Effects of Raindrop-Size Distribution Variation within the Radar Scattering Volume on Radar Observables

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ABSTRACT

Dual linear polarization weather radars measure as primary observables the mean power $\bar{P}_H$ and $\bar{P}_V$, corresponding to returns at horizontal and vertical polarizations, respectively. Differential reflectivity Z_{DR} is defined as the ratio between these two measurements. Under the assumption of an exponential drop-size distribution, characterized by the two parameters N_0 and D_0, it has been shown that Z_{DR} may be used to estimate the median volume diameter D_0, following which the parameter N_0 and, therefore, other drop-size distribution-dependent quantities, may be determined from the horizontal reflectivity Z_H.

In this paper the effects of reflectivity gradients, due to the variation of the drop-size distribution within the radar scattering volume, on the radar observables (Z_H, Z_{DR}) and derived rainfall rates are examined for radar observations with a stationary antenna. The bias of the estimates, their standard errors, and the optimum receiver response are computed for power law and logarithmic receivers. Finally, for the special case of the square law receiver, the contrasting effects due to either similar or opposing signs of the gradients of the parameters N_0 and D_0 are evaluated.

1. Introduction

Dual linear polarization weather radars usually measure the mean received powers $\bar{P}_H$ and $\bar{P}_V$ by alternately integrating a number of returns in the space–time domain at horizontal and vertical polarizations. Differential reflectivity Z_{DR} is defined as the ratio between the two measurements. Under the assumption of an exponential drop-size distribution, characterized by the two parameters N_0 and D_0, Seliga and Bringi (1976) showed that Z_{DR} may be used to estimate the median volume diameter D_0, following which the parameter N_0 and an improved estimate of rainfall rate R may be determined from the horizontal reflectivity factor.

Rogers (1971), Zawadzki (1982), and Scarchilli et al. (1986) have shown that reflectivity gradients within the radar measurement cell can significantly bias reflectivity measurements. This bias or error depends on the receiver transfer function as well as the number of averaged samples used for estimating the reflectivities. Moreover, depending on the reflectivity gradient, there is an optimum receiver transfer function for estimating the mean input power.

This paper extends the previous studies to analyze the effects of reflectivity gradients due to the variation of the drop size distribution within the radar scattering volume on the radar observables (Z_H, Z_V) and on derived parameters, such as differential reflectivity Z_{DR} and rainfall rate R. The bias of the estimates and their corresponding standard errors or fractional standard deviations as a function of receiver response characteristics are evaluated by considering radar observations obtained with a stationary antenna.

2. Model formulation

Assume that rain is characterized by an exponential drop size distribution (DSD)

$$N(D) = N_0 \exp(-3.67(D/D_0)) \quad (1)$$

where N_0 ($\mathrm{mm^{-1}\,m^{-3}}$) defines the intensity of the distribution, D_0 (mm) is the median volume diameter, and D (mm) is the spherical equivolume diameter of oblate spheroidal-shaped raindrops. At S-band and for $N(D)$ given by (1), very good approximations for the

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horizontal reflectivity factor Z_H (mm^6 m^{-3}) and differential reflectivity Z_{DR} (dB) have been obtained by Ulbrich and Atlas (1984)

$$Z_H = 0.08 N_0 D_0^7 \tag{2}$$

$$Z_{DR} = 10 \log(Z_H/Z_V) = 0.76 D_0^{1.55}. \tag{3}$$

Rainfall rate R (mm h^{-1}) is defined by

$$R = 0.6\pi 10^{-3} \int_0^\infty D^3 v(D) N(D) dD \tag{4}$$

where $v(D)$ (m s^{-1}) is the terminal fallspeed in still air of raindrops of size D and is approximated by Atlas and Ulbrich (1977)

$$v(D) = 3.77 D^{0.67}. \tag{5}$$

The above equations may be combined to give

$$R = 0.24 \times 10^{-3} N_0 D_0^{4.67} \tag{6}$$

$$R = 1.93 \times 10^{-3} Z_H (Z_{DR})^{-1.50}. \tag{7}$$

Since raindrops are essentially Rayleigh scatterers at both S- and C-band wavelengths, the above equations are considered good approximations at $\lambda = 5.4$ cm, the wavelength used for the fractional standard deviation sample computations presented in this paper (see section 5). This is also the wavelength of the Polar 55-C radar that will be used in Italy to experimentally evaluate the results of this paper.

In order to improve precipitation estimates derived from radar measurements, radar observables are often integrated along range, azimuth, and/or elevation to increase the sample size and thus reduce the variance of the estimates. As commonly observed experimentally, the radar reflectivity factor Z expressed in dBZ (i.e., 10 log$Z$) often varies linearly in space within distances larger than, but comparable to, the beamwidth (Zawadzki 1982). The analysis in this paper assumes that the variation of ($Z_{H,V}$, Z_{DR}) occurs along a single Cartesian coordinate x. If 10 logZ_H varies linearly in the space, then both N_0 and D_0 must vary exponentially, as one can see from (2)

$$N_0(x) = N_0^* \exp(0.23 G_{N_0} x) \tag{8a}$$

$$D_0(x) = D_0^* \exp(0.23 G_{D_0} x) \tag{8b}$$

where N_0^* and D_0^* represent the values of N_0 and D_0 at the center ($x = 0$) of the measurement cell and (G_{N_0}, G_{D_0}) are the corresponding reflectivity gradient parameters in dB km^{-1}, which are related to the horizontal reflectivity gradient G_H in dB km^{-1} by: $G_H = G_{N_0} + 7G_{D_0}$.

From a general view point we should analyze four possible cases corresponding to the following different signs of the gradients of the microphysical parameters: 1) $G_{N_0} > 0$, $G_{D_0} > 0$; 2) $G_{N_0} < 0$, $G_{D_0} < 0$; 3) $G_{N_0} > 0$, $G_{D_0} < 0$; and 4) $G_{N_0} < 0$, $G_{D_0} > 0$. However, as it will be demonstrated in the following section, cases 1) and 2) produce identical results and cases 3) and 4) likewise produce similar results. Thus, there are two distinct cases of interest, one with gradients of like signs and the other with gradients of opposite signs. The theory includes all cases. However, computations will focus on the case of like signs with the case of opposite signs demonstrated in a separate section.

In this analysis the antenna gain is assumed to be uniform across the beam, and the precipitation field is taken to vary in range only. It should be noted that under these same precipitation conditions (with a linear variation of dBZ in range) a Gaussian radiation pattern would yield, except for a constant, the same radar reflectivity observations (Zawadzki 1982). Radar signal attenuation along the propagation path that can affect the precipitation measurements mostly for C- and X-band weather radars is not considered. Problems associated with antenna rotation will be examined in a future paper.

3. Radar observables

In the stationary antenna case, the same number N of meteorological echoes are integrated in each resolution cell. Estimating the horizontal mean input power $\bar{P}_H$ obtained through a power law receiver with exponent b, where the output Y is related to the input power by

$$Y = P_H{}^b, \tag{9}$$

produces a bias due to the signal fluctuation,

$$B(b) = \frac{10}{b} \log \Gamma(b + 1), \tag{10}$$

where Γ is the gamma function, given by Scarchilli et al. (1986).

