial equation $\mathbf{CB} = A$ in which $\mathbf{C}$ depends on
the observation locations $(x_j, y_j, z_j)$, on the various ra-
dars locations $(x_i, y_i, z_i)$, on the base orthonormal
functions, and on the advection speed. Thus, $\mathbf{C}$ is ob-
tained by including, as input data of the $\mathbf{C}$-building
software, the various locations of the observations and
of the radars, after the used advection speed and the
expansion base are inserted. Additional conditions
(continuity equation, boundary condition, etc.) are
expressed in the same matricial form and are intro-
duced in the retrieval process by simply adding them
to $\mathbf{C}$. Thus, any additional constraint is taken into ac-
count by a simple matricial addition. This is the first
element of the flexibility of the method. Moreover,
from an operational point of view, note that any change
of base functions consists in a simple change of de-
ition of the $P$, $Q$, and $R$ functions at the program
input, and therefore requires no change in the retrieval
software. This is possible because this process is not
built on special characteristics of a particular base. This
is the second element of the method's flexibility. The
use of the MANDOP orthogonal functions base allows
the addition of higher order terms in the expansion
without changing the expansion coefficients of lower
order. Inversely, if the expansion is performed at a
higher order than the phenomenon itself, the coeffi-
cients of these higher order terms are found to be null.
Note that if the wind components are expressed in
terms of orthogonal functions $F_K = (PQR)_K$, the radial
wind is expressed in terms of $G_K = (LPQR)_K$ which
are not orthogonal. Thus, the retrieval of the coeffi-
cients $b_K$ cannot be simplified by using methods
founded upon the orthogonality properties of the base
functions, allowing direct retrieval of coefficients or
removal of many terms in the matrix $\mathbf{C}$. An alternative
possibility could be to replace the radial winds $(u_j$ and
$v_j)$ in the minimization process [Eq. (13)] with the
product of radial winds by radial distances $r_j$ and di-
rection cosines $L_{ij}$ with products of radial distances by
direction cosines. In this case, it may be shown that
the radial winds are expressed in terms of orthogonal function pro-
products. Thus, in the case of a full domain, one is able to
drop many terms from the matrix $\mathbf{C}$, rendering the
software more efficient. However, this approach is only
applicable to polynomial function bases (allowing to
express $r_jL_{ij}$ in terms of the $P$, $Q$ or $R$ functions)
and to a full and especially small domain. Moreover,
this domain must be relatively far from the radars, oth-
otherwise the farthest radial winds would have a stronger
weight than the closest ones in the minimization pro-
cess. Thus, this approach is adapted to the retrieval of
the wind field in a convective domain, but not in a
mesoscale domain.

Two additional advantages of the MANDOP anal-
alysis are that any type of radar scanning can be used
without reformulation of the retrieval process, and that
any dataset (eventually inhomogeneous and without
data on the retrieval domain limits) can be processed,
as discussed later. Moreover, any number of radars
greater than or equal to two can be used without
reformulation of the process. Among these radars, air-
borne radar data can also be included in the retrieval
process. The only difference between ground-based and
airborne radar is that radar location is fixed for a
ground-based radar, while it varies with each obser-
vation for an airborne radar. Thus, in the first case, the
input data $(x_j, y_j, z_j)$ are constant for all data from
each radar, while in the second case, the input data
$(x_j, y_j, z_j)$ vary for each observation, but this implies
no change in the retrieval software. Finally, the retrieval
process permits to bypass the interpolation, and to use
data points where they occur (without defining a grid
mesh) as densely or sparsely as they occur, and to per-
form a sequential pass through all the data in any order,
by simply accumulating terms in the matrices.

In conclusion, this formulation of the method is the
most general one, allowing its application to all possible
situations (dataset, number and types of radars, types
of scannings, of bases, etc.).

3. Application to simulated and real data

These applications are an illustration of the capa-
bilities of the MANDOP analysis to study the dynamical
processes involved in mesoscale precipitating sys-
tems. Numerical tests are carried out to demonstrate
the role of each constraint in the present analytical
variational analysis and to substantiate the theoretically
predicted filtering characteristics. Simulated winds were
chosen in order to represent atmospheric waves or rolls.
Indeed, theoretical studies have shown that the most
probable mesoscale physical mechanisms [ducted
gravity waves (Lindzen and Tung 1976); conditional
symmetric instability (Bennetts and Hoskins 1979);
symmetric CISK in a baroclinic flow (Emanuel 1982)]
manifest themselves with this mesoscale pattern. The
interest of the analytical and variational approach used
in the MANDOP analysis will be particularly evidenced
by comparison with classical simpler methods. Quality
of the detailed wind retrieval will be also estimated by
comparison with data from other instruments.

This application is carried out in the particular case
of a Legendre polynomial base. This base is simply
selected to illustrate the application. Let us recall that
the MANDOP analysis is not linked to a particular
base, and any other orthogonal function base may be used without reformulation of the analysis. In particular, Fourier expansion could be used if a wavelike character is suspected from the wind field structure. Nevertheless, note that if there is no periodic character of the wind field, the use of the Fourier base would lead to an expansion up to a very high order for the wind field to be represented, and thus to the retrieval of a huge number of terms. Then in a mesoscale domain, we may expect to have the contribution of larger scale wind fields (for example, within the large cloud band resulting from frontogenetic effects or within stratiform parts of squall lines) that may be described in the domain in terms of linear variations and not in terms of periodic variations. Inversely, if in the domain, unsuspected wavelike structures are present, they can be represented (as shown in the paper) by Legendre polynomials of sufficient order.

The application to real data will be performed using conical scans since they allow extension of the retrieval domain to the mesoscale. We have limited the polynomial expansion to order 5 in order to filter out the convective scale motions [as shown in next subsection 2a(3)]. Let us recall that the present method can be used for any type of radar scanning without reformulation in particular for fast scanning, such as coplane for which the internal time evolution associated with convective motions can be considered as weak. Since we truncate the expression in order to filter out small scale motions (less than 15 km), it is not necessary to process all the real data. One solution, for example, could involve the selection of one radial measurement among, for example, five, but in this case this would have resulted in insufficient coverage at the boundaries. Therefore, we averaged quasi-instantaneous (using a time classification process) data from each radar within a grid mesh with a grid resolution \((5 \times 5 \times 0.350 \text{ km}^3)\) consistent with the filtered small scale motions. The chosen procedure (space average on quasi-instantaneous measurements from each radar in each grid mesh that leads to a single datum in each grid mesh at the mean position of these data) leads to a reduced dataset that is almost regularly distributed in the retrieval domain for each radar. But this is, of course, a particular application of the MANDOP analysis. As pointed out previously, the normal way to use the MANDOP analysis is to process all data in any order, without defining a grid mesh, by simply accumulating terms in the matrices.

For simulated data, it is assumed that the three radars considered are located as in the LANDES-FRONTs 84 experiment (Chalon 1987) (Fig. 2). Two of these radars (R1 and R2) are C-band Doppler radars of the Ronsard system (Nutten et al. 1979), and the third, RABELAIS (RA), is a Ka-band Doppler radar (Sauvageot 1982). It is also assumed that the zone in which the simulated winds are defined and the zone where the retrieved winds are obtained are identical; i.e., the winds are simulated by one point in the center of each grid mesh.

a. Application to simulated data

As previously specified, data of any number of radars may be analyzed simultaneously without reformulating the problem. We specify in appendix E the coefficients of the wind expansion retrieved without ambiguity by the method according to this number. We also give theoretically predicted characteristics of the MANDOP analysis, namely velocity components' accuracy (2), filtering characteristics (3 and 4), and, finally we applied MANDOP analysis to simulated data (5 and 6).

