A NEW APPROACH TO THE PROBLEM OF TURBULENT MIXING

By Harrison E. Cramer

Massachusetts Institute of Technology[1]

(Manuscript received 28 February 1952)

ABSTRACT

The principle of dynamic entrainment is applied to the turbulent-mixing problem. With use of the velocity fluctuations characteristic of turbulent flow and the equation of continuity, an expression is derived for the mass exchange between an accelerated fluid and its environment, under steady-state conditions. A two-stage diffusion mechanism is postulated, in which it is held that the dilution and spreading of aerosols occur in consequence of the inflow and outflow required by continuity in regions of accelerated motion. It is assumed that the inflow and outflow are orderly, that the atmosphere is incompressible, that changes in density are negligible, and that the entrained air is uniformly mixed with the aerosol. Several elementary models are described, and the results obtained by numerical integration in selected cases are discussed. It is indicated that the turbulent-mixing process depends upon the scale of the velocity fluctuations, and upon the ratio of the amplitude of these fluctuations to the mean wind speed.

1. Introduction

It is generally recognized that our present knowledge of the physical processes involved in turbulent mixing is severely limited. Workers in this field have had to rely principally upon analogies borrowed from the kinetic theory of gases [3]. Progress has largely been made through the application of statistical methods that do not depend upon a detailed knowledge of the mechanisms involved [6;7]. Although it may never be possible, nor necessary, to describe turbulent-mixing processes completely, there is a clear need for the formulation of physical principles upon which the mathematicians and fluid dynamicists may base further developments. The purpose of this paper is to outline a new approach to the problem of turbulent mixing and to discuss the applicability of this approach in terms of a few simple models. No attempt has been made to take into account all of the factors that may be significant in the mixing process; rather, attention has been directed toward a single mechanism that is believed to be of considerable importance. The treatment has purposely been confined to elementary considerations, since it seemed desirable to regulate the degree of complexity of the analysis by the extent to which the physical principles involved were understood, and by the extent to which the structure of turbulence has been established empirically.

Velocity fluctuations are a basic feature of turbulent flow; these fluctuations are, of course, the result of accelerated motion. A necessary consequence of accelerated motion in an incompressible fluid is an exchange of mass, required by continuity, between the moving fluid and its environment. This mechanism, called dynamic entrainment, has been applied to the study of convection currents [4], but it applies equally well in the case of accelerated horizontal motions. For example, when positive accelerations occur, the fluid tends to be stretched or rarefied; this rarefaction results in large pressure forces, directed inward, that bring about a compensating inflow of mass from the environment. In the case of negative accelerations, the fluid tends to be compressed and the consequent outward-directed pressure forces bring about a compensating outflow of mass. This mass exchange may be effected by an ordered flow, by a disorderly (turbulent) flow, or by a combination of the two. The dynamic-entrainment mechanism does not specify the type of flow, but merely the mass that must be added or subtracted to maintain continuity. It appears reasonable to expect that both types of flow occur, but the relative importance of each cannot be evaluated on the basis of present information. From the dynamic-entrainment point of view, turbulent motion may be grossly described as a series of rarefactions and compressions that tend to produce a complete mixing of the fluid properties. Also, the entrainment mechanism implies the existence of sources and sinks of variable intensity within a turbulent fluid or gas; the sources correspond to regions of negative acceleration (outflow), and the sinks to regions of positive acceleration (inflow). The existence of sources and sinks, in turn, suggests a vortex-type motion; for most cases of atmospheric turbulence, this would result in the superposition of a rotation on a translatory motion. This is a commonly observed feature of turbulent flow.

In view of the dependence of the turbulent-mixing process described above upon fluctuations in wind

[1] Research sponsored by the Geophysics Research Division of the Air Force Cambridge Research Center under Contract No. AF28(099)-7.

velocity, it seems appropriate to comment briefly on the origin of these fluctuations. There are two factors that are generally regarded as significant in this respect: the roughness of the underlying surface and the thermal stratification of the atmosphere. Over land surfaces, the roughness elements are usually of sufficient importance to ensure turbulent (fully rough) flow in the air layers next to the ground, regardless of the thermal stratification. The extent to which these elements affect the flow depends, of course, upon their dimensions and spatial distribution as well as upon the stability of the atmosphere and the wind speed. Since roughness elements vary in size from tiny blades of grass to such topographical features as mountain ranges, this factor is frequently of very great significance. The role of thermal stratification appears to be largely that of encouraging or discouraging the growth of eddies that have had their inception in the flow over rough surfaces. At suitable heights above the ground and under favorable conditions, the thermal factor is frequently in almost complete control of the growth and dissipation of eddies. It is very difficult, in the air layers near the ground, to separate the contributions made to turbulence by each of these factors. Only in extreme cases is it possible to state with assurance that the fluctuations in wind velocity are principally the result of either mechanical or convective influences.