This bias B is zero for the square law receiver ($b = 1$), and, for the special well-known case of a logarithmic receiver ($Y = \log P_H$), the bias is equal to -2.5 dB, which can be obtained by setting $b = 0$ in (10) as has been demonstrated by Scarchilli et al. (1986). Thus, the average values of a reflectivity factor parameter Z [with the bias $B(b)$ due to signal fluctuations removed] within a measurement cell of length d for the logarithmic and the power law receivers are given by

$$\overline{(10 \log Z_H)} + 2.5 = \frac{1}{d} \int_{-d/2}^{d/2} 10 \log Z_H dx + 2.5 \tag{11a}$$

and

$$(10 \log \overline{Z_H}) - B(b) = \frac{10}{b} \log\left(\frac{1}{d} \int_{-d/2}^{d/2} Z_H{}^b dx\right) - B(b). \tag{11b}$$

These equations may be used to determine the dependence of the average values of (Z_H, Z_V) on $N(D)$ and its variability within the measurement cell by appropriate combinations of (2), (3), and (8). Taking account of (2) we can substitute (8) into (11) and perform the integration yielding

$$\overline{(10\log Z_{\rm H})} + 2.5 = 10\log[0.08N_0^*(D_0^*)^7] + 2.5 \quad (12a)$$

$$(10\log\overline{Z_{\rm H}}) - B(b) = 10\log[0.08N_0^*(D_0^*)^7] + \frac{10}{b}\log(\sinh\alpha/\alpha) - B(b) \quad (12b)$$

where

$$\alpha = 0.23 \times b(G_{N_0} + 7G_{D_0})d/2. \quad (13)$$

Equation (12a) gives the horizontal reflectivity expressed in dBZ at the center of the measurement cell. Equations (12) represent the estimates of the mean horizontal reflectivity, all of which are biased because of the reflectivity gradient, except for case of the square law receiver (Rogers 1971).

If the reflectivity field of interest is taken to be $Z_{\rm H}$ and is allowed to vary exponentially in space, then $Z_{\rm V}$ would not in general vary in the same way since it is related to $Z_{\rm H}$ through (2) and (3). Thus,

$$\overline{(10\log Z_{\rm V})} + 2.5 = 10\log[0.08N_0^*(D_0^*)^7] - 0.76(D_0^*)^{1.55}\frac{\sinh\beta}{\beta} + 2.5 \quad (14a)$$

$$(10\log\overline{Z_{\rm V}}) - B(b) = \frac{10}{b}\log\left\{\overline{Z_{\rm H}}^b + [N_0^*(D_0^*)^7]^b \times \sum_{k=1}^{\infty}(-1)^k\Delta(k)\frac{\sinh\gamma(k)}{\gamma(k)}\right\} - B(b) \quad (14b)$$

where

$$\beta = 0.36G_{D_0} \times d/2 \quad (15)$$

$$\gamma(k) = \alpha + 0.36kG_{D_0} \times d/2 \quad (16)$$

$$\Delta(k) = [0.175b(D_0^*)^{1.55}]^k/k!. \quad (17)$$

Equations (14) represent the biased estimates of the mean vertical reflectivity; generally, these biases due to the reflectivity gradient are different from those corresponding to the estimates given by (12). Note that the square law receiver case always gives an unbiased estimate for both $Z_{\rm H}$ and $Z_{\rm V}$.

Since the differential reflectivity $Z_{\rm DR}$ is a parameter derived from the ratio of the measurements between the horizontal and vertical reflectivities (3), a power law receiver estimator of the mean differential reflectivity $\overline{Z_{\rm DR}}$ can be defined as

$$\hat{Z}_{\rm DR}^{P_1} = \frac{10}{b}\log\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{Z_{\rm H}}{Z_{\rm V}}\right)^b\right]_i \quad (18)$$

where n is the number of spatially independent samples. The mean value of (18) can be obtained by replacing the summation by an integration over the spatial variable x,

$$Z_{\rm DR}^{P_1}({\rm dB}) = \frac{10}{b}\log\left[\frac{1}{d}\int_{-d/2}^{+d/2}\left(\frac{Z_{\rm H}}{Z_{\rm V}}\right)^b dx\right] = \frac{10}{b}\log\left\{1 + \sum_{k=1}^{\infty}\Delta(k)\frac{\sinh b[\gamma(k)-\alpha]}{b[\gamma(k)-\alpha]}\right\}. \quad (19)$$

Note that $\hat{Z}_{\rm DR}^{P_1}$ is a function of D_0^* and G_{D_0} only and, therefore, is a highly biased estimator of the median volume diameter $(D_0)_t$ of the DSD within the total measurement cell. Indeed, $(D_0)_t$ can be obtained from

$$\sum_{i=1}^{n}\int_0^{(D_0)_t}(N_0)_iD^3\exp\left[-3.67\frac{D}{(D_0)_i}\right]dD = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\infty}(N_0)_iD^3\exp\left[-3.67\frac{D}{(D_0)_i}\right]dD \quad (20)$$

where $(N_0)_i$ and $(D_0)_i$ are the characteristic parameters of the DSD in the ith resolution cell.

By replacing the summation with an integration and introducing (8), it is easily observed that $(D_0)_t$ is a nonlinear function of D_0^*, N_0^*, G_{D_0}, and G_{N_0} within the measurement cell. Moreover, in the limiting case when the average in each resolution cell is obtained from a single H–V pulse pair sample, (18) becomes the ratio estimator described by Bringi et al. (1983) that is highly biased and characterized by an indeterminate variance. For these reasons, the estimator of $\overline{Z_{\rm DR}}$ based on the difference of the estimates in dB of the mean horizontal and vertical reflectivities, which is the most commonly used form of averaging, is preferred.

Here

$$\hat{Z}_{\rm DR}^{P_2} = \frac{10}{b}\left[\log\frac{1}{n}\sum_{i=1}^{n}(Z_{\rm H}{}^b)_i - \log\frac{1}{n}\sum_{i=1}^{n}(Z_{\rm V}{}^b)_i\right], \quad (21)$$

note that $\hat{Z}_{\rm DR}^{P_2}$ depends on D_0^*, N_0^*, G_{D_0}, and G_{N_0} and gives an unbiased estimate of $\overline{Z_{\rm DR}}$ for the square law receiver. The logarithmic receiver response for this estimate can be obtained from the difference between (12a) and (14a)

$$Z_{\rm DR}^{L} = 0.76(D_0^*)^{1.55}(\sinh\beta/\beta). \quad (22)$$

In contrast to $Z_{\rm DR}^{P_2}$, $Z_{\rm DR}^{L}$ as well as $Z_{\rm DR}^{P_1}$ are functions of D_0^* and G_{D_0} only.

In order to evaluate the dependence of the different estimators of $\overline{Z_{\rm DR}}$ on reflectivity gradients, computations were performed for the power law receiver with b ranging from 0.1 to 1.0 in steps of 0.1, and for the logarithmic receiver by choosing $D_0^* = 1.58$ mm; $N_0^* = 2000$ mm^{-1} m^{-3}; G_{D_0} ranging from 0.1 to 3.7 dB km^{-1} in 0.9 dB steps; and G_{N_0} from 0 to 20 dB km^{-1} in 5 dB steps. For this purpose, the errors

$\epsilon(Z_{DR}^{P_2})$ and $\epsilon(Z_{DR}^{L})$ are defined with reference to the average value of Z_{DR} given by $Z_{DR}^{P_2}(b = 1)$ as follows

$$\epsilon(Z_{DR}^{P_2}) = Z_{DR}^{P_2} - Z_{DR}^{P_2}(b = 1) \quad (23a)$$

$$\epsilon(Z_{DR}^{L}) = Z_{DR}^{L} - Z_{DR}^{P_2}(b = 1). \quad (23b)$$

As shown in Fig. 1 the absolute value of the bias, corresponding to the estimate $\hat{Z}_{DR}^{P_2}$, increases with G_{D_0} and decreases with b. It can be easily demonstrated that the bias increases also with G_{N_0}; however, in all cases the square law receiver ($b = 1$) gives an unbiased estimate as expected from the definition of $\epsilon(Z_{DR}^{P_2})$. Figure 2 shows that the absolute errors, similar to ones expressed by (23) and corresponding to the estimate $\hat{Z}_{DR}^{P_1}$, also increase with G_{D_0} and G_{N_0}, are generally much greater than those for $\hat{Z}_{DR}^{P_2}$ shown in Fig. 1, and are smallest for the logarithmic receiver.