1) NUMBER AND IDENTIFICATION OF THE RETRIEvable COEFFICIENTS

Let us now turn back to the linear system (14) without additional conditions. The solution of such a three-dimensional system in the absence of the continuity equation implies that the equations of the system are linearly independent. Then, three radars should be necessary for a complete retrieval of the wind. Let us show how many and which coefficients are obtained when one or two radars are operating and when no constraints are imposed in the retrieval process. The purpose of this calculation is to illustrate that it is possible to identify the uncalculable coefficients in these unfavorable conditions, and to thus determine the wind field by imposing the values of the undetermined coefficients. We shall only present the results. More details and demonstrations are provided in appendix E. For simplicity, we assume in the following that each wind component will be developed on each coordinate in a series of Legendre polynomials up to the same order \(n\). Then, \(3n^2\) coefficients have to be determined. The choice of a Legendre base does not preclude general-
ization to other orthonormal bases (as shown in appendix E).

If we consider that only one radar is available, the case \( n = 1 \) (where the three components reduce to three constants) is trivial: the three components are readily obtainable.

The case \( n = 2 \) is studied in detail in appendix E. It appears that among a total of 24 coefficients, 12 derived from "high order terms" are readily obtainable. The other 12 are related through eight equations, of which seven, at most, are independent. At least five coefficients have to be known independently. The case \( n = 3 \) also gives the same results: of a total of 81 coefficients, 27 may readily be derived; the other 54 are related by, at most, 26 independent equations. These results are general for any value of \( n \): on a total of \( 3n^3 \) coefficients, \( 3n^2 \) are readily obtainable, the other \( 3n^3 - 3n^2 \) are related by, at most, \((n^2 - 1)\) independent equations.

If we now consider that two radars are available, in the case \( n = 2 \), seven more equations are obtainable than with only one radar, four of them being independent so that, finally, one coefficient cannot be retrieved. A generalization to any value of \( n \) is possible, and the result is that for a total of \( 3n^3 \) coefficients, \( 3n^2 \) are directly derivable, the other \((3n^3 - 3n^2)\) being related by, at most, \((2n^3 - 5)\) independent relations.

As a conclusion, with one or two scanning radars without physical constraints, additional information is necessary for \( n > 1 \) in order to retrieve the coefficients not directly obtainable. It may be imposed by physical considerations. On this basis, a computer program has been developed in order to determine which coefficients can be retrieved directly for each value of the parameter \( n \), using one or two scanning radars.

2) ACCURACY ON THE HORIZONTAL AND VERTICAL COMPONENTS ESTIMATE

In order to test this method, we simulated radial air velocities constructed from components \( V_i \) through Eq. (9). Components \( V_i \) were in turn simulated as products of three terms, each of them being an expansion on each coordinate in a series of Legendre polynomials up to fifth order [Eqs. (1), (2), and (3)]. The coefficients of the expansion were imposed. Neither the continuity equation nor other constraints were introduced in these first simulations. Three radars were then necessary for complete retrieval of the wind. Retrieval is also performed on Legendre developments up to order \( n = 5 \) on the three coordinates. Simulated as well as retrieved winds were supposed to be defined by one point at the center of each grid element. The corresponding wind fields are not presented here since they appear identical. Certainly, the analytical determination of a wind on the same base and at the same grid points as the one on which it was analytically defined must be as good. However, the ill-conditioned \( C \) matrix (resulting from the poor contribution of the vertical velocity to the radial velocity at low elevation scan as explained in §2c and appendix B) increases the errors on the development coefficients of the vertical velocity. Indeed, the observed errors are within 1% while those on the horizontal coefficients are observed to be within 0.01%. These coefficient errors lead to relative errors of about 0.1 and 0.001 on the vertical and horizontal components, respectively, as shown in the following estimation.

Since each component \( U_i \) has an analytical form \( V_i \) given by Eq. (4), the error on \( V_i \) is

\[
\delta V_i = \sum_{K=1}^{N_i} g_K \delta b_K (N_i = n^3 \text{ in the present case}) \tag{34}
\]

If \( \delta b_K = \delta_i \), only depending on the \( i \)th component, we may write

\[
\delta V_i^2 = \delta_i^2 \left( \sum_{K=1}^{N_i} g_K \right)^2 = \delta_i^2 \left( \sum_{K=1}^{N_i} \sum_{K'=1}^{N_i} g_K g_{K'} \right) \tag{35}
\]

Integrating this expression over the retrieval volume yields

\[
\delta V_i \leq N_i^{0.5} \delta_i \tag{36}
\]

since \( g_K \) are products of orthonormal functions. This provides an upper limit for the errors.

For \( n = 5 \), we obtain

\[\delta V_1 = \delta V_2 = 125^{0.5} \delta_1 = 0.001 V_1\]

\[\delta V_3 = 125^{0.5} \delta_3 = 0.1 V_3\]

For stratiform conditions, \( V_3 \) is typically \( 20 \text{ cm s}^{-1} \) implying that \( \delta V_3 \) belongs to the order of \( 2 \text{ cm s}^{-1} \). Thus, when simulated winds determined from components expanded up to order \( n = 3 \) on the three coordinates are retrieved on base functions up to the order \( n = 5 \), the coefficients corresponding to orders 4 and 5 are found to be zero within the error bars given above.

3) CHARACTERISTIC SCALE OF THE OBSERVED PHENOMENA AND CUTOFF (MINIMAL) WAVELENGTH OF THE METHOD FOR A LEGENDRE POLYNOMIAL BASE

The cutoff wavelength of the method \( \lambda_\text{c} \) is due, on the one hand, to the horizontal resolution of the analysis, and, on the other hand, to the development order of the analytical winds in the given domain size. As to the horizontal resolution, if the spatial sample resolution is \( \Delta X_i \), it means that no structure with a wavelength smaller than \( \lambda_{\text{c1}} = 2\Delta X_i \) will be observed, according to Shannon's theorem. Concerning the order of development, since components are developed up to the order \( n \) along each coordinates, a wave along each axis reaches zero at most \( (n-1) \) times, thus implying a minimal wavelength \( \lambda_{\text{c2}} \) of about \( 2/(n-1) \) times the total explored domain. Therefore, when applied to

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real cases, parameters $n$, $L$, and $\Delta X_1$ will be chosen such that $\lambda_{c1} \ll \lambda_{c2}$. For the present application, the horizontal resolution $\Delta X_1 = 5$ km leads to a minimum observation wavelength $\lambda_{c1} = 10$ km, and a development up to the order $n = 5$ on the domain size $D_1$ of 100 km leads to a $\lambda_{c2}$ of 50 km. Thus, the minimum observable wavelength is indeed imposed by the order of the development. Corresponding simulations [described in subsection 3a(5)] confirm these results. Note that if the study necessitates smaller scale retrieval, one could simply reduce the sample spatial resolution; similarly, however, this implies reduction of the total retrieval zone. If one maintains the same total retrieval domain, it is then necessary to increase the order of expansion.

Let us recall that if the method induces a cutoff wavelength $\lambda_{c}$, it means that the minimum observable scale is $\ell = 0.26\lambda_{c}$ (corresponding to half the power of the phenomenon, represented by, for example, a gaussian function, passing through the window resulting from the method cutoff in Fourier space). In the present case, the minimum observed scale $\ell$, is fixed by the development order and is found to be 13 km.

4) REPRESENTATIVENESS OF THE ANALYTICAL FORMULATION OF WIND DERIVATIVES

The analytical development of the wind components $V_i$ (up to order $n$) implies the existence and continuity of successive space derivatives (up to order $n - 1$). The consistent agreement between experimental and theoretical gain curves confirms this point [see §3a(5)]. On the other hand, the development of the wind up to the order $n$ implies a retrieval with a cutoff wavelength of about $2D_i/(n - 1)$ ($D_i$ being the domain width in the $X_i$ direction) since $(n - 1)$ roots exist for the equation: $V_i = 0$. Similarly, the first (second, etc.) derivatives are represented by a development up to the order $n - 1$ $(n - 2$, etc.) implying retrieval with a cutoff wavelength of $2D_i/(n - 2)$ $(2D_i/(n - 3)$, etc.). In the present, particular case of Legendre development up to the order 5, the successive cutoff wavelengths are, for $D_i = 100$ km: 50 km for the wind components $V_i$, 66 km for the first derivative $dV_i/dX_i$, 100 km for the second derivative $d^2V_i/dX_i^2$, etc. To conclude this point, the calculated derivatives provide a good representation of the true space derivatives at a scale greater than the wind itself.