2. The mass-exchange equation

The exchange of mass that takes place between a moving fluid and its environment as a consequence of accelerated motion may be expressed in very simple terms. Consider a filament of an incompressible fluid; the velocity u within the filament is in the direction of flow, and the filament cross-section is assumed invariant. For the steady-state condition, and in the absence of significant changes in density, the equation of continuity for the volume element of fig. 1 is[2]

$$\rho u\, d\sigma + dm\, dl\, d\sigma = \rho(u + du)\, d\sigma, \tag{1}$$

where dl is the length of the volume element, $d\sigma$ the cross section of the element, dm the mass entrained per second per unit volume of the element, and ρ is the density. When (1) is expanded, we get

$$dm = \rho\, du/dl. \tag{2}$$

To obtain an expression for the total mass of entrained fluid dM, consider a thin fluid plate of mass M that traverses the distance dl with average velocity $\bar{u}$. The volume of this plate is equivalent to M/ρ cm^3, and the time required for the plate to traverse the distance dl is equivalent to $dl/\bar{u}$ sec. Thus, we may write

$$dM = dm\, M\, dl/\rho\bar{u}. \tag{3}$$

When the value for dm given by (2) is inserted in (3), the result is

$$dM/M = du/\bar{u}. \tag{4}$$

The mass of fluid within the plate after it has traveled a finite distance along the filament is obtained by integration of (4),

$$\int_{M_1}^{M_2} dM/M = (1/\bar{u}) \int_{u_1}^{u_2} du,$$

where the limits refer to the values of the mass and the velocity at points L_1 and L_2 along the filament. The result of the integration may be written as

$$M_2 = M_1 \exp(\Delta u/\bar{u}), \tag{5}$$

where $\Delta u = (u_2 - u_1)$. This equation may be inter-

[2] For simplicity, this expression is written as though the velocity varied only in the direction of flow. This is not strictly true, since the entrainment mechanism implies the existence of rotational motion.

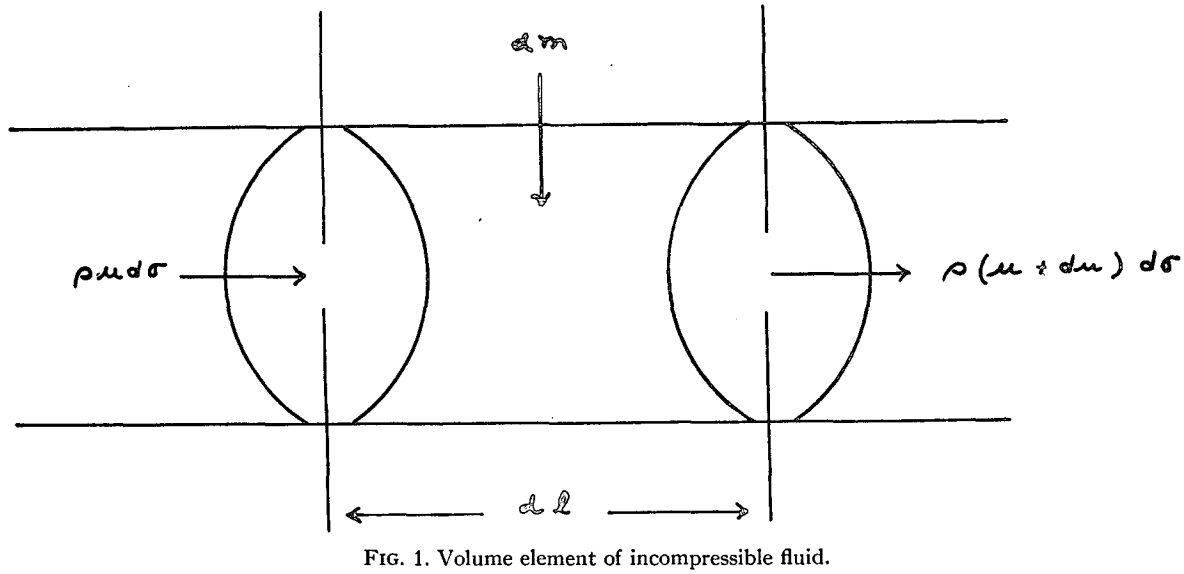

FIG. 1. Volume element of incompressible fluid.

preted in the following manner: As a thin fluid plate of initial mass M_1 and constant cross-section traverses a region in which the velocity of flow increases from u_1 to u_2, the mass of the plate increases to M_2; the amount of the increase depends upon the ratio $(u_2 - u_1)/\bar{u}$. In view of the assumption of constant cross-section, the increase in mass must be reflected by a corresponding increase in plate thickness. For regions of negative acceleration ($u_1 > u_2$), the mass of the plate decreases and this is reflected by a shrinkage of the plate thickness. Since the density of the fluid has been assumed constant, we may substitute "volume" for "mass" in the above interpretation. These phenomena will be discussed in greater detail in the following section.