4. Rainfall rate

The mean rainfall rate $\bar{R}$ can be found by integrating (6) along the measurement cell, which gives

$$\bar{R} = 0.242 \times 10^{-3} N_0^*(D_0^*)^{4.67}(\sinh\delta/\delta) \quad (24)$$

where

$$\delta = 0.23(G_{N_0} + 4.67 G_{D_0})(d/2). \quad (25)$$

Since R is derived from (7), the correct reference estimate of $\bar{R}$ is obtained with the square law receiver response that yields

$$\hat{R} = 1.93 \times 10^{-3} \frac{1}{n} \sum_{i=1}^{n} (Z_H)_i (Z_{DR})_i^{-1.50} \quad (26)$$

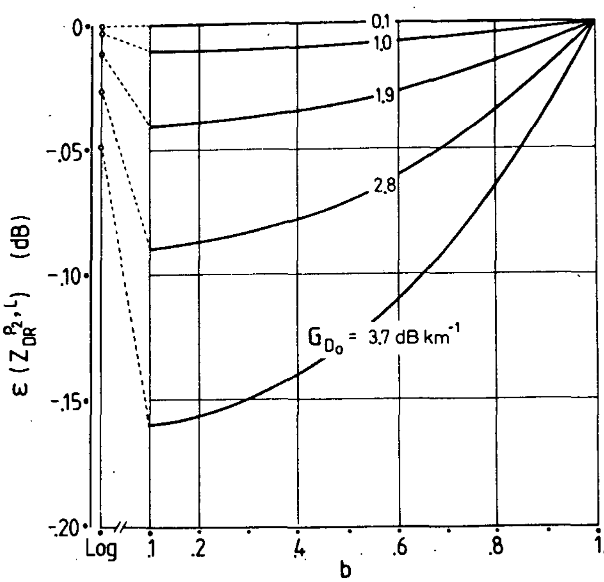

FIG. 1. The biases $\epsilon(Z_{DR}^{P_2})$ and $\epsilon(Z_{DR}^{L})$ as defined in (23a) and (23b) in correspondence with the estimates $\hat{Z}_{DR}^{P_2}$ and $\hat{Z}_{DR}^{L}$, for different values of G_{D_0} (dB km^{-1}) with G_{N_0} = 0 dB km^{-1}; the parameter b characterizes the power law receiver.

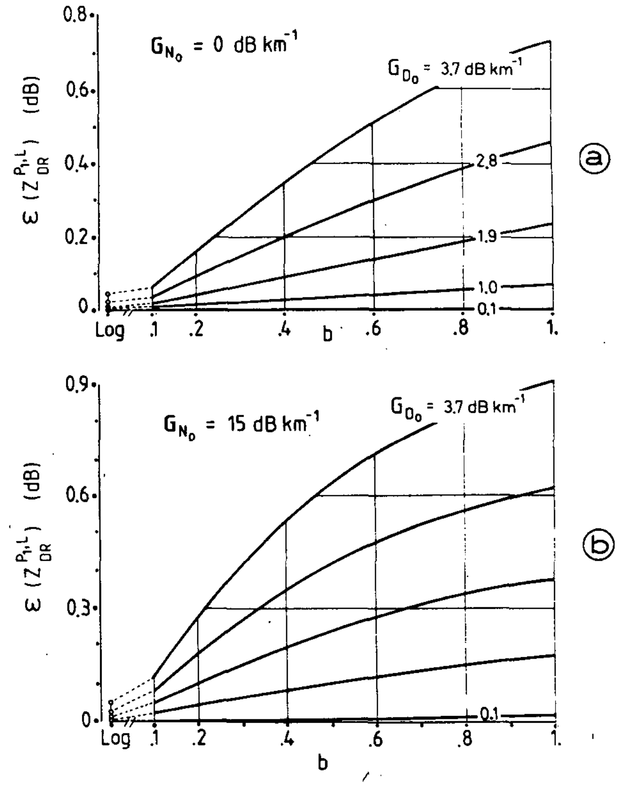

FIG. 2. The biases $\epsilon(Z_{DR}^{P_1})$ and $\epsilon(Z_{DR}^{L})$ corresponding to the estimates $\hat{Z}_{DR}^{P_1}$ and $\hat{Z}_{DR}^{L}$, for different values of G_{D_0} (dB km^{-1}) with (a) G_{N_0} = 0 dB km^{-1} and (b) G_{N_0} = 15 dB km^{-1}; the parameter b characterizes the power law receiver.

where $(Z_H)_i$ and $(Z_{DR})_i$ correspond to values measured in the ith resolution cell and Z_H (mm^6 m^{-3}) and Z_{DR} (dB). The estimate (26) is unbiased for all G_{D_0} and G_{N_0}. To evaluate effects due to the variation of the receiver characteristic along the entire dynamic range of input signal, the following estimators are defined by

$$\hat{R}^{P_2} = 1.93 \times 10^{-3} \left[\frac{1}{n} \sum_{i=1}^{n} (Z_H{}^b)_i\right]^{1/b} (\hat{Z}_{DR}^{P_2})^{-1.50} \quad (27)$$

$$\hat{R}^{L} = 1.93 \times 10^{-3} \times \exp\left[0.23 \frac{1}{n} \sum_{i=1}^{n} (10 \log Z_H)_i\right] (\hat{Z}_{DR}^{L})^{-1.50}. \quad (28)$$

The relative errors due to the estimators $\hat{R}^{P_2}$ and $\hat{R}^{L}$ are defined as

$$\epsilon(\hat{R}^{P_2}) = (\hat{R}^{P_2} - \bar{R})/\bar{R} \quad (29)$$

$$\epsilon(\hat{R}^{L}) = (\hat{R}^{L} - \bar{R})/\bar{R} \quad (30)$$

where the reference value $\bar{R}$ is given in (24).

Figure 3 illustrates the relative errors associated with the average of (27) and (28) for different values of G_{D_0} and for G_{N_0} = 0 and 15 dB km^{-1}. These errors are largest for small b with the greatest errors occurring for the logarithmic receiver and the least at $b = 1$. The

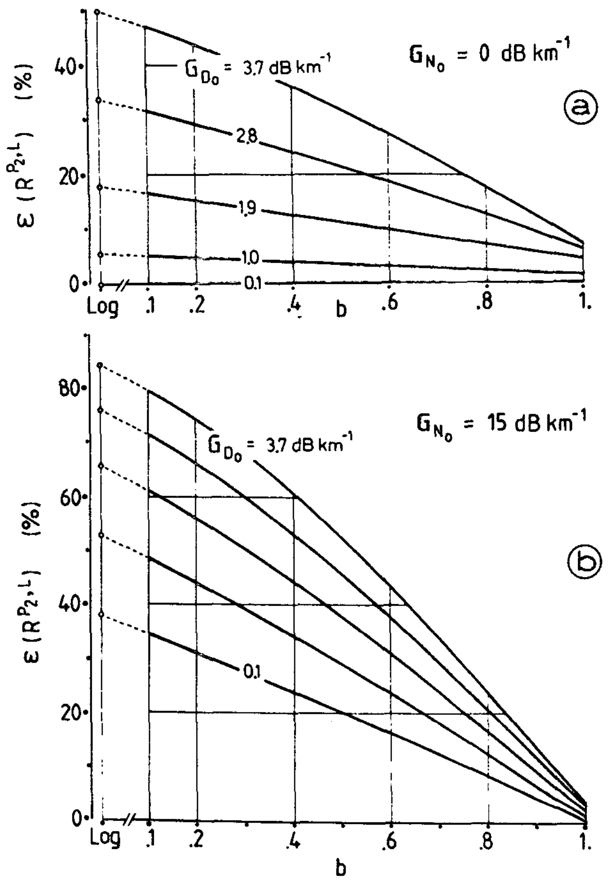

FIG. 3. Relative errors $\epsilon(R^{P_2})$ and $\epsilon(R^L)$ as defined in (29) and (30) in correspondence with the estimates $\hat{R}^{P_2}$ and $\hat{R}^L$, for different values of G_{D_0} (dB km^{-1}) with (a) G_{N_0} = 0 dB km^{-1} and (b) G_{N_0} = 15 dB km^{-1}; the parameter b characterizes the power law receiver.

errors in R^{P_2} also increase with increasing G_{D_0} and G_{N_0}. These sample computations indicate that large biases are expected when log-averaging is used, and that linear power-averaging is relatively insensitive to gradients, produces small errors in $\bar{R}$, and is, therefore, preferred.