5) RETRIEVAL WITHOUT ADDITIONAL CONDITIONS AND CUTOFF WAVELENGTH OF THE METHOD

A Legendre polynomial base is appropriate for polynomial-like winds retrieval as it appears in section 3a(2). The accuracy on horizontal components is very good, and if vertical components provide a poorer retrieval, this problem, due to the $C$ matrix being ill-conditioned (see appendix B), can be addressed by introducing the density continuity equation (appendix C), and the boundary condition at ground-level (appendix D). Both are expressed in a matricial form [see also results in section 3a(6)].

In order to determine the cutoff wavelength and, at the same time, to examine the adequacy of the Legendre base to wavelike motions, we shall consider a wave, referred to as wave A along $X_1$, defined by its “observed” components $U_2 - U_3$ as:

$$U_1 = A \exp [i(k_1 X_1 + \phi_{01})]$$
$$U_2 = 0$$
$$U_3 = 0$$

where $A$ is the wave amplitude, $k_1$ is the wave vector along the $X_1$ axis, and $\phi_{01}$ is the corresponding phase. As explained in section 3a(3), if wave A is developed to fifth order on the $OX_1$ axis, it will undergo four zero crossings in the domain’s width (100 km), thus allowing a retrieval of a minimum wavelength 50 km; i.e., an observable minimum scale of about 13 km.

The first simulation is built from wave A, in which wavelength $\lambda_1 (=2\pi/k_1)$ varies between 50 and 200 km. The simulated radial velocity is calculated from Eq. (8) in which $U_1-U_2$ are given by Eqs. (37a–c) and $V_i$ is taken to be zero. Coefficients then deduced from the MANDOP analysis allow a retrieval of wind components $V_1-V_3$.

The gain $G_{exp}$ of the analysis is defined by $G_{exp} = V_i/U_1$, and the corresponding phaseshift is $\Delta \phi = \phi_{01} - \phi_{0i}$, if $\phi_{0i}$ is defined as the phase of the retrieved wind. Results are given in Figs. 3a–b.

Figure 3a shows $G_{exp}$ as a function of $\lambda_1$. It evidences a cutoff (i.e., the value taken by $\lambda_1$ for $G_{exp} = 0.5$) for $\lambda_1 = 57$ km ($\ell = 15$ km, about three elementary grid meshes), in good agreement with previous theoretical considerations. The phaseshift $\Delta \phi$ is less than $5^\circ$ for passing wavelengths (Fig. 3b). These considerations may be extended to the vertical axis, leading, in that case, to a minimum observable scale about 1 km, with the present chosen grid mesh dimension (350 m). The use of Legendre polynomials (or of other bases) implies that the analytical representation of the wind is continuous and differentiable.

Six theoretical gain curves given by expression:

$$G_{th} = 1/(1 + \mu(2\pi/\lambda_1)^{2n_d})$$

deduced from Tostud and Chong’s (1983) analysis, for the order $n_d$ derivative, are also given in Fig. 3a when a type A wave is simulated. The Lagrange parameter $\mu$, which expresses the relative importance of the derivative constraint with respect to the least-squares fitting term, controls the cutoff wavelength. $\mu$ is chosen from Eq. (38) in order that $\lambda_1 = \lambda_c$ for $G_{th} = 0.5$. As expected, the best agreement with experimental gain $G_{exp}$ is found for a value of $n_d$ of about 5. These theoretical gain curves (respectively corresponding to $n_d$
= 1 to 6), show the sharper cutoff as \( n_d \) increases. They show that the cutoff slope is well marked and does not change too much for values of \( n_d \) greater than 4. Thus, the value \( n_d = 4 \) should provide a good filter for the method. However, as we want to resolve characteristic scales of at least 10 km, \( n_d = 5 \) seems to be a good trade-off.

6) RETRIEVAL IMPROVEMENT WITH MATRIX-EXPRESSED CONDITIONS

The wind presently simulated, referred to as wave B, is a longitudinal wave obeying the density continuity equation and the boundary condition at ground level:

\[
U_1 = A_1(\lambda_1/2\Pi)[2/H \sin[2\Pi/\lambda_2(z-z_0)] + (2\Pi/\lambda_2) \cos[2\Pi/\lambda_2(z-z_0)] \times \exp(-(z-z_0)/H) \sin(2\Pi X_1/\lambda_1 + \phi_1) \] (39a)

\[
U_2 = 0 \quad (39b)
\]

\[
U_3 = -A_3 \sin[2\Pi(z-z_0)/\lambda_2] \exp[-(z-z_0)/H] \times \cos(2\Pi X_1/\lambda_1 + \phi_1) \] (39c)

This introduces a vertical component and vertical variations on the horizontal component. The horizontal wavelength \( \lambda_1 \) and vertical wavelength \( \lambda_2 \) are respectively chosen to be 100 and 14 \text{ km}; \( z_0 \) is the ground level.

Figure 4 shows a simulated wind vertical section (4a) compared with vertical sections of winds retrieved under various conditions (4b–g). The simulated wind field defined by Eqs. (39a–c) consists of rolls, horizontally separated by updrafts and downdrafts with a maximum of about 3 m \text{s}^{-1}. Horizontal velocities range from 0 to 15 m \text{s}^{-1}. The main characteristics of the simulated wind field are reproduced on all the retrieved wind fields, in spite of the fact that Legendre polynomials are not the ideal base for wavelike wind field description. The regions that appear to be retrieved less accurately are located on the edges of the domain, which may be due to geometrical effects.

Figures 4b–d exhibit wind fields retrieved with three radars, R1, R2 and Ra. In Fig. 4b, an example with no imposed constraints is given. Strong discrepancies appear, especially at low altitudes, due to quasi-horizontal scanning providing little information on the vertical motion and evidencing the absence of constraints at ground level.

The introduction of a boundary condition at ground level, in Fig. 4c, provides for much better agreement between original and retrieved winds. Corresponding errors are about 0.6 m \text{s}^{-1} on the horizontal components and 0.5 m \text{s}^{-1} on the vertical component.

Adding the density continuity equation constraint, in Fig. 4d, leads to improved results (0.5 and 0.3 m \text{s}^{-1} for horizontal and vertical components, respectively).
Figures 4e–g show the wind field retrieval with two radars, R1 and R2. The continuity equation constraint now necessary and is therefore introduced, just as the ground-level condition.

In Fig. 4e, the retrieved wind field is found to be very close to the original wind. Corresponding errors are about 0.3 and 0.1 m s⁻¹ for the horizontal and vertical wind components, respectively. Notice that including the continuity equation constraint improves the vertical velocity retrieval (see §3b). Indeed, the vertical component appears close to zero at the top level, implying no numerical errors related to the clas-
physical upward integration of the continuity equation. On the other hand, the relative error is equally dispatched between the three wind components.

The addition of a random number of 0.7 m s\(^{-1}\) standard deviation simulating the radar statistical error (which is nearly the same as the estimated value for the dual Doppler Ronsard facility) to the simulated wind shows that a satisfactory filtering of the noise is performed by the method (Fig. 4f) and induces wind components errors comparable to the previous ones. A substantial degradation (not shown here) occurs when a noise greater than 1.5 m s\(^{-1}\) is added. We notice that results are better with two rather than three radars due to the insufficient weight given to the continuity

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**Fig. 4e.** Wind field retrieved with two radars (R1 and R2), ground level condition and continuity equation.