3. The basic diffusion mechanism

Although the development of the mass-exchange equation of the preceding section is in terms of an ideal fluid, the results apply equally well to gases and vapors that satisfy the requirements of incompressibility and constant density. The atmosphere may be considered as meeting these requirements to a sufficient degree, since the flow velocities with which we shall be concerned are much lower than the speed of sound, and changes in density appear to be of minor significance. An idealized longitudinal section of an aerosol plume, formed by continuous emission from an elevated source, is shown in fig. 2. This section is divided into three stages: stages 1 and 3 are regions of positive acceleration, in which the velocity u in the direction of flow increases in a regular manner; stage 2 is a region of negative acceleration, in which a regular decrease of u occurs. Although the plume may contain rotation, it is assumed that this fact need not be considered in the continuity argument that follows. Consider the passage through the section, under steady-state conditions, of a thin aerosol plate of initial cross-section σ and initial thickness τ. As the plate traverses the distance L_1, the velocity increases from u_1 to u_2 and the volume (or mass) of the plate

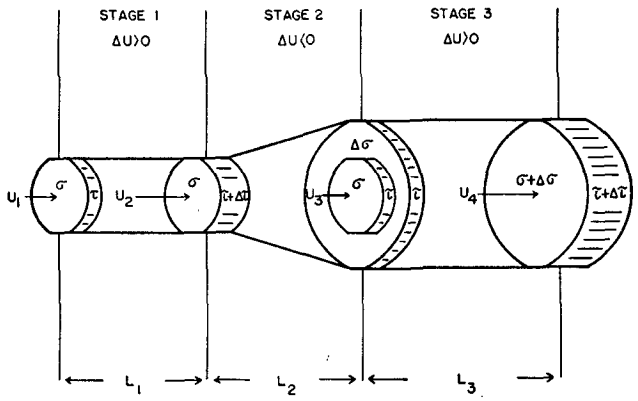

FIG. 2. Idealized section of aerosol plume, showing basic diffusion mechanism.

increases as a result of entrainment from the environment; for simplicity, it is assumed that the inflow from the environment is orderly. In this stage of positive acceleration, the cross section σ is held constant and the increase in volume is associated with an increase in plate thickness to $\tau + \Delta\tau$. It is also assumed, again for reasons of simplicity, that the entrained air is completely mixed with the aerosol initially contained within the plate. Thus, at the end of stage 1, the aerosol concentration has been reduced, the thickness of the plate has increased, but the dimensions of the plume are unchanged. The volume V_1 of the plate at the end of stage 1 is given by the expression

$$V_1 = V_0 \exp (u_2 - u_1)/\bar{u}, \tag{6}$$

where the zero subscript denotes the initial value. In view of the assumption of complete mixing, the aerosol concentration within the plate at the end of stage 1 is written as

$$\chi_1 = \chi_0 \exp - (u_2 - u_1)/\bar{u}, \tag{7}$$

where the subscripts have the same significance as before.

As the plate traverses the distance L_2, the velocity decreases from u_2 to u_3, and the plate volume also decreases as a result of an orderly outflow into the environment; this decrease is reflected in a shrinkage of the plate thickness from $\tau + \Delta\tau$ to τ. For simplicity, it is assumed that no mixing occurs between the outflowing aerosol and the environment. As a consequence of the orderly outflow, there is an increase in the effective aerosol cross-section from σ to $\sigma + \Delta\sigma$. Thus, at the end of stage 2, the thickness of the plate has decreased, the aerosol concentration is unchanged from its value at the end of stage 1, but the effective cross-section of the aerosol plume has increased. In view of the assumptions that have been made, it is clear that the volume of the aerosol plate at the beginning and end of stage 2 must be the same. Therefore, the relationship between the cross sections of the plate is given by the equation

$$\sigma + \Delta\sigma = \sigma \exp - (u_3 - u_2)/\bar{u}. \tag{8}$$

The percentage increase in the cross section of the aerosol plume as a result of the orderly outflow is identical with the percentage increase in plate volume that occurred in stage 1, provided the quantity $|\Delta u/\bar{u}|$ is the same for both stages.

Stage 3 is essentially a repetition of stage 1, except that the initial plate volume and cross section are larger. As the aerosol plate of cross section $\sigma + \Delta\sigma$ traverses the distance L_3, the plate volume again increases as a result of an orderly inflow from the environment; the aerosol concentration within the plate again decreases, due to the mixture between the entrained air and the aerosol within the plate. Therefore, at the end of stage 3, the expression for

the concentration may be written as

$$\chi_3 = \chi_1 \exp - (u_4 - u_3)/\bar{u}, \tag{9}$$

since no change in concentration occurred in stage 2.

The dilution and spreading of the aerosol plate that take place in consequence of the orderly inflow and outflow in regions of accelerated motion are regarded as the basic diffusion mechanism. The mechanism consists of two stages: a positive acceleration stage, $\Delta u > 0$, in which the aerosol concentration is reduced as a result of mixing with the entrained air; and a negative acceleration stage, $\Delta u < 0$, in which the aerosol plume spreads symmetrically away from the plume axis as a result of the outflow required by continuity. These stages do not have to occur in any definite order or sequence; however, according to the assumptions, decreases in concentration occur only in regions of positive acceleration, and increases in plume cross-section occur only in regions of negative acceleration. It may be pointed out that, if the assumption of orderly inflow and outflow is abandoned, (5) still gives the correct net change in the mass of the aerosol plate. Presumably, as a result of what may be termed turbulent (disorderly) exchange, the actual changes in concentration and plume cross-section that occur will exceed the values predicted by the equations described above. Thus, the present treatment, which, for simplicity, assumes an ordered motion, appears to give a minimum estimate of the effectiveness of the mechanism.