5. Standard errors

Generally, the variances of the radar observables M_r depends on both the number of time averaged samples N and the number of resolution cells n over which the respective temporal and spatial average is performed. In the stationary antenna case, time and spatial averages can be separated. Assuming that the resolution cells are independent of each other, the variance of the estimate $\hat{M}_r$ can be written as

$$\mathrm{var}(\hat{M}_r) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{var}(\hat{M}_r)_i \qquad (31)$$

where $(\hat{M}_r)_i$ is the estimate of M_r in the ith cell. Then, the variance of the estimate $\hat{Z}_H{}^b$ of the mean horizontal reflectivity at the output of the power law receiver can be written as

$$\mathrm{var}(\hat{Z}_H{}^b) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{(N_I)_P}[\Gamma(2b+1) - \Gamma^2(b+1)](Z_H)_i^{2b} \qquad (32)$$

where $(N_I)_P$ is the number of independent samples at the receiver output. Following and extending the analysis of Davenport and Root (1958), $(N_I)_P$ can be obtained from

$$\left(\frac{1}{N_I}\right)_P = \sum_{m=-(N-1)}^{N-1}\left[\frac{\Gamma(2b+1)}{\Gamma^2(b+1)} - 1\right]^{-1}\frac{N-|m|}{N^2} \times \sum_{k=1}^{\infty}\left[\frac{b(1-b)(2-b)\cdots(k-1-b)}{k!}\right]^2 \times \rho^{2k}(mT_s) \qquad (33)$$

where ρ is the autocorrelation of the meteorological input signal that is assumed to be of gaussian shape and given by

$$\rho(mT_s) = \exp\left[-8\left(\frac{\pi\sigma_v mT_s}{\lambda}\right)^2\right], \qquad (34)$$

where T_s is the pulse repetition period, σ_v the Doppler width and λ the wavelength (taken to be 5.4 cm here). Rearranging and replacing the summation by an integral in (32) yields

$$\mathrm{var}(\hat{Z}_H{}^b) = \frac{\Gamma(2b+1) - \Gamma^2(b+1)}{n(N_I)_P} \times \left[N_0^* D_0^{*7}\frac{\Gamma(7)}{(3.67)^7}\right]^{2b}\frac{\sinh 2\alpha}{2\alpha}. \qquad (35)$$

The fractional standard deviation FSD of the estimate of the mean horizontal reflectivity expressed in dB can then be approximated by (Papoulis 1967)

$$\mathrm{FSD}(\hat{Z}_H{}^P) \approx \frac{4.34}{b}\left[\frac{\mathrm{var}(\hat{Z}_H{}^b)}{(\overline{Z_H{}^b})^2}\right]^{1/2}. \qquad (36)$$

Substituting (35) and (12a) into (36), it can easily be seen that by increasing the reflectivity gradient, the parameter b, which characterizes the power law receiver, is greater, and the FSD of the estimate of Z_H is increased. The variance of the estimate of $Z_V{}^b$ obtained by the power law receiver can be computed in the same way as in (32) by noting that

$$\frac{1}{d}\int_{-d/2}^{+d/2} Z_V{}^{2b}dx = \left[\frac{\Gamma(7)}{3.67^7}N_0^*(D_0^*)^7\right]^{2b} \times \sum_{k=0}^{\infty}(-2)^k\Delta(k)\frac{\sinh[\gamma(k)+\alpha]}{\gamma(k)+\alpha}. \qquad (37)$$

When utilizing the logarithmic receiver, the variance of the estimates of the horizontal and vertical reflectivities (obtained from sampling of signals whose echo power is exponentially distributed) is independent of Z_H and Z_V in each resolution cell and is given by

$$\text{var}(\hat{Z}^L_{\text{H,V}}) = (5.56)^2/n(N_I)_L \qquad (38)$$

where $(N_I)_L$ is the number of independent samples at the output of the logarithmic receiver,

$$\left(\frac{1}{N_I}\right)_L = \sum_{m=-(N-1)}^{N-1} \frac{6}{\pi^2} \frac{N-|m|}{N^2} \times \sum_{k=1}^{\infty} k^{-2}\rho^{2k}(mT_s), \qquad (39)$$

given by Walker et al. (1980).

As in the previous section, the FSD's of the radar observables and derived parameters were computed for the same values of N_0^* and D_0^* and the same ranges of b, G_{N_0}, and G_{D_0} for a typical number of available samples $N = 128$ and Doppler widths σ_v ranging from 1 to 6 m s^{-1} in intervals of 1 m s^{-1}. Note that to obtain FSD's for n resolution cells, the FSD results given in Figs. 4–10 should be divided by the square root of n.

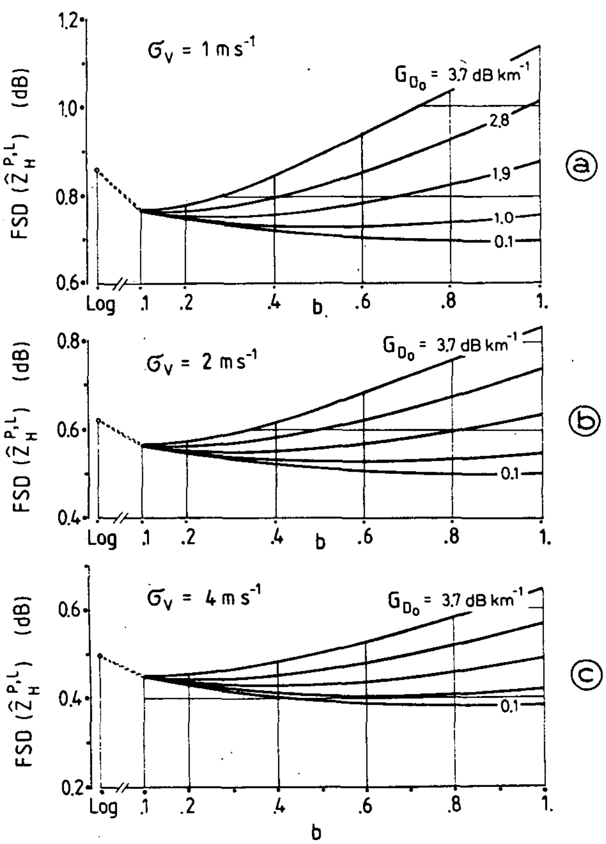

FIG. 4. FSD in dB of the estimates $\hat{Z}_H{}^P$ and $\hat{Z}_H{}^L$, obtained respectively by the power law and logarithmic receiver, for different values of G_{D_0} (dB km^{-1}), G_{N_0} = 0 dB km^{-1}, and the number of averaged samples N = 128 with (a) σ_v = 1 m s^{-1}, (b) σ_v = 2 m s^{-1}, and (c) σ_v = 4 m s^{-1}; the parameter b characterizes the power law receiver.