**Fig. 4f.** Same retrieval as Fig. 4e, but a noise 0.7 m s\(^{-1}\) standard deviation was added to the simulated wind.

**Fig. 4g.** Same case as Fig. 4f, but the retrieval was limited to the zone effectively observed by the two radars.
equation. This in turn results from the fact that the number of data considered for estimating the density continuity equation and the boundary condition contributions are not the exact number of data derived from all radars taken into account in the calculation of matrix C, but are the grid point data. Thus, account for data from three instead of two radars gives lower weight to the additional constraints. The restriction of the retrieval zone to that effectively explored in the present case by conical scannings indicates that the wind is properly retrieved (Fig. 4g).

b. Application to real data

The purpose of this subsection is to show the consistency of the retrieved wind field. The data used in the present section were extracted from the LANDFRONTS 84 experiment data bank. Experimental testing of the method requires measurements performed at close times and spacing with various instruments. This had led us to select the following observations:

- vertical profiles of wind components at low levels deduced from acoustic soundings performed by a sodar located close to radar R2;
- vertical profiles of the horizontal wind deduced from upper air soundings (RS);
- reflectivity and Doppler information coming from the C band (wavelength 5.5 cm) dual Doppler radars of the RONSARD system (R1, R2).

The scans used for this MANDOP analysis application are coordinated conical scannings. The characteristics of these scans are as follows:

- elevation sampling from 0°5–42°5 by 1° (between 0°5 and 3°5) and 2.5° (between 6° and 42°5);
- sampling in azimuth from 0° to 360° by step of 1.5°;
- distance range from 5–56.2 km covered by 128 gates spaced 400 m apart.

The propagation speed (100 degrees east from north, 16 m s⁻¹) used in the procedure for correcting for advection is estimated from visual correlation of successive radar reflectivity patterns. Considering the volume scanning time (10 min), this leads to a translation of the data within the volume of 7 km.

The contribution of the terminal fall velocity of hydrometeors $V_t$ to the radial Doppler velocity is estimated through an empirical relation between particle fall speed and deduced radar reflectivity factor. Then the MANDOP analysis is applied. Let us recall that the analytical forms of radial winds are variationally adjusted to the observed radial velocities (affected by a mean sampling time $t$ relative to $T_0$) according to the anelastic continuity equation and to the lower kinematic boundary condition satisfied by the analytical form of the three wind components. The boundary condition used will be $w = 0$ at $Z = 190$ m (mean...
altitude of radars) assuming flat ground over the whole domain (in the present case orography is horizontally filtered at the domain resolution).

We present wind fields from the MANDOP analysis and also from conventional VAD (Browning and Wexler 1968) and dual DVAD (Scialom and Testud 1986) analyses that are consecutively applied. Comparisons will be made between these analyses and with radiosounding and sodar measurements.

1) Description of the MANDOP Wind Field

The deduced three-dimensional wind field is illustrated in Figs. 5, 6, and 7. Figure 5b exhibits the horizontal cross-section wind field at 1100 m altitude. The corresponding reflectivity pattern (deduced from a 100 km range PPI at 1.5° elevation made just before the onset of dual Doppler observations) appears in Fig. 5a. Selected horizontal and vertical sections are shown in Figs. 6 and 7, respectively, to illustrate the horizontal

Fig. 6. Airflow in the cross motion reference frame on horizontal cross-sections at 0.9 km altitude (a), 2 km (b), 3 km (c), 5 km (d), illustrating the various airflows acting in the system.
relative velocity and the vertical motions. The orientation of cross sections (given in Fig. 7) with respect to the horizontal grid is indicated in Fig. 6a with letters labeling end columns of each cross section. Cross section $II'$ illustrates the updraft in the southern core and its precise colocation with the reflectivity core. Cross section $JJ'$ is in the along-core direction through the stratiform part of the northern core. All velocities in the reference frame are linked to the cores. The $Y$ and $X$ horizontal axes are in southeast–northwest and southwest–northeast directions, respectively (see Fig. 1).

Two convective regions can be identified from the reflectivity data (Fig. 5a). In the SW half-area the southern convective region appears as an organized narrow elongated rain core. It is associated with a narrow region of gust from convergence (Fig. 5b) and cyclonic shear between a westerly low level flow coming from the rear of the convective region and an inflow resulting essentially from the propagation of the system (its intensity is equal to the system motion). Strong

Fig. 7. Airflow in the cross motion reference frame and reflectivity contours at 40, 30 and 20 dBZ on vertical cross sections in the vertical plane (a) $II'$ and (b) $JJ'$ (as indicated in Fig. 6a). Arrows in the upper left corner of the figures represent 15 m s$^{-1}$ velocities.

Fig. 8. Temporal evolution of the absolute wind directions obtained from sodar data (at 200 m altitude).
Fig. 9. Horizontal field of absolute wind deduced from MANDOP analysis at: (a) 0.5 km altitude. Sodar place (close to R2) is figured by S. Wind directions measured by sodar are about 300 and 260 degrees at 1040 and 1200 UTC, respectively, and appear in the upper right corner. (b) 0.9 km, (c) 2 km, (d) 3 km, (e) 4 km and (f) 5 km. Radiosounding location is figured by RS for 1130 UTC. A translation of the radiosounding location has been performed using a space-time conversion along the advection speed direction of the cores. Corresponding radiosounding wind directions, also appearing in Fig. 10, are displayed in the upper right corner. The distance between two grid points represents a velocity vector scale of 15 m s⁻¹.
ascending motions are located along this gust front confluence (Fig. 7a,b). In the NE half-area the convective region also shows an organized structure but new convective cells are present ahead of it.

The stratiform part of the system, while relatively uniform in the SW half-area (associated with the southern rain core), appears as a region of clear patchy echoes in the NE half-area.

The low level air enters the anvil from the front and apparently ascends along a sloping surface of zero relative velocity located above the frontward low level flow. We observe a mesoscale ascending motion of several tens of centimeters per second in the upper regions of the stratiform part of the precipitating system. In the area immediately rearward of the leading convective region, downward air motions extend from the ground to the iso 0°C altitude.

2) COMPARISON WITH SODAR DATA

Figure 8 shows the wind direction of absolute wind at 300 m altitude given by the sodar station located close to radar R2. It shows a rotation of the wind from 300° to 260° between 1000 and 1200 UTC. A space–time conversion along the advection speed of the system crossing the sodar station shows that the corresponding length scale variation is about 85 km. These wind directions deduced from the sodar are plotted on Fig. 9a, which also gives the horizontal absolute wind field at 0.5 km altitude obtained from the MANDOP analysis. These two independent kinds of data are observed to be in good agreement.

3) COMPARISON WITH RAWINDSONDE DATA

Figure 10 presents the hodograph of the wind for the closest sounding of 1130 UTC. The location of the rawindsonde station is indicated by RS in Fig. 9b. Typically, it shows a horizontal wind–height profile with a southwesterly flow (230°, 16 m s⁻¹ at the top of the flow and 12 m s⁻¹ at its base) in the 4–7 km slab above a westerly flow of 19 m s⁻¹. This southwesterly flow is capped by a westerly upper-level jet streak. The absolute horizontal wind fields deduced from the MANDOP analysis are displayed in Fig. 9b–f for levels 0.9 km, 2 km, 3 km, 4 km and 5 km. In the same figure, the location of the radiosounding, and the corresponding wind directions deduced from the hodograph, are also plotted and show reasonable agreement with the MANDOP results. They are characterized by the previous distinct horizontal flows:

- the low level westerly flow (270°, 20 m s⁻¹)
- the southwesterly flow (230°, 12 m s⁻¹ at 4 km and 17 m s⁻¹ above).