4. Model I

An elementary model, based upon the turbulent-mixing process discussed above, will now be described. The model is essentially an extension of the mechanism of fig. 2 for the case in which $|\Delta u/\bar{u}|$ is constant. Consider a rectangular coordinate system with x directed along the mean wind, y directed normal to the mean wind, and z directed vertically. The wind velocity u is assumed to vary in accordance with a simple triangular wave form of constant amplitude Δu and constant wavelength $2L$. The regular variation in wind velocity is restricted to the x direction; thus, the fluctuations are uniform and of infinite extent in both the y and z directions. This set of assumptions may be interpreted as defining an "effective eddy" [6] of finite longitudinal dimension and of infinite transverse and vertical extent. The model consists of a series of alternating positive and negative regions of acceleration. An aerosol plate of initial cross-section σ_0 and with initial concentration χ_0 travels through this field of effective eddies; there is a constant percentage increase in plume cross-section for each region of negative acceleration traversed by the plate, and a constant percentage decrease in aerosol concentration for each region of positive acceleration traversed. Thus, if the number of such regions traversed is N, the expressions for the concentration and cross section become

$$\chi_N = \chi_0 \exp - (\Delta u/\bar{u})N; \tag{10}$$

and

$$\sigma_N = \sigma_0 \exp (\Delta u/\bar{u})N. \tag{11}$$

Although it is not absolutely essential to the present discussion, it is of interest to discuss the adaptation of the model to a continuous source. The discharge of vapors or gases into the atmosphere from a continuous source is, in most instances, marked by a relatively short transition zone in which the effluent behaves as a turbulent jet. In this transition zone, the cross section of the effluent increases and the velocity of flow decreases until it is approximately equal to the mean wind speed. The diffusion that occurs in this phase is principally due to the turbulent-jet properties of the aerosol. Therefore, the proper reference point for studying the role of the wind in diffusing the effluent is at the end of the transition stage. This feature of continuous sources has been noted by Barad [1], and the transition zone has been described as the aerodynamic phase. The equation of continuity shows the relationship, under steady-state conditions and in the case of uniform mixture, between the strength of a continuous source and the properties of the effluent at the end of the aerodynamic phase. The continuity expression may be written

$$Q = \bar{u}\chi_0\sigma_0, \tag{12}$$

where Q is the source strength in g/sec, χ_0 the aerosol concentration in g/cm^3, and σ_0 is the cross section of the aerosol plume in cm^2. It appears that χ_0 and σ_0, the initial values found in (10) and (11), must be determined empirically. One might conclude that the aerodynamic phase, on account of its short length, is of little significance. However, this is not the correct interpretation, since very large decreases in concentration are usually effected over this distance.

The quantity $\Delta u/\bar{u}$ that appears in the diffusion equations is a gustiness ratio; it is very similar to the intensity of turbulence which is defined as the ratio of the root-mean-square of the fluctuations and the mean wind speed, $\overline{(u'^2)}^{\frac{1}{2}}/\bar{u}$. The approximate range of $\Delta u/\bar{u}$ in the lower atmosphere is from 0.05 to 0.40, when the mean wind speeds are measured over periods of at least several minutes duration [2].

For purposes of obtaining a numerical example of the model described above, the following values were assigned:

$$Q = 1 \text{ g/sec}, \quad \Delta u/\bar{u} = 0.25, \quad \sigma_0 = 20 \text{ cm}^2,$$
$$\bar{u} = 5 \text{ m/sec}, \quad \chi_0 = 10^{-4} \text{ g/cm}^3, \quad L = 1 \text{ m}.$$

The distance L is the length of each acceleration zone; thus, the wavelength of the triangular velocity

TABLE 1. Summary of the computations for model I.

N	$\exp(\Delta u/\bar{u})N$	$\exp -(\Delta u/\bar{u})N$	χ (g/cm³)	σ (cm²)
0	1.0	1.0	10^{-4}	20
1	1.284	0.7788	7.79×10^{-5}	25.68
2	1.649	0.6065	6.06	32.97
3	2.117	0.4723	4.72	42.34
4	2.718	0.3679	3.68	54.36
5	3.490	0.2865	2.86	69.81
6	4.482	0.2231	2.23	89.63
7	5.755	0.1738	1.74	115.1
8	7.389	0.1353	1.35	147.8
9	9.488	0.1054	1.05	189.7
10	12.182	0.0821	8.21×10^{-6}	243.6

fluctuations is 2 m. The distance from the continuous point source to the end of the aerodynamic phase was arbitrarily set at 50 cm. For convenience in computing, the aerosol plume is assumed to have circular symmetry. It has also been assumed, for simplicity, that complete mixing takes place between the aerosol and the entrained air. The results of the computations are summarized in table 1 and are presented in fig. 3. The outline of the plume was obtained from the cross sections; on account of the assumption of circular symmetry, this is a plot of the plume diameter. It is evident from both the equations and the results that the changes in plume dimension and concentration are exponential in character. A basic criticism of this model is that the rate of increase in plume diameter and the associated rate of decrease of aerosol concentration soon become much larger than observation would indicate desirable.