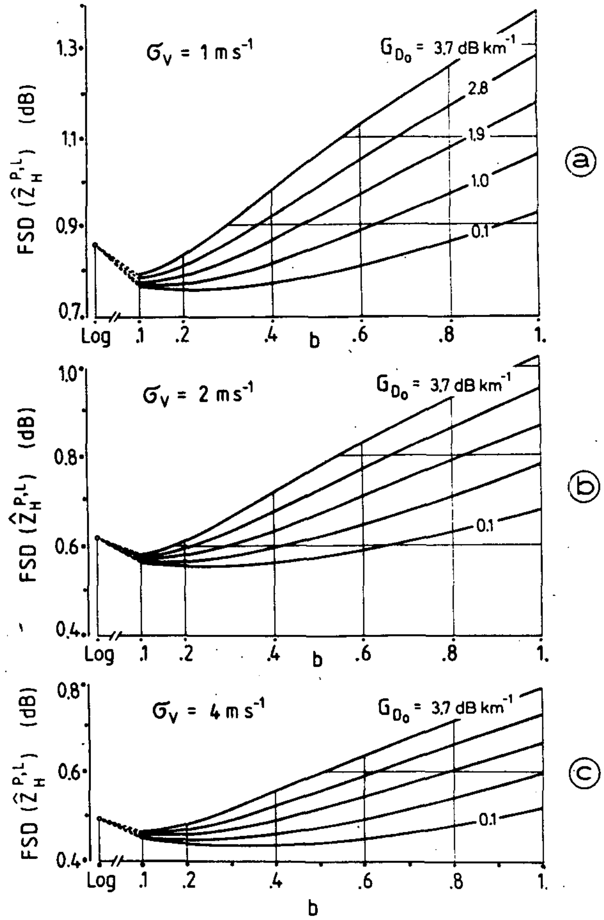

FIG. 5. Same as Fig. 4 except for G_{N_0} = 15 dB km^{-1}.

Figures 4 and 5 show that, as expected, the square law receiver has the minimum FSD of the mean horizontal reflectivity estimate for a uniform reflectivity (i.e., G_{N_0} = 0 and G_{D_0} = 0, here the lowest G_{D_0} = 0.1 dB km^{-1} shows this trend). As the reflectivity gradient increases and the Doppler width decreases, the minimum FSD moves towards smaller values of b.

In the case of alternate sampling and utilizing the logarithmic receiver, the variance of the estimate $\hat{Z}^L_{\text{DR}}$ of the mean differential reflectivity is given by Chandrasekar et al. (1986),

$$\text{var}(\hat{Z}^L_{\text{DR}}) = \frac{(5.56)^2}{n} 2 \sum_{m=-(N-1)}^{N-1} \frac{(N-|m|)}{N^2} \times [|\rho_L(2m)| - |\rho_L(2m+1)||\rho^2_{\text{HV}}(0)|], \qquad (40)$$

where $\rho_{\text{H,V}}(0)$ is the sample pair correlation at zero lag, which is equal to 0.99 in the computations (Sachidananda and Zrnic 1986); ρ_L is the signal autocorrelation at the output of the logarithmic receiver that is given by (Walker et al. 1980)

$$\rho_L = \frac{6}{\pi^2} \sum_{k=1}^{\infty} k^{-2}\rho^{2k}. \qquad (41)$$

On the other hand, the FSD of the estimate $\hat{Z}_{\rm DR}^{P_2}$ can be expressed as

$$\mathrm{FSD}(\hat{Z}_{\rm DR}^{P_2}) = \frac{4.34}{b}\left[\frac{\mathrm{var}(\hat{Z}_{\rm H}{}^{b})}{(\overline{Z_{\rm H}{}^{b}})^2} + \frac{\mathrm{var}(\hat{Z}_{\rm V}{}^{b})}{(\overline{Z_{\rm V}{}^{b}})^2} - \frac{2\,\mathrm{cov}(\hat{Z}_{\rm H}{}^{b}, \hat{Z}_{\rm V}{}^{b})}{\overline{Z_{\rm H}{}^{b}}\,\overline{Z_{\rm V}{}^{b}}}\right]^{1/2} \quad (42)$$

where $\mathrm{cov}(\hat{Z}_{\rm H}{}^{b}, \hat{Z}_{\rm V}{}^{b})$ is the covariance between the estimates of the mean horizontal and vertical reflectivities.

Figure 6 illustrates that in the case of uniform reflectivity the minimum FSD of $\hat{Z}_{\rm DR}^{P_2}$ is obtained with a square law receiver and occurs near $\sigma_v = 2$ m s^{-1}; moreover, Figs. 6 and 7 show that the FSD of $\hat{Z}_{\rm DR}^{P_2}$ is much more sensitive to G_{D_0} and G_{N_0}, and in this case the minimum moves towards values of b around 0.5 for $\sigma_v = 2$ m s^{-1} with increasing gradients.

The fractional standard deviation of the estimate $\hat{R}$

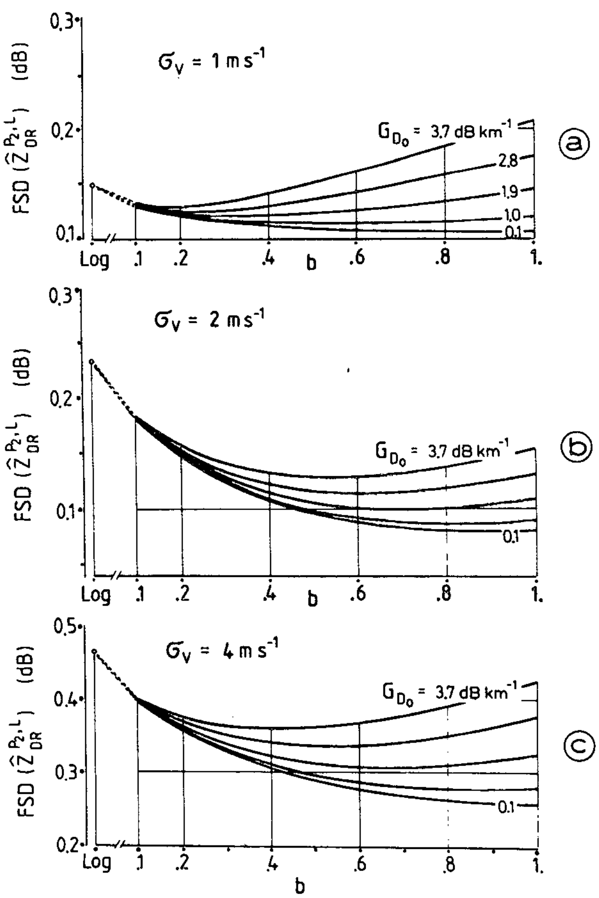

FIG. 6. FSD in dB of the estimates $\hat{Z}_{\rm DR}^{P_2}$ and $\hat{Z}_{\rm DR}^{L}$, obtained respectively by the power law and logarithmic receiver, for different values of G_{D_0} (dB km^{-1}), $G_{N_0} = 0$ dB km^{-1}, the number of averaged samples = 128, and the cross-correlation coefficient $\rho_{\rm HV} = 0.99$ with (a) $\sigma_v = 1$ m s^{-1}, (b) $\sigma_v = 2$ m s^{-1}, and (c) $\sigma_v = 4$ m s^{-1}; the parameter b characterizes the power law receiver.