The comparison of the different horizontal cross sections shows that the southwesterly flow increasingly penetrates northward in the retrieval zone as the cross section rises. This results from the rising of the flow during its northward propagation.

4) COMPARISON WITH VAD ANALYSIS RESULTS

In the VAD methodology, the horizontal components of the wind are supposed to be represented by their first-order Taylor expansion (or linear expansion) around the radar site. This linearity hypothesis implies that the variation in azimuth of the radial velocity, obtained in a particular radar range gate and for a given elevation of the conical scan, is fully represented by its second-order Fourier series expansion. These Fourier coefficients linked to the horizontal wind characteristics are estimated from least squares fitting to the radial winds (Testud et al. 1980). The vertical velocity is then derived through an integration of the air mass continuity equation under the anelastic assumption (Cgura and Phillips 1962).  

Figure 11 shows a comparison between results from these two very different (MANDOP and VAD) analyses. Mean horizontal cross- and along-core velocities are respectively displayed in Figs. 11a and 11b, while mean vertical velocities appear in Fig. 11c. Solid lines show the MANDOP processing, while dotted and dashed lines represent a VAD analysis applied to R1 and R2 data, respectively, all for 1040 UTC. Also displayed in Fig. 11a are R1(+) and R2 (dash-dotted) VAD results for 1053 UTC evidencing mesoscale wind stationarity. The MANDOP profiles are obtained by averaging the analytical form of the three wind com-
ponents over the common volume zone explored by R1 and R2. Vertical profiles deduced from this averaging process are in good agreement with those of the VAD analysis, though both analyses are not performed on the same retrieval volume (as seen on Fig. 2).

Notice that, at a first approximation, the MANDOP profiles must be compared with the mean profiles given by the two VAD analyses, since the MANDOP is performed in the common zone explored by both radars for VAD processing. This implies that each radar explores a dynamically homogeneous area. On the other hand, the MANDOP results (see Fig. 9) show, in the common zone, the existence of a transition between two relatively homogeneous regions. Thus again, the MANDOP profiles are expected to be intermediate between the two VAD profiles.

Concerning the horizontal winds (Figs. 11a–b), the MANDOP results effectively lay between R1 and R2 VAD results (U' and V' components are along the WE and SN axes, respectively. U' component appears greater for R1 than for R2, as expected from the MANDOP horizontal wind retrieval (Fig. 9). The same remark holds for V', except at low levels (below 2 km) where the MANDOP profile is closer to R1 profile. This latter aspect is consistent with the MANDOP observations in which the northern area wind field regime appears dominant. At upper levels, the MANDOP and VAD profiles appear very close to each other since the wind transition is not so well marked at low levels.

Concerning the vertical wind (Fig. 11c), the main characteristics of VAD profiles are observed in the MANDOP profiles with weak descending motions below the 0°C isotherm (1.9 km) and mesoscale updraft above. The more pronounced updraft in the R2 vicinity may be explained by the strong vertical motions located in this area (see cross section II' given in Fig. 7). However, the VAD profiles seem to fit the MANDOP profile, and they filter out the small-scale vertical structure. This may be easily explained. It should be recalled that the determination of horizontal wind divergence \( \text{div}V_H \), in the VAD analysis, is influenced by the horizontal scale \( r_H \) of the VAD scannings and therefore, in the present case, essentially contains the contribution of the horizontal mesoscale field. Thus, vertical phenomena with a scale less than 5 km are filtered off as shown in the following. Specifically, assuming the anelastic hypothesis, the horizontal divergence term may be written as:

\[
\frac{\partial w}{\partial z} - \frac{w}{H_0} = \text{div}V_H, \tag{40}
\]

\( H_0 \) being the scale height of the atmosphere. If \( K_H \) and \( K_Z \) are respectively the horizontal and vertical wavenumbers (\( w = e^{ik_Z z}, V_H = e^{ik_H h} \) with \( K_Z = 2\pi/\lambda_Z \) and \( K_H = 2\pi/\lambda_H \)), (40) is equivalent to

\[
(iK_Z - 1/H_0)w = iK_H V_H. \tag{41}
\]

Then

\[
\{\frac{\partial w}{\partial V_H}\} = K_H/\{K_Z^2 + (1/H_0)^2\}^{0.5} \tag{42}
\]

and the observed values (see Fig. 11c) \( \partial w \approx 0.3 \text{ m s}^{-1} \) and \( \partial V_H \approx 4 \text{ m s}^{-1} \) yield

\[
\frac{\partial w}{\partial V_H} \approx \lambda_Z/\lambda_H \approx 1/10.
\]

In the present case, \( r_H \approx \lambda_H \approx 50 \text{ km} \) leads to the observed cutoff wavelength \( \lambda_Z \approx 5 \text{ km} \).

5) COMPARISON WITH DVAD ANALYSIS RESULTS

The DVAD analysis gives a complete representation of the horizontal wind field under the classical hypothesis of the VAD analysis that the variation of the horizontal wind components is linear. In the first step, this method uses the VAD analysis performed for two spaced radars, simultaneously operating in order to provide an estimation of the horizontal wind at each radar site, and of the three combinations (divergence, stretching, and shearing deformations) of the first-order spatial derivatives of the horizontal wind field. Two of these derivatives are thus readily obtained. The other two cannot be obtained with only one radar. These unknown derivatives are then analytically expressed in terms of the known parameters coming from both radars and derived by a least-square fitting procedure. Therefore, all first-order horizontal derivatives of the horizontal wind field are estimated.

The DVAD wind fields to be compared with those from the MANDOP analysis are shown in Fig. 12. The agreement between the two analyses (and also with the VAD analysis) gives further support to the validity of the MANDOP analysis method. The wind-field linearity assumed in the DVAD analysis is clearly apparent. While at upper levels, for relatively homogeneous wind fields, the DVAD and the MANDOP provide nearly similar results, at low levels where a more marked transition is observed the DVAD linearity does not give an accurate representation of the wind field. Moreover, although the MANDOP uses the same basic data set as the DVAD, it provides a more detailed description of the horizontal wind field. In particular, it allows vertical velocity to be obtained in the whole retrieval domain, while the DVAD provides its mean value on each horizontal plane.

Let us examine in more detail the retrieved wind field at 2 km altitude. It is characterized by some discrepancies between the wind fields provided by both methods (Figs. 9c and 12c). The DVAD winds appear stronger than the northerly MANDOP winds, and are oriented differently for the southerly winds. This results from two main reasons. On the one hand, the DVAD provides a good wind estimate at the radar location but linearly extrapolates the wind to the domain limits. On the other hand, the DVAD does not allow shorter
scale variations of the wind to be obtained due to the linear form of the wind.

4. Possible extensions

a. Retrieval of air parcel and particle trajectories

The analytical representation of the wind allows motion of air parcels in the three-dimensional precipitating system to be readily obtained. Indeed, as a parcel moves along its path, its next position is simply calculated from the wind components directly given by the value of their analytical representation at the current point and from the timestep length which may be as short as is required. Particle trajectories are obtained from wind components to include the particle vertical terminal velocity (deduced from the reflectivity factor through an empirical relation).
b. Pressure and temperature retrieval

In addition to classical parameters directly related to the wind field, such as vorticity, it is possible to retrieve physical parameters such as pressure or temperature perturbations allowing to describe the thermodynamic mechanisms implied in the mesoscale circulations. The analytical formulation of the wind field allows direct derivation of analytical form of thermodynamic perturbation fields. Thus, the procedure to obtain pressure and temperature only differs from that described in previous papers (e.g., Gal-chen 1978) in that the input data are not a three-dimensional wind field given at grid points of a regular grid mesh, but an analytical wind field obtained through the elaborate processing MANDOP of the radial velocity fields observed in a multiple Doppler radar experiment.