5. Model II

There are various modifications that might be made in model I, to overcome the extreme spreading and dilution that occur in the aerosol plume for large values of N. It seems desirable, at least for the present, to retain the effective-eddy concept; this concept not only results in a very considerable simplification of the computation procedure, but our present knowledge of the structure of turbulence appears inadequate to suggest an alternative scheme. The assumption of complete mixing also facilitates the treatment of the model operation and, as will be shown later, need not preclude the development of a concentration profile. A logical improvement in the previous model would seem to be the introduction of a transverse and vertical scale factor. It is known with certainty that wind speed fluctuations are generally of limited horizontal and vertical extent. A diffusion model that retains the principal features of model I and, in addition, specifies a transverse and vertical scale of turbulence will now be described.

Model II is composed basically of a field of effective eddies of uniform size; as before, the longitudinal scale of these eddies is specified by half the wavelength of the velocity fluctuations. For convenience, the transverse and vertical scales are set equal and, to simplify the mass-exchange process, the individual eddies are assumed to have square cross-sections. The length of the longitudinal scale relative to the others is arbitrary; observations indicate that the along-wind dimension of eddies is frequently longer than the cross-wind dimension. A schematic representation of a section of the proposed eddy field is presented in fig. 4; the field is so constructed that each unit is enclosed by four neighboring units in which the sign of the acceleration is opposed to that of the central unit. As in model I, the sign of the acceleration is alternately positive and negative in the direction of flow. In fig. 4, regions of positive acceleration are denoted by (+), and regions of negative acceleration are denoted by (−); the vectors indicate the direction of mean flow and show qualitatively the variations in speed that occur within the field.

The adaptation of this model to a continuous point source follows closely the procedure outlined previously. At the end of the aerodynamic phase, the effluent is assumed to behave in accordance with the mechanism of model I, until the plume cross-section attains the dimensions of the effective eddy cross-section. Once this stage has been reached, the mixing

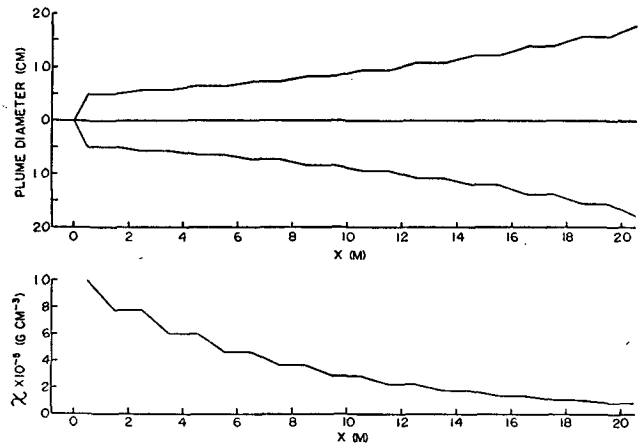

FIG. 3. Plume diameter and aerosol concentration for model I.

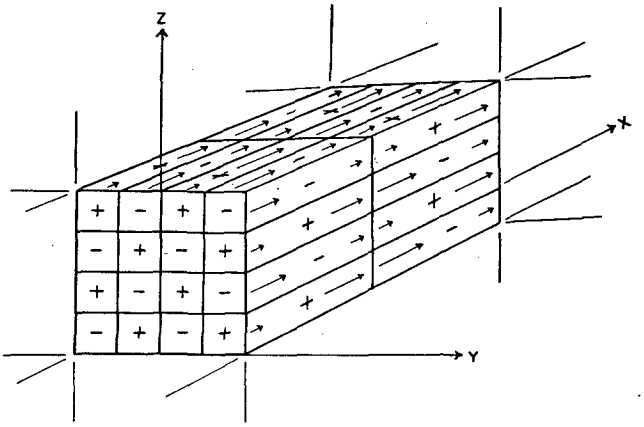

FIG. 4. Structure of turbulent flow for model II.