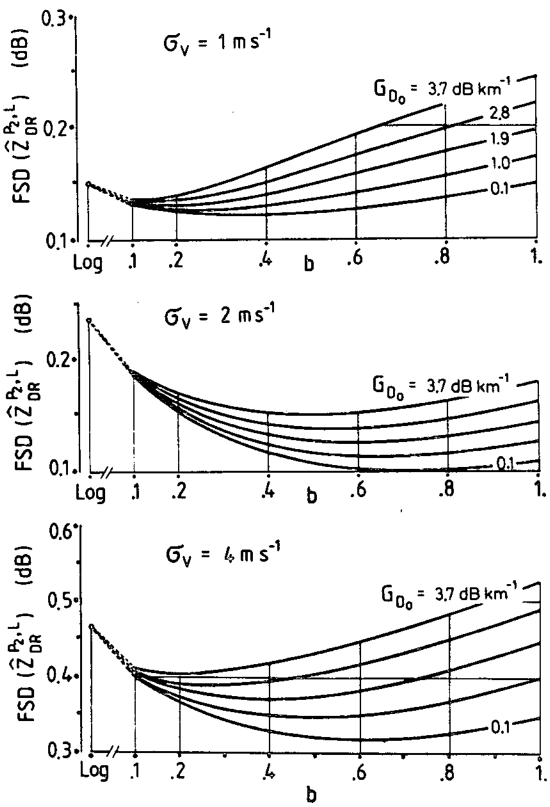

FIG. 7. Same as Fig. 6 except for $G_{N_0} = 15$ dB km^{-1}.

can be obtained from (26) where the estimates $\hat{Z}_{\rm H}$ and $\hat{Z}_{\rm DR}$ are statistically decorrelated in each cell, as it is demonstrated by Chandrasekar et al. (1986),

$$\mathrm{FSD}(\hat{R}) = \bar{R}^{-1}[\mathrm{var}\hat{Z}_{\rm H}(\bar{Z}_{\rm DR})^{-3} + 2.25(\bar{Z}_{\rm DR})^{-5} \times \exp(0.46 \times \overline{10\log Z_{\rm H}})\,\mathrm{var}(\hat{Z}_{\rm DR})]^{1/2}. \quad (43)$$

Figure 8 shows that the minimum of (43) occurs near $\sigma_v = 2$ m s^{-1} for all G_{N_0}'s and G_{D_0}'s. By increasing the Doppler width, the FSD of $\hat{R}$ becomes less sensitive to the gradient G_{N_0}; this results from the FSD of $\hat{Z}_{\rm H}$ decreasing in comparison with the FSD of $\hat{Z}_{\rm DR}$, which is independent of the gradient G_{N_0}.

The FSD of the estimates $\hat{R}^{P_2}$ and $\hat{R}^{L}$ of the mean rainfall rate can be obtained from (27) and (28)

$$\mathrm{FSD}(\hat{R}^{P_2}) = \{[b\mathrm{FSD}(\hat{Z}_{\rm H}{}^{P})/4.34]^2 + [\mathrm{FSD}(\hat{Z}_{\rm DR}^{P_2})1.50/Z_{\rm DR}^{P_2}]^2\}^{1/2} \quad (44)$$

$$\mathrm{var}(\hat{R}^{L}) = \mathrm{var}(\hat{Z}_{\rm H}{}^{L}) + 2.25\,\mathrm{var}(\hat{Z}_{\rm DR}^{L})/(Z_{\rm DR}^{L})^2. \quad (45)$$

Examination of (44) and (45) shows that whenever the gradient G_{D_0} increases, the minimum FSD moves near $b = 0.8$ for $\sigma_v = 4$ m s^{-1} and near $b = 0.3$ for $\sigma_v = 1$ m s^{-1} as illustrated in Fig. 9. Figure 10 shows that as G_{N_0} increases, the value of b decreases corresponding

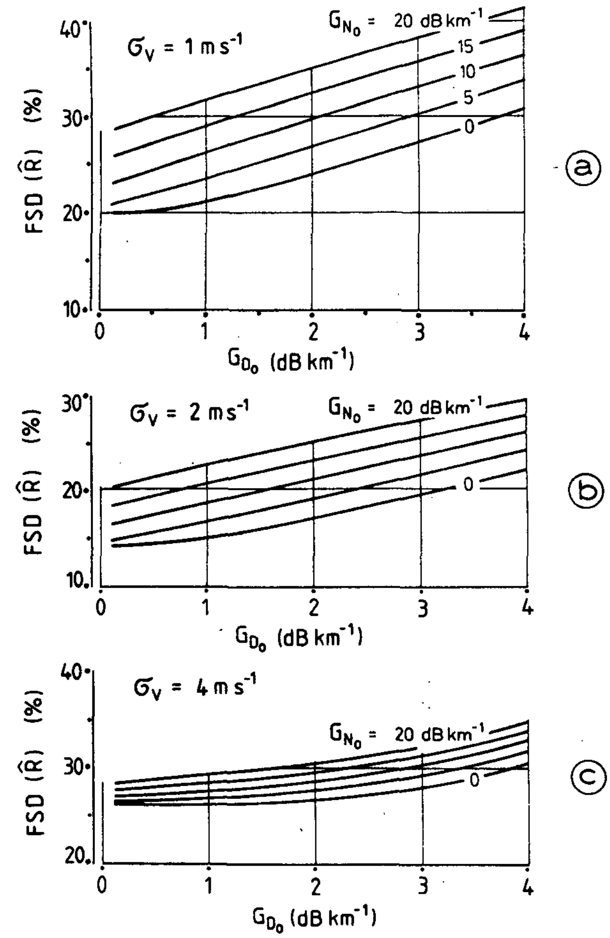

FIG. 8. FSD of the estimate $\hat{R}$ as a function of the gradient G_{D_0} (dB km^{-1}) for different values of G_{N_0} (dB km^{-1}), the number of averaged samples $N = 128$, and the cross-correlation coefficient $\rho_{HV} = 0.99$ with (a) $\sigma_v = 1$ m s^{-1}, (b) $\sigma_v = 2$ m s^{-1}, and (c) $\sigma_v = 4$ m s^{-1}.

to the minimum FSD (this is especially clear for $\sigma_v = 2$ m s^{-1}).

6. Effects due to opposite signs of the gradients

In order to evaluate the effects due to opposite signs of the variation of the microphysical parameters N_0 and D_0 around the values at the center of the measurement cell, the response of the square law receiver for the estimator (21) with G_{N_0} ranging from -20 to 20 dB km^{-1} in 1 dB steps and G_{D_0} from -3 to 3 dB km^{-1} in 0.1 dB steps was examined.

As can be seen from (12b) and (14b) and from Fig. 11, the estimates of the mean differential reflectivity are equal for the same sign of the couple (G_{N_0}, G_{D_0}), while for opposite signs the estimate corresponding to $G_{N_0} > 0$, $G_{D_0} < 0$ coincides with one described by $G_{N_0} < 0$, $G_{D_0} > 0$. The figure shows that for each value of G_{D_0} and symmetrical values of G_{N_0}, the estimates are quite different; also, this difference increases with increasing G_{N_0}. As expected, when G_{N_0} and G_{D_0} are of opposite sign, the Z_{DR} estimate (21) decreases below the value of Z_{DR} at the center of the measurement cell and exhibits a minimum that lies along a parabolic-type curve that passes through the point $G_{N_0} = G_{D_0} = 0$, corresponding to the uniform reflectivity condition.

More meaningfully from a physical viewpoint, Fig. 12 shows $Z_{DR}^{P_2}$ as a function of the parameter G_{D_0} with the horizontal reflectivity gradient G_H ranging from -20 to 20 dB km^{-1} in 5 dB steps. Generally, the error increases as $|G_{D_0}|$ increases. For similar signs and increasing values of G_H and G_{D_0}, the estimate increases; note that for $|G_H| \sim 1$ dB km^{-1} (not shown in Fig. 12) and dissimilar signs of the gradients, $Z_{DR}^{P_2}$ is nearly constant and approximately equal to the value at the center of the measurement cell.