5. Conclusion

The study previously developed describes a new analysis, MANDOP, for the retrieval of the three-dimensional mesoscale wind field from observations by multiple Doppler radar. It gives the mathematical principle of the wind retrieval. This analysis, based on a variational concept, relies upon the expressions of the wind components as products of expansions in series of orthonormal functions on each axis, thus allowing the radial wind to be analytically expressed. The analytical form of the radial wind is then variationally adjusted to the observed wind by including in the minimization process the additional conditions satisfied by the wind, such as mass conservation and boundary condition at ground level. The coefficients allowing a complete description of the wind are thus derived by the process. The variational process allows the vertical wind to be retrieved with good accuracy. The filtering characteristics related to the order of expansion of the orthonormal functions are found to be adapted to mesoscale studies in practical cases.

The present paper also deals with the application of the method to real cases and specifies how to operate in these cases where a great number of data with time scatter have to be taken into account. It illustrates how the analytical form of the wind implies data filtering and interpolating.

This application to real data shows that the wind field obtained through the MANDOP analysis is in good agreement with VAD and DVAD linear analyses, and with wind field profiles obtained by means of sodar and rawinsonde. Moreover, although it uses the DVAD dataset, it provides a more detailed description, especially for vertical motions, of the observed situation.

Theoretical works made these last ten years have evidenced the importance of theoretical instabilities (conditional symmetric instability, symmetric CISK in a baroclinic flow, and wave CISK) for producing vertical motions at the mesoscale, in addition to those resulting from synoptic mechanisms (baroclinic instability). Therefore, in conclusion, the MANDOP analysis constitutes a new tool now available to detect the circulations acting at convective scale, but also at mesoscale, to test in particular the instabilities responsible for the organization at this scale of the precipitation field.
Fig. 12. Horizontal field of absolute wind deduced from DVAD analysis at 2 km altitude (a), 3 km (b), and 5 km (c). The distance between two grid points represents a velocity vector scale of 15 m s⁻¹.
Acknowledgments. The LANDES-FRONT S 84 experiment is the result of a cooperative work involving several French laboratories and in particular the Etablissement d'Etudes de et Recherches Météorologiques (EERM) which provided radiosounding data of this study. In addition to the contributions of the participating Institutes, financial support was provided by the Institut National des Sciences de l'Univers (Programme Atmosphère Météorologique). Special recognition must be made to Dr. Peter Golé, Dr. Paul Amayenc, Dr. Gérard Belmont, Dr. Catherine Gloaguen and Dr. Françoise LeCa for their helpful comments.

APPENDIX A

List of Symbols

Indices
index $i$ refers to wind components; $i = 1–3$
index $j$ refers to radars; $j = 1–3$ in FRONTS 84 field experiment
index $\mid$ refers to coordinates; $\mid = 1–3$, 3 referring to vertical coordinate
index $k$ is the current index for the order of development

$U_i$ wind components
$V_i$ analytical representation of the wind components
$u_j$ radial velocity observed from radar $j$, which is the projection of the hydrometeors' velocity on the radar direction of observation, since radar waves are scattered by hydrometeors; the horizontal air wind component is identical to the hydrometeors' horizontal velocity; the vertical component of the observed hydrometeors' velocity is the sum of the vertical air wind and of the terminal fall velocity of the hydrometeors

$V_j$ analytical representation of the radial velocity observed from radar $j$
$u_j$ radial air wind
$v_j$ analytical expression of the radial air wind
$V_i$ terminal fall velocity of the hydrometeors
$X_1$, $X_2$, $X_3$ coordinates; altitude $X_3$ is also noted $z$
$X_{min}$, $X_{max}$ limit coordinates of the retrieved zone

$X_j$ radar $j$ coordinates
$r_j$ radial range relative to the radar $j$: $r_j = \{ \sum_{j=1}^{10} (X_j - X_{ij})^2 \}^{0.5}$
$X_{sl} = 0.5(X_{max} + X_{min})$
$X_{dl} = 0.5(X_{max} - X_{min})$
$x_1$ reduced coordinates. Reduced coordinates are chosen in order to have a base of orthonormal functions in the retrieval domain (e.g., interval $(-1, +1)$ for Legendre polynomials):

$\alpha_j = X_{dl}/(1.5)^{0.5}$
$\beta_j = X_{sl} - X_{ij}$
$L_{ij}$ direction cosines of the components $U_i$ for the considered radar $j$:
$L_{ij} = (X_i - X_{ij})/r_j$
$\Sigma_c$ means extension of sum to experimental points in the whole retrieval domain;
$\Sigma_g$ means extension of sum to experimental points at ground;
$\Sigma_i$ means sum over the number of radars;
$\Sigma_k$ means sum over $K$, with $K$ varying between 1 and $N_i$, if $i = 1$, $N_1 + 1$ and $N_1 + N_2$, if $i = 2$,
$N_1 + N_2 + 1$ and $N$, if $i = 3$

APPENDIX B

System Matrix Condition

The contribution of each radar $j$ to matrix $C$ terms $C_{Kj}$ [see Eqs. (15a–c) and (17)] have orders of magnitude given by Table B1, as a function of $K$ and $K'$.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$N_1$</th>
<th>$N_1 + N_2$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
</tr>
<tr>
<td>$N_1$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
</tr>
<tr>
<td>$N_1 + N_2$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
</tr>
<tr>
<td>$N$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
<td>$L_{ij}L_{ij}$</td>
</tr>
</tbody>
</table>

$L_{ij}$ are direction cosines of velocity components for the considered radar $j$. $L_{ij}$ is about ten times less than $L_{ij}$ or $L_{ij}$, matrix $C$ appears to be ill-conditioned. Three regions may be defined in Table B:

- the region where $K$ and $K'$ are both comprised between 1 and $N_1 + N_2$ is characterized by magnitude 1;
- the italic written regions where $K$ (or $K'$) is comprised between 1 and $N_1 + N_2$, and $K'$ (or $K$) is greater than $N_1 + N_2$, are characterized by magnitude 1/10;
- the relief written region where both $K$ and $K'$ are greater than $N_1 + N_2$ is characterized by magnitude 1/100.

Terms including $L_{ij}$ are then not so well retrieved as the other. A first possibility could be changing vertical variable $W (=V_j)$ by $W/10$, which would induce changing $L_{ij}$ by 10 $L_{ij}$. Corresponding $C$ matrix would be better conditioned.

Adding constraints constitutes another possibility. Appendices C and D show that matrices $C'$ and $C''$
respective expressing equation constraint and ground-level condition lead to matrices better conditioned than matrix \( \mathbf{C} \).

**APPENDIX C**

**Matricial Expression of Density Continuity Equation**

It can be shown that Eq. 22 [see section 2c(1)] is equivalent to

\[
\sum_{K'} C'_{KK'} b_{K'} = 0 \quad \text{for} \quad K = 1, 2, \ldots, N
\]

where \( C'_{KK'} \) is the element of a matrix \( \mathbf{C'} \), sum of four matrices \( \mathbf{E}, \mathbf{F}, \mathbf{F}' \) and \( \mathbf{G} \), the analytical elements of which are

\[
E_{KK'} = \sum_{e} \left\{ 4 \rho^2 / d(K)d(K') \right\} g_k g_{k'}
\]

for any \( K \) and \( K' \)

\[
F_{KK'} = \sum_{e} \left\{ -\rho / H \right\} \{ 2 \rho / d(K') \} g_k g_{k'}
\]

for \( K > N1 + N2 \) and any \( K' \)

\[
G_{KK'} = \sum_{e} \left\{ \rho^2 / H^2 \right\} g_k g_{k'}
\]

for \( K' > N1 + N2 \) and any \( K \)

\( d(K) \) (also noted \( dX_i = 2X_0d_i \)) is the retrieval domain width along \( X_i \) direction: its value is \( dX_1, dX_2 \) or \( dX_3 \) according as \( K \) is less than \( N_1 + 1 \), comprised between \( N_1 + 1 \) and \( N_1 + N_2 \), or greater than \( N_1 + N_2 \). Density \( \rho \) is given by \( \rho = \rho_0 \exp[-(z - z_0)/H] \), \( H \) being the corresponding scale height.