and spreading processes are carried out in terms of the individual eddies. This mechanism is shown schematically in fig. 5; the solid lines indicate the portion of the field that contains an aerosol concentration (the aerosol plume), the symbols (+) and (−) denote regions of positive and negative accelerations, respectively, and the vectors show the direction in which the mass exchange required by continuity is taking place. In stage 1, the plume cross-section is identified with the cross section of a single eddy; since the sign of the acceleration is positive for this eddy, the concentration is reduced in this stage as a result of mixture with the air entrained from the four neighboring cells. It is assumed that each neighbor supplies one fourth of the required entrainment. Therefore, at the end of stage 1, the plume cross-section is unchanged and the aerosol concentration has been reduced. During stage 2, the sign of the acceleration is negative in the eddy that initially contains the aerosol plate; as a result of the orderly outflow from this cell to the four neighboring cells, the effluent is, at the end of this stage, found in five units. It is assumed that the four neighbors share equally in the outflow, and that complete mixing occurs in each cell. The aerosol concentration within the central cell is unchanged during this stage. In stage 3, the mass-transfer mechanism produces a finite concentration in eight additional neighbor cells. The four cells enclosing the central cell have no change in concentration during this stage, while the concentration within the central cell itself is further reduced by the entrainment process. In stage 4, twelve additional cells are added to the plume as a result of the spreading from perimeter cells with negative accelerations. Changes in concentration within the plume are easily explained in terms of the inflow to cells with positive accelerations. It should be pointed out that the bulk of the effluent is still located within the central cell, and that the term "plume" has been used in a very general sense to include all cells in which even a trace of the effluent may be found.

The adaptation of model II to a continuous line source of infinite cross-wind extent is shown in fig. 6. As in the previous example, the solid lines enclose cells in which a finite aerosol concentration exists, and the small vectors show the direction in which the mass transfer required by continuity is taking place. At the start of stage 1, the effluent is contained in a single row of eddy cells denoted by C in the figure; this is the state of plume development after the aerodynamic phase and after a simple mixing phase (see model I) of sufficient extent to produce a vertical plume dimension equal to the effective eddy diameter. During stage 1, the effluent spreads to the A cells of row 1 from the A cells of row C; the concentration of the B cells in row C is reduced, due to the inflow from the B cells of row 1. During stage 2, the aerosol spreads to the B cells in row 1 and to the A cells of row 2; the concentration in the A cells of row C decreases, as a result of mixture with the outflow from the A cells of row 1. The changes in plume dimension and aerosol concentration that occur in stages 3 and 4 are readily explained in similar terms. For each new stage in model development, a new row of cells is added to the top and bottom of the plume; each row contains type A and B cells that exchange mass with each other and with corresponding A or B cells in adjacent rows. As the model develops, differences in concentration between A and B cells in the same row are, in general, negligible.

Two numerical examples of the model have been worked out; the first example is for a continuous point source, and the second is for a continuous line source of infinite cross-wind extent. In each case, the ratio $\Delta u/\bar{u}$ was held constant and set equal to 0.182; this value was selected because it results in a percentage increase in plate volume (20 per cent) for each stage of positive acceleration that is readily divisible by four. The concentration in each cell of the aerosol plume was computed from the approximate mixture formula,

$$\chi_1 = (\chi_0 V_0 + \Delta V \chi)/(V_0 + \Delta V), \qquad (13)$$

where χ_0 and V_0 are the initial concentration and volume, respectively, ΔV is the increment in volume,

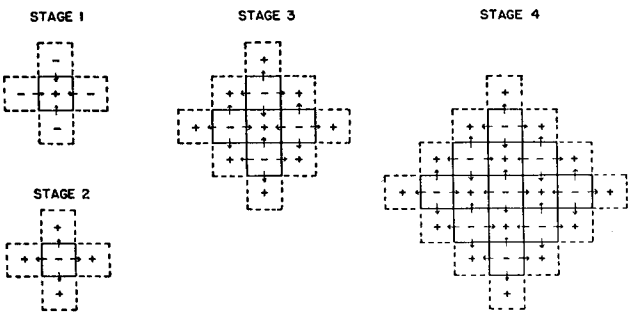

FIG. 5. Turbulent-mixing process for model II (point source).

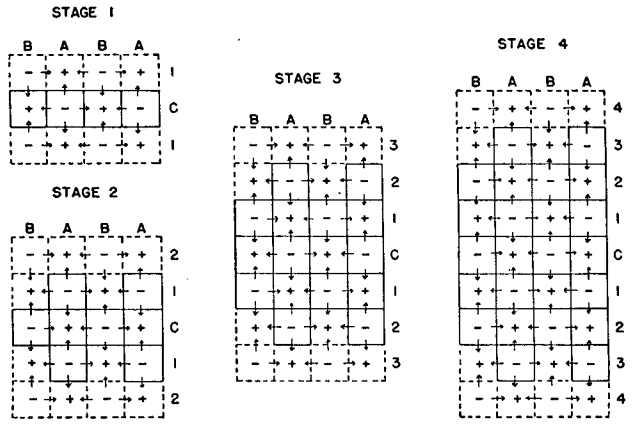

FIG. 6. Turbulent-mixing process for model II (line source).