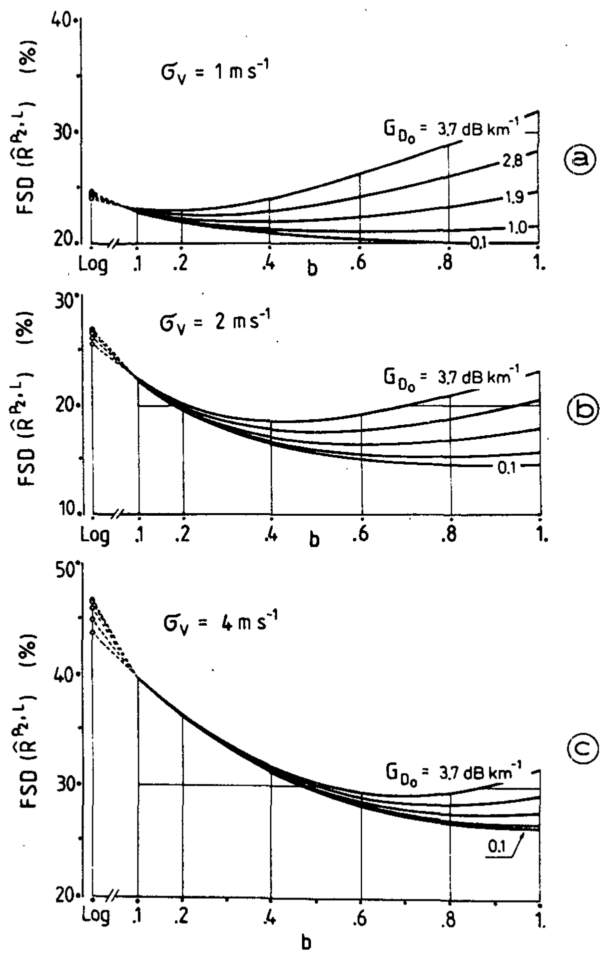

FIG. 9. FSD of the estimates $\hat{R}^{P_2}$ and $\hat{R}^{L}$, obtained by the power law and logarithmic receiver, for different values of G_{D_0} (dB km^{-1}), $G_{N_0} = 0$ dB km^{-1}, the number of averaged samples $N = 128$, and the cross-correlation coefficient $\rho_{HV} = 0.99$ with (a) $\sigma_v = 1$ m s^{-1}, (b) $\sigma_v = 2$ m s^{-1}, and (c) $\sigma_v = 4$ m s^{-1}.

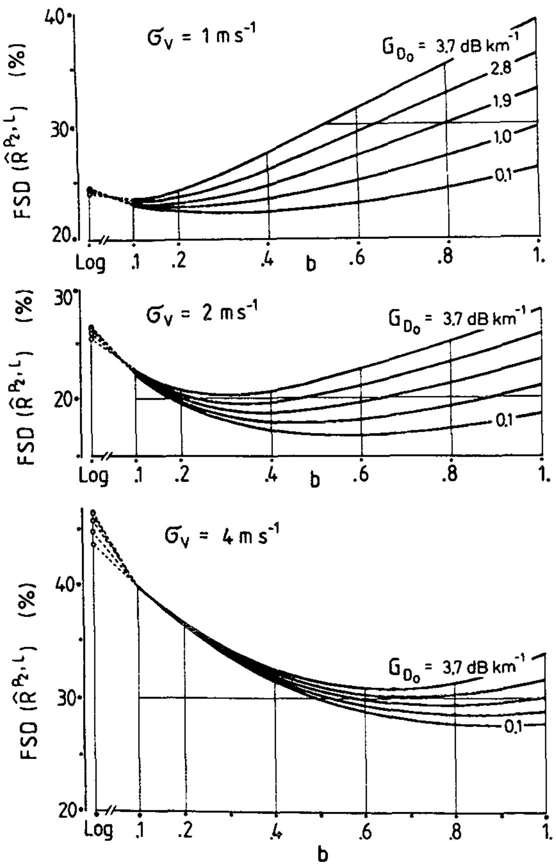

FIG. 10. Same as Fig. 9 except for G_{N_0} = 15 dB km^{-1}.

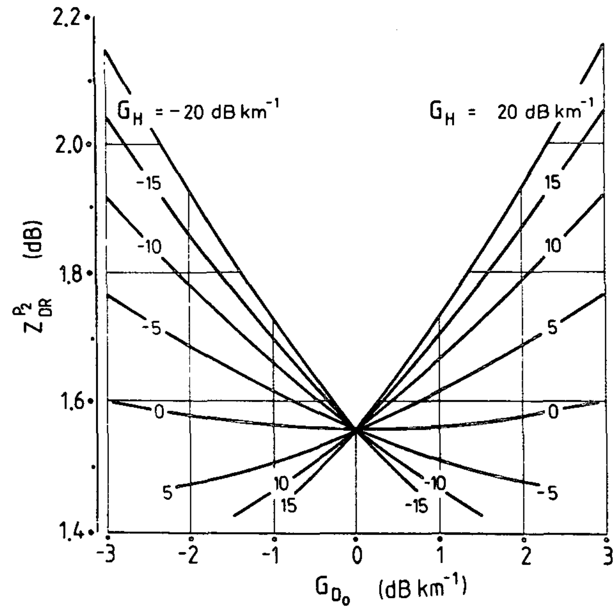

FIG. 12. Same as Fig. 11 except for different values of the horizontal reflectivity gradient G_H (dB km^{-1}).

Figure 13 shows the FSD of $\hat{Z}_{DR}^{P_2}$ for the alternate sampling scheme and a Doppler width σ_v = 1 m s^{-1} with the averaged sample number N = 128 and the cross-correlation coefficient at zero lag $\rho_{HV}(0)$ = 0.99. Note that the FSD increases with increasing $|G_H|$ and $|G_{D_0}|$, as expected; it also exhibits a minimum for uni-

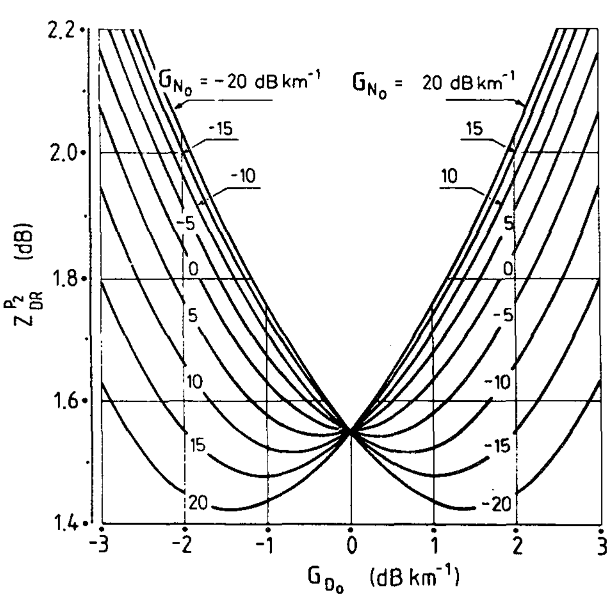

FIG. 11. Average value of the estimate $\hat{Z}_{DR}^{P_2}$ obtained by the square law receiver as a function of G_{D_0} (dB km^{-1}) for different values of G_{N_0} (dB km^{-1}).