The four matrices \( \mathbf{E}, \mathbf{F}, \mathbf{F}' \) and \( \mathbf{G} \) have the following structure in which hatched zones indicate useful (non-zero) parts.

\[
\begin{array}{c c c c}
K & K' \\
\hline
E & F & K' & G
\end{array}
\]

\( \mathbf{E}, \mathbf{F} + \mathbf{F}' \) and \( \mathbf{G} \) are symmetric matrices. Adding them [after multiplying by a factor \( \lambda'; \) see Eq. (31)] to matrix \( (\lambda \mathbf{C} + \lambda' \mathbf{C}') \) improves total system matrix condition since it reinforces terms implying the vertical direction \( X_3 \) with the following respective weighting contributions:

<table>
<thead>
<tr>
<th>1/(dX_1 dX_1)</th>
<th>1/(dX_1 dX_2)</th>
<th>1/(dX_1 dX_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/(dX_2 dX_1)</td>
<td>1/(dX_2 dX_2)</td>
<td>1/(dX_2 dX_3)</td>
</tr>
<tr>
<td>1/(dX_3 dX_1)</td>
<td>1/(dX_3 dX_2)</td>
<td>1/(dX_3 dX_3)</td>
</tr>
</tbody>
</table>

\( K' \) for \( \mathbf{E} \)

<table>
<thead>
<tr>
<th>0</th>
<th>0</th>
<th>-2\rho^2/(H dX_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-2\rho^2/(H dX_2)</td>
</tr>
<tr>
<td>-2\rho^2/(H dX_1)</td>
<td>-2\rho^2/(H dX_2)</td>
<td>-4\rho^2/(H dX_3)</td>
</tr>
</tbody>
</table>

\( K' \) for \( \mathbf{F} + \mathbf{F}' \)

<table>
<thead>
<tr>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>\rho^2/H^2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>\rho^2/H^2</td>
</tr>
</tbody>
</table>

\( K' \) for \( \mathbf{G} \)
APPENDIX D

Matricial Expression of Ground Level Condition

Condition (26) [see §2c(2)] is equivalent to

$$\sum_{K'} C_{K'K} b_{K'} = 0, \text{ for } K = 1, 2, \cdots, N.$$  

$C_{K'K}$ is the element of matrix $C^*$ the expression of which is

$$C_{K'K}^* = \sum_{g} O_{K'K} g_{K} g_{K'}.$$  

The following Table C gives $O_{K'K}$ as a function of $K$ and $K'$:

| $K$  | 1 \ldots N_{1} | \ldots | N_{1} + N_{2} | \ldots | N |
|-----|----------------|------|-------------|------|
| $N_{1}$ | $C_{a}^{2}$ | $C_{a} S_{a}$ | $-C_{a}$ |
| $N_{1} + N_{2}$ | $C_{a} S_{a}$ | $S_{a}^{2}$ | $-S_{a}$ |
| $N$ | $-C_{a}$ | $-S_{a}$ | 1 |

where $C_{a}$ and $S_{a}$ are given by $C_{a} = \cos \xi \tan \alpha$ and $S_{a} = \sin \xi \tan \alpha$, and where $\alpha$ and $\xi$ are orography information defined in section 2c.

Matrix $C_{K'K}^*$ is symmetric. Table C shows that $C_{a}$ and $S_{a}$ are less than 1 if slope $\tan \alpha$ is less than 45°: in this case, matrix $C^*$ condition tends to compensate matrix $C$ condition. If orography is not taken into account, $C_{a}$ and $S_{a}$ are null. In both cases, adding $\lambda C^*$ to $\lambda C$ should provide better condition for system matrix ($\lambda C + \lambda C^*$).

APPENDIX E

Analytical Determination of the Wind: Coefficients Independently Retrieved

If the three wind components are expressed on the three axes as Legendre polynomials up to the same order $n$, the number of parameters to be determined is $N = 3 n^3$. We shall see in detail, on the simple example of $n = 2$, how many and which components are directly retrievable when data originate from only one, or two radars, in the absence of physical conditions on the wind; i.e., without the continuity equation being taken into account. A generalization to any order $n$ will then appear as straightforward. Equation (10) may be transformed into

$$r_{p} \nu_{j} = \sum_{p} \sum_{q} \sum_{r} \left[ (\alpha_{1} P_{2} + \beta_{1}) P_{p} Q_{q} R_{r} b_{l} + (\alpha_{2} Q_{2} + \beta_{2}) P_{p} Q_{q} R_{r} b_{l} + (\alpha_{3} R_{2} + \beta_{3}) P_{p} Q_{q} R_{r} b_{l} \right]$$  

(E1)

with the following definition for index $J_{i}$: each triplet $(P_{p} Q_{q} R_{r})$ is associated with three values of index $J_{i}$ respectively corresponding to each component $V_{i}$ through the expression:

$$J_{i} = J(p, q, r, i) = r + (q - 1)n + (p - 1)n^2 + (i - 1)n^3 \quad \text{(E2)}$$

1. Case $n = 2$ with one radar

In the case $n = 2$, equation (E1) may be completely developed as follows:

$$r_{p} \nu_{j} = s_{1p} + s_{2p} Q_{2} + s_{3p} Q_{2} + s_{4p} R_{2} + s_{5p} Q_{2} Q_{2} + s_{6p} Q_{2} R_{2} + s_{7p} P_{2} R_{2}$$

$$+ s_{8p} P_{2} Q_{2} + s_{9p} P_{2} Q_{2} + s_{10p} P_{2} R_{2} + s_{11p} P_{2} Q_{2} Q_{2} + s_{12p} P_{2} Q_{2} R_{2} + s_{13p} P_{2} R_{2} R_{2}$$

$$+ s_{14p} P_{2} Q_{2} Q_{2} + s_{15p} P_{2} Q_{2} R_{2} + s_{16p} P_{2} R_{2} R_{2}$$

$$\quad + s_{17p} P_{2} Q_{2} Q_{2} + s_{18p} P_{2} Q_{2} R_{2} + s_{19p} P_{2} R_{2} R_{2}$$

(E3)

in which $s_{ij}$ are known expressions.

The twelve coefficients $b_{1}, b_{2}, b_{3}, b_{4}, b_{5}, b_{8}, b_{11}, b_{12}, b_{15}, b_{16}, b_{18}, b_{20}, b_{22}$ and $b_{24}$ are readily determined since they are implied in the "highest order" terms (those containing $a_{1} P_{2}, a_{2} Q_{2}$, and $a_{3} R_{2}$) which are derived without ambiguity. The other twelve coefficients are related by the following eight equations, in which the expressions $s_{ij}$ are known, since they are functions of the already determined first twelve coefficients and of $s_{ij}$:

$$\beta_{11} b_{1} + \beta_{21} b_{9} + \beta_{31} b_{17} = s_{11} \quad \text{constant term} \quad \text{(E4)}$$

$$\alpha_{1} b_{1} + \beta_{21} b_{3} + \beta_{31} b_{21} = s_{21} \quad P_{2} \text{ term} \quad \text{(E5)}$$

$$\alpha_{2} b_{4} + \beta_{11} b_{5} + \beta_{31} b_{19} = s_{31} \quad Q_{2} \text{ term} \quad \text{(E6)}$$

$$\alpha_{3} b_{17} + \beta_{11} b_{2} + \beta_{21} b_{10} = s_{41} \quad R_{2} \text{ term} \quad \text{(E7)}$$

$$\alpha_{4} b_{1} + \alpha_{2} b_{13} + \beta_{31} b_{23} = s_{51} \quad P_{2} Q_{2} \text{ term} \quad \text{(E8)}$$

$$\alpha_{2} b_{10} + \alpha_{3} b_{19} + \beta_{11} b_{4} = s_{61} \quad Q_{2} R_{2} \text{ term} \quad \text{(E9)}$$

$$\alpha_{3} b_{21} + \alpha_{2} b_{21} + \beta_{11} b_{14} = s_{71} \quad P_{2} R_{2} \text{ term} \quad \text{(E10)}$$

$$\alpha_{4} b_{4} + \alpha_{2} b_{14} + \alpha_{3} b_{23} = s_{81} \quad P_{2} Q_{2} R_{2} \text{ term} \quad \text{(E11)}$$