and χ is the concentration of the increment. In the present examples, $(V_0 + \Delta V)^{-1}$ is constant and is equal to 0.833. Perhaps the procedure followed in the mixing computations will be clarified if the equations for the first few stages of the point-source model are written down. In stage 1 (fig. 5), the aerosol is contained in a single cell with concentration χ_0; at the end of this stage, the concentration $\chi_{c,1}$ is simply $0.833\ \chi_0$. In stage 2, there is an orderly outflow from the central cell to the four neighboring cells. The mixing equation for each of these cells may be written as

$$\chi_{1,2} = 0.833(0.05\ \chi_{c,1}) = 0.0347\ \chi_0. \quad (14)$$

The first subscript in the above expression refers to the distance from the plume axis in cell diameters; the second subscript is the stage number. In stage 3, the mixing equation for the central cell becomes

$$\chi_{c,3} = 0.833(\chi_{c,1} + 0.20\ \chi_{1,2}) = 0.699\ \chi_0. \quad (15)$$

Two mixing equations are required for the perimeter cells that are added during this stage; the four cells located on the transverse and vertical axes (exterior) each have but one neighbor with a finite concentration, while the four intermediate (interior) cells each have two such neighbors. The mixing equations are

$$\chi_{2i,3} = 0.833(0.10\ \chi_{1,2}) = 2.89 \times 10^{-3}\ \chi_0, \quad (16)$$

and

$$\chi_{2e,3} = 0.833(0.05\ \chi_{1,2}) = 1.45 \times 10^{-3}\ \chi_0. \quad (17)$$

The subscripts i and e denote interior and exterior, respectively. As successive stages are encountered, the model structure becomes increasingly complex; the changes in concentration, however, are computed in the manner outlined above.

The mixing computations were carried out for eighteen complete stages of model growth, in both the point-source case and the infinite line-source case; after this point, attention was directed to the central portion of the plume and the computations were sufficiently extended to permit the determination of the concentration in the central cell for twelve additional stages. It should be pointed out, in defense of this procedure, that only very minute quantities of the aerosol are found at distances greater than a few cell diameters from the plume axis. This feature of the model is shown in fig. 7, where the concentration profiles for stage 25 ($N \approx 13$) are presented. Although the plume is 25 cell diameters in radius, appreciable concentrations are present only in the four or five cells nearest the plume axis. As might be expected, the peak concentration of the line-source model at this stage is about two and one half times greater than that of the point-source model.

The concentrations along the plume axis were extended through one hundred stages of positive acceleration ($N = 100$) by the following procedure. The ratio χ_N/χ_{N-1}, which relates the concentration at any two successive stages, has an initial value of 0.833 and approaches a limiting value of 1.0 for large values of N. An expression of the form

$$y = a + b \exp(Nx)$$

was fitted, by a standard method, to the fifteen values of this ratio obtained by direct computation. The equations obtained from these data, for both the point source and the line source, are

$$y = \chi_N/\chi_{N-1} = 1 - 0.1774 \exp(-0.0508\ N), \quad (18)$$

and

$$y = \chi_N/\chi_{N-1} = 1 - 0.0895 \exp(-0.0514\ N), \quad (19)$$

where the latter refers to the line source. There is excellent agreement between the computed values of the concentration and the extrapolated values obtained from the above expressions, for the first fifteen values of N. The concentrations obtained by means of these expressions are shown in fig. 8; the outstanding feature of these curves is that the peak concentration becomes practically constant for large values of N ($N > 45$). A similar type of behavior has been observed in smoke plumes under stable stratification [1]. The effectiveness of introducing a transverse and vertical scale factor to overcome the tendency, noted in model I, for extreme changes in concentration with large values of N is shown in fig. 9. The extrapolated peak concentrations described above have

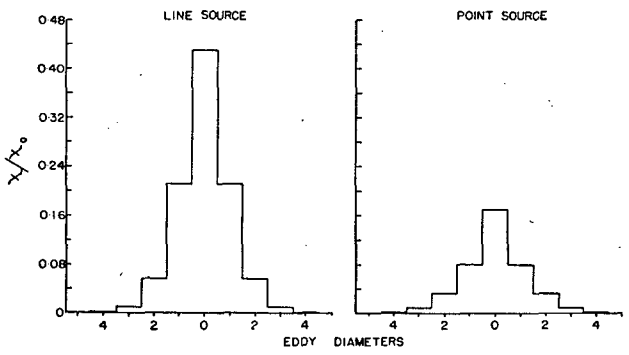

FIG. 7. Concentration profiles of model II for stage 25.

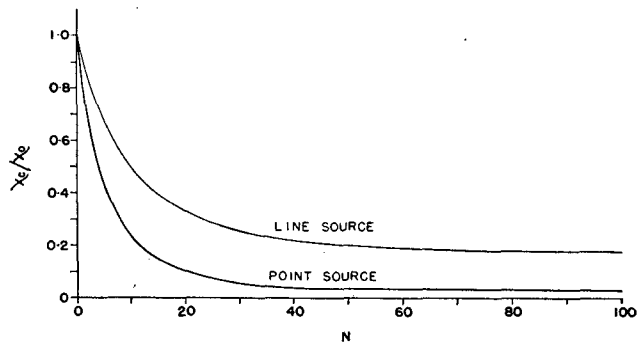

FIG. 8. Peak concentrations for model II, obtained by extrapolation.