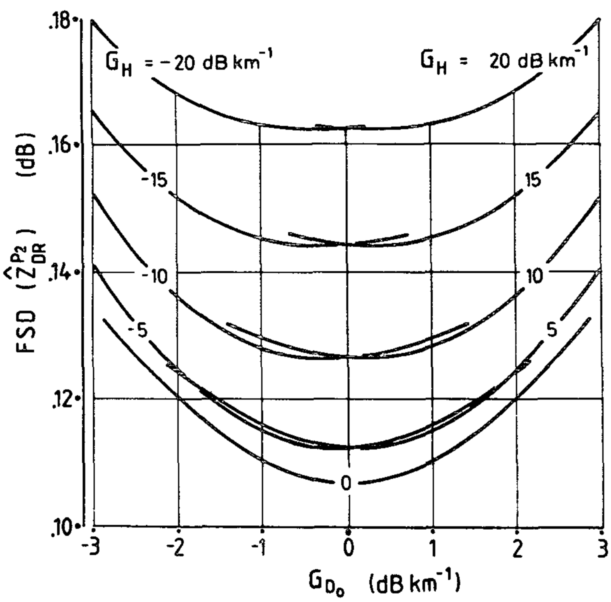

FIG. 13. FSD in dB of the estimate $\hat{Z}_{DR}^{P_2}$ as a function of G_{D_0} (dB km^{-1}) for different values of G_H (dB km^{-1}), the number of averaged samples N = 128, the cross-correlation coefficient $\rho_{H,V}$ = 0.99, and σ_v = 1 m s^{-1}.

form reflectivity and the same symmetry as for the estimate $Z_{DR}^{P_2}$. Moreover, for $|G_H| < 20$ dB km^{-1} the FSD's corresponding to the couple (G_H, G_{D_0}) are very nearly the same as for $(G_H, -G_{D_0})$.

Effects of opposite signs of the gradients on the estimate of rainfall rate were also examined. Figure 14 shows the error of the estimate $\hat{R}^{P_2}$ of the mean rainfall rate within the measurement cell. A similar symmetry of the results occurs with G_{D_0} as noted previously for the results given in Figs. 11–13. Note that for each value of G_H, the same sign of G_{D_0} makes the error of the estimate slightly less than the case with opposite signs. Generally, it should be pointed out that the greatest errors occur for small variations of horizontal reflectivity and large excursions of the differential reflectivity.

Finally, Fig. 15 illustrates the FSD of the estimate $\hat{R}^{P_2}$, which shows an increasing error with increasing gradients. As expected, the minimum FSD occurs for the uniform reflectivity case and is around 22%. The FSD of $\hat{R}^{P_2}$ corresponding to the couple (G_H, G_{D_0}) is always greater than the one corresponding to $(G_H, -G_{D_0})$. In any case, the increase in the FSD over the gradient sizes anticipated is limited to around 0.1 over that for the uniform reflectivity case; this is not a very significant increase.

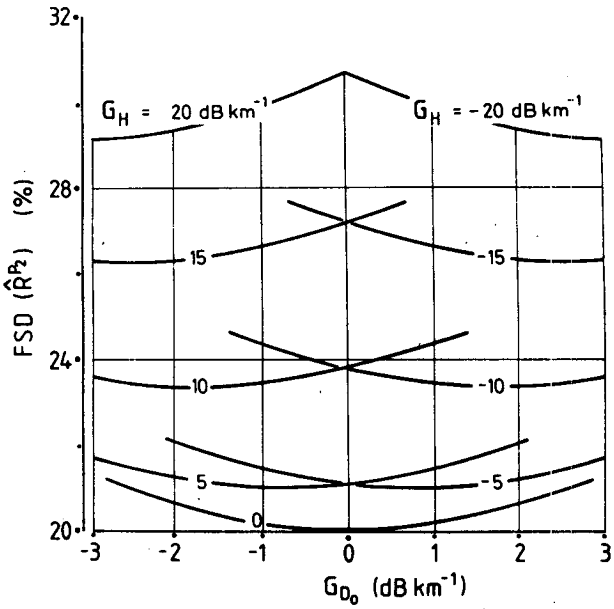

FIG. 15. FSD of the estimate $\hat{R}^{P_2}$ as a function of G_{D_0} (dB km^{-1}) for different values of G_H (dB km^{-1}), the number of averaged samples $N = 128$, the cross-correlation coefficient $\rho_{HV} = 0.99$, and $\sigma_v = 1$ m s^{-1}.

7. Conclusions

In this paper the effects of an exponential spatial variation in the parameters (N_0, D_0) of an exponential DSD on the radar observables (Z_H, Z_{DR}) and derived rainfall rates are examined for radar observations with a stationary antenna. The bias of the estimates, their FSD, and the optimum receiver response are found for power law and logarithmic receivers.

The results indicate a strong dependence on receiver type and signal processing. In particular, two averaging forms have been analyzed. When the estimate of Z_{DR} is based on the logarithm of the average of the ratios of $Z_{H,V}$, it has been shown that the estimate of the differential reflectivity within the composite measurement cell is highly biased, as it is expected. When averaging both $Z_{H,V}$ prior to determining Z_{DR} from $Z_{DR} = 10 \log(Z_H/Z_V)$, the minimum error in the estimate of the differential reflectivity and the rainfall rate within the measurement cell is obtained with the square law receiver for all values of the gradients G_{N_0} and G_{D_0}. For a uniform reflectivity, the minimum FSD of both Z_{DR} and R is obtained for a square law receiver response and Doppler width σ_v near 2 m s^{-1}; in that case the maximum variance corresponds to the logarithmic receiver response. On the other hand, it has been shown that by increasing the reflectivity gradients and decreasing the Doppler width, the parameter b, which characterizes the power law receiver, is greater, and the FSD of both Z_{DR} and R is increased. This means that the optimum receiver response varies, in general, with the signal processing and the meteorological situation; however, the results suggest that for Doppler width $\sigma_v \geq 2$ m s^{-1} and horizontal reflectivity gradient $G_H \leq 30$

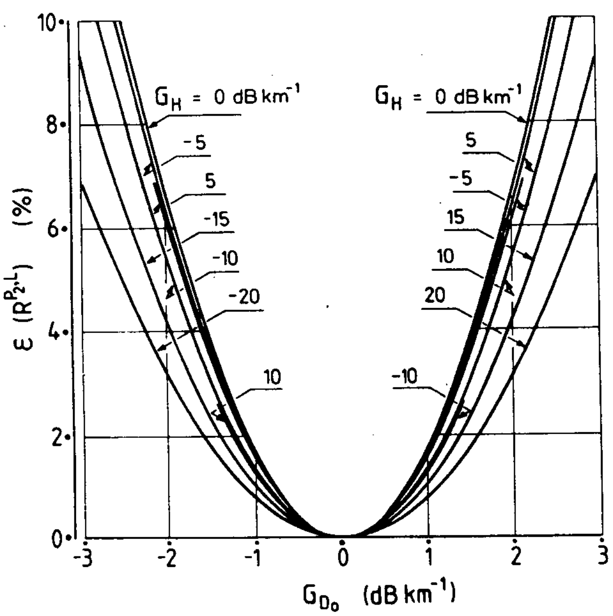

FIG. 14. Relative error $\epsilon(R^{P_2})$ as defined in (29) for the square law receiver, as a function of G_{D_0} (dB km^{-1}) for different values of G_H (dB km^{-1}).

dB km^{-1} the optimum response is obtained with the square law receiver.

The effects of the similar and opposite signs of the gradients were investigated for the special case of a square law receiver. Errors in the estimate of the mean rainfall rate and its FSD decrease for the case of opposite signs between the horizontal reflectivity gradient and the gradient G_{D_0}.

In order to evaluate the results of this analysis and to reduce errors in the estimate of the rainfall rate, the gradients G_{N_0} and G_{D_0}, the values of N_0 and D_0 at the center of the measurement cell, and the Doppler width σ_v have to be estimated. From a practical point of view these estimates can be obtained by combining suitably the measurements of horizontal and vertical reflectivity at the output of the square law and logarithmic receivers, in similar way as for estimating the reflectivity gradients through bias measurements (Scarchilli et al. 1986); as is well known, the estimate of the parameter σ_v is given by Doppler measurements. On this purpose, related experimental studies are in progress to test hypotheses on how to deal with the retrieval of parameters that characterize the spatial variation of rainfall parameters such as N_0, D_0, and R.

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