These eight equations are not independent, since they are in turn linked by the relation:

$$\alpha_{3} \beta_{21} (a_{1} s_{31} - \beta_{11} s_{51}) + \alpha_{1} \beta_{31} (a_{2} s_{41} - \beta_{21} s_{61})$$

$$+ \alpha_{2} \beta_{11} (a_{3} s_{21} - \beta_{31} s_{71}) - \alpha_{1} a_{2} a_{3} s_{11} + \beta_{11} \beta_{21} s_{18} = 0 \quad \text{(E12)}$$

A maximum of seven independent equations exist then between the 12 unknown coefficients $b_{1}, b_{2}, b_{3},$
\[ b_4, b_9, b_{10}, b_{13}, b_{14}, b_{17}, b_{19}, b_{21}, \text{and } b_{23}, \text{so that at least 5 of them have to be derived independently. As an example, if } b_1, b_{10}, b_{13}, b_{17} \text{ and } b_{23} \text{ are fixed, the other coefficients } (b_2, b_3, b_4, b_5, b_{14}, b_{19} \text{ and } b_{21}) \text{ can be retrieved.}

Remark: In some cases, the choice of five fixed coefficients leads to seven nonindependent coefficients, for example, if \( b_1, b_2, b_3, b_4 \) and \( b_5 \) are fixed, it is possible to calculate \( b_{10}, b_{17}, \) and \( b_{19}, \) but \( b_{13}, b_{14}, b_{21} \) and \( b_{23} \) are linked; in this case only six coefficients could be independently derived. To summarize the results at order \( n = 2 \) with one radar, on a total of \( 3n^3 = 24 \) coefficients, it appears from equation (E1) that the high order terms, which are \( 3n^2 = 12 \) in number are readily computable, the other twelve being related by, at most, \( (n^3 - 1) = 7 \) independent equations, so that at least five coefficients must be fixed.

2. Case \( n = 2 \) with two radars

When two radars are available, the first 12 high order coefficients are derivable as before (since \( \alpha_i \) does not depend on the radar), and the other 12 coefficients are related by 8 more equations similar to (E4)–(E11):

\[
\begin{align*}
\beta_{12}b_1 + \beta_{23}b_2 + \beta_{32}b_17 &= s_{12} \\
\vdots & \vdots \\
\alpha_1b_4 + \alpha_2b_{14} + \alpha_3b_{23} &= s_{82}
\end{align*}
\]  
(E13)

(E14)

Equations E13–E20 are related by a relation similar to Eq. E12, yielding a maximum of 14 equations between 12 unknowns. However, these 14 equations are not independent but are in turn linked by the three relations:

\[
s_{81} = s_{82} 
\]

(E21)

\[
\begin{align*}
\alpha_3\frac{s_{51} - s_{52}}{\beta_{31} - \beta_{32}} + \alpha_1\frac{s_{61} - s_{62}}{\beta_{11} - \beta_{12}} + \\
\alpha_2\frac{s_{71} - s_{72}}{\beta_{21} - \beta_{22}} &= s_{81} = s_{82}
\end{align*}
\]  
(E22)

\[
\begin{align*}
\alpha_3\gamma_1^2\gamma_2\left[\alpha_2\gamma_3(s_{51} - s_{52}) + \gamma_2(\beta_{32}s_{21} - \beta_{31}s_{22})\right] + \\
+ \alpha_1\gamma_2^2\gamma_3\left[\alpha_3\gamma_1(s_{61} - s_{62}) + \gamma_3(\beta_{13}s_{61} - \beta_{23}s_{62})\right] + \\
+ \alpha_2\gamma_3^2\gamma_1\left[\alpha_1\gamma_2(s_{61} - s_{62}) + \\
+ \gamma_1(\beta_{23}s_{71} - \beta_{21}s_{72})\right] &= 0
\end{align*}
\]  
(E23)

with the additional definition:

\[
\gamma_i = \beta_{ii} - \beta_{ij}
\]

(E24)

Finally, 11 independent equations relate twelve unknowns, so that one of them has to be fixed.

3. Generalization to any value of \( n \) (with one radar)

In the general case, the equation (E1) may be rewritten as:

\[
\begin{align*}
r_jv_j &= \alpha_1P_2 \sum_r \sum_{q_r} P_r b_{j1} \sum_{q_r} Q_r R_r \\
+ \alpha_2Q_2 \sum_r \sum_{q_r} Q_r b_{j2} \sum_{q_r} P_r R_r \\
+ \alpha_3R_2 \sum_r \sum_{q_r} R_r b_{j3} \sum_{q_r} P_r Q_r \\
+ \beta_{j3} \sum_r \sum_{q_r} R_r b_{j3} \sum_{q_r} P_r Q_4 R_r.
\end{align*}
\]  
(E25)

It is clear from Eq. (E25) that the "highest order" terms are the terms in \( \alpha_1, \alpha_2, \alpha_3; Q_1, \alpha_3 R_2, \) their number is obviously \( 3 \times n^3/n = 3n^2 \), they immediately provide the derivation of \( 3n^2 \) coefficients \( b_{ii} \), since each of these coefficients only appears in one of these terms. The other coefficients are related by at most \( n^3 \) relations coming from the identification of the other terms of the development of \( r_jv_j \). If those \( n^3 \) relations were independent, reasoning step by step in the same way would provide eight independent relations at the order 2, which is not true, as demonstrated.

In conclusion, at order \( n \), at most \( (n^3 - 1) \) independent relations exist between the \( (3n^3 - 3n^2) \) coefficients not directly derivable when data come from only one radar.

4. Generalization to any value of \( n \) with two radars

Turning back to the case \( n = 2 \), it appears that the same twelve high order coefficients are directly derivable as with one radar. As to the other twelve coefficients, they are related by at most 11 \( (= 2n^3 - 5) \) independent relations, so that at least one coefficient has to be known. Generalization to any value of \( n \) may be done as before: if there were more than \( (2n^3 - 5) \) independent relations at order \( n \), between the \( (3n^3 - 3n^2) \) not directly computable coefficients, it would also be the case for the orders \( n - 1, n - 2 \), etc. until the order \( n = 2 \), which is not true, as demonstrated. In conclusion, at order \( n \), at most \( (2n^3 - 5) \) independent relations exist between the \( (3n^3 - 3n^2) \) coefficients not directly derivable, when data come from only two radars in the absence of an additional condition. Then, at least \( (n^3 - 3n^2 + 5) \) coefficients must be independently known in this case.

5. Other orthonormal bases

The same conclusions hold for other orthonormal bases. Since the terms of the development of \( r_jv_j \) were, on a Legendre base, of the form:

\[
\alpha_1P_2 P_r Q_4 R_r + \beta_{j3} R_r P_r Q_4 R_r
\]
Now, they are on any base of the more general form:

$$\alpha_1 P_2 p_\theta d r + \beta_{ij} p_\theta d r,$$

We may take the example of the order $n = 2$ without loss of generality: in this case, if the system of equations (E4) to (E11) contains $N_{\text{ind}}$ linearly independent equations in a Legendre development, it will contain at least $N_{\text{ind}}$ linearly independent equations in any development.

REFERENCES