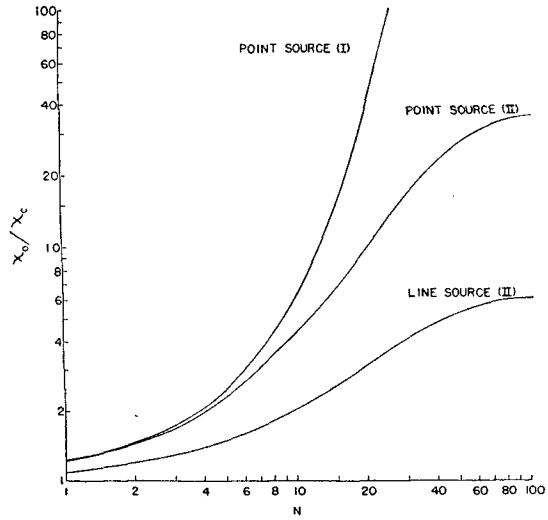

FIG. 9. Comparison between peak concentrations of model II and model I.

been plotted as functions of χ_0 and N on logarithmic scales. The corresponding values of χ_0/χ_c obtained from a mixing process based on model I are also entered for purposes of comparison.

6. Discussion

As stated previously, the purpose of this paper is to describe a new approach to the problem of turbulent mixing, and to indicate the potential usefulness of this approach by obtaining numerical results for several rudimentary models. Although these results appear promising, it is evident that a rigorous test of the validity of this approach awaits the collection and analysis of extensive field data on the structure of turbulent flow. An observational program designed to meet this requirement, at least in part, is already in progress at the Round Hill Field Station of the Massachusetts Institute of Technology. The development of the equations describing the dilution and spreading of aerosols has purposely been halted at an elementary stage; further progress in this direction would appear to depend, also, upon the results of additional field observations. No attempt has been made to take into account all of the factors that may play a significant role in the atmospheric-diffusion process. It is felt that additional factors may be introduced when the diffusion process itself is better understood, and when the need for these factors is clearly defined.

It should be pointed out that none of the assumptions made in deriving the mass-exchange equation, or in the numerical integrations that followed, is requisite for a solution. All of these assumptions may be altered at the expense of increased complexity. One of the basic assumptions is that of a steady state; while turbulent motion is admittedly not steady, it would be difficult to prescribe a realistic non-steady state. For simplicity, it has been assumed that the velocity fluctuations have an elementary triangular wave form; this is clearly not realistic, and future observations may demonstrate that this simplification is untenable. Similarly, the assumption of an effective eddy of uniform size appears to be of limited usefulness; this is particularly true, of course, when a wide range of eddy sizes is actively involved in the mixing process [5]. This difficulty may be overcome by specifying an effective eddy size that is either a function of plume cross-section or of distance from the source. Although no allowance has been made for the shifts in wind direction that regularly occur, it appears that this factor may be taken into consideration either by increasing the size of the initial area source, or by increasing the gustiness ratio $\Delta u/\bar{u}$. The assumption of complete mixing is probably not entirely valid, even when the mixing process is restricted to eddies of small size. However, departures from complete mixing may be minimized by limiting the size of the basic units involved in the mixing process, and by making the necessary adjustments in the gustiness ratio. As mentioned earlier, the assumption of an orderly inflow and outflow apparently leads to a minimum estimate of the dilution and spreading of the aerosol plume. The significance of this factor may be investigated empirically, and suitable modifications may be made in the gustiness ratio.

Acknowledgments.—The writer is indebted to Prof. H. G. Houghton, who first suggested the concept of dynamic entrainment to him. Valuable criticism and suggestions have been derived from numerous discussions with Dr. A. Fleisher, Dr. E. W. Hewson, and Dr. F. A. Record. Mr. W. R. Hindle, Jr. and Mr. George Fontes assisted in the computations, and the latter also completed the figures.

REFERENCES

1. Barad, M. L., 1951: Diffusion of stack gases in very stable atmospheres. *Meteor. Monogr.*, **1**, No. 4, 9–14.
2. Cramer, H. E., 1953: Preliminary results of a program for measuring the structure of turbulent flow near the ground. In "International symposium on atmospheric turbulence in the boundary layer," *Geophys. Res. Pap., A. F. Cambridge Res. Center* (to be published).
3. Goldstein, S., 1938: *Modern developments in fluid dynamics.* Vol. 1, Oxford, Clarendon Press, 205–206.
4. Houghton, H. G., and H. E. Cramer, 1951: A theory of entrainment in convective currents. *J. Meteor.*, **8**, 95–102.
5. Lowry, P. H., 1951: Microclimate factors in smoke pollution from tall stacks. *Meteor. Monogr.*, **1**, No. 4, 24–29.
6. Sutton, O. G., 1932: A theory of eddy diffusion in the atmosphere. *Proc. roy. Soc. London, A*, **135**, 143–165.
7. ——, 1949: *Atmospheric turbulence.* London, Methuen and Co., 44–87.