POWER SPECTRA OF THE EDDY-VELOCITY COMPONENTS

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(Manuscript received 21 June 1954)

ABSTRACT

Power spectra of the eddy-velocity components have been determined at four levels within the layer from 2 to 12 meters under varying conditions of mean wind speed, trajectory and thermal stability. A filtering technique suggested by J. W. Tukey has been used to obtain rough estimates of contributions to the total variance for seven consecutive frequency intervals within the range from about 0.5 to 0.005 cycles per second. At the higher frequencies studied, variances for all three components are approximately equal and equipartition of turbulent energy is thus indicated. Spectra for the u- and v-components appear to be invariant with frequency at the lowest level, and tend to increase slowly with decreasing frequency at the higher levels. The w-spectra at all levels fall off sharply with decreasing frequency, contributions to the vertically-directed energy becoming almost negligible at the lowest frequencies investigated.

1. Introduction

The determination of the spectrum of turbulent energy is one of the basic unsolved problems in the physics of the earth's boundary layer. Since turbulence is defined in terms of deviations or departures from a mean flow, the statistical quantity of variance is a convenient measure of the power or energy contained in turbulent fluctuations. The mean flow in the atmosphere is usually a function of the averaging time, and is therefore somewhat arbitrary. To arrive at a satisfactory understanding of turbulent diffusion processes, it is necessary to specify the spectrum or variation with frequency of many of the physical properties of atmospheric turbulence. Required basic knowledge includes a description of the partition of turbulent energy for the three orthogonal components of the wind velocity. Early investigators, such as Best [1], Scrase [10] and others, were hampered by lack of suitable instrumentation and appropriate analytical techniques. Although these difficulties have not been completely surmounted, sufficient progress has recently been made to permit a description of some of the gross features of the partition of eddy energy [4; 5; 6; 7; 8; 9]. Before the problem can be satisfactorily resolved, however, many additional observations as well as further improvements in instrumentation are clearly required. The purpose of this article is to present the results of a series of determinations of the power spectra of the eddy-velocity components in the layer from 2 to 12 m. The frequency interval investigated ranges from about 0.5 to 0.005 cycles/sec.

2. Observations

Measurements of the eddy-velocity components, u', v' and w', on which this study is based, were obtained from the records of heated-thermocouple anemometers and sensitive bivanes mounted at four levels on a mobile tower: 11.9, 6.4, 3.7 and 2.3 m. Since detailed descriptions of this instrumentation and the conditions under which the observations were made are presented elsewhere [2; 3], only a brief account need be given here. Response characteristics of the bivanes and anemometers are very closely matched. When the mean wind speed is 2.5 m/sec, both instruments resolve sinusoidal fluctuations with a period of 2 sec with an accuracy greater than 90 per cent. For a mean wind speed of 5.5 m/sec, the same fluctuations are resolved with an accuracy greater than 96 per cent. The response characteristics of the bivanes indicated a period of resonance which is approximately equal to $5/\bar{u}$ sec, where $\bar{u}$ is the mean wind speed in meters per second. Under average conditions, the resonance factor may result in computed values for the v and w eddy-velocity components that deviate approximately 10 per cent from the true values; the effect on the u' values is considerably smaller. It should be pointed out that the procedure by which individual eddy velocities are computed is itself probably subject to uncertainties of the same order of magnitude.

Wind-speed and wind-direction data were recorded by taking lapse-time photographs of indicating meters at the rate of one exposure per second; the average sampling time was about 11 min. Five sets of measurements were obtained in a large field containing long grass and low brush, during the period 10 to 11 June 1952. The average height of roughness elements in the immediate vicinity of the site was estimated to be 50 cm. The weather during this period was generally fair in the presence of a fresh polar-continental air mass. Trajectories for the 11 June observations were entirely over a land surface; the trajectory for 10 June was principally over water, except for a distance of

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[1] The research reported in this article was sponsored by the Geophysics Research Directorate, Air Force Cambridge Research Center, under Contract No. AF 19(604)-145.

about 150 m directly upwind from the point of measurement. The data for 20 August 1952 were collected at a seaplane ramp, also during a period of fair weather; the trajectory was entirely over water, except for a distance of about 8 m from the edge of the ramp to a point directly beneath the instrumentation mounted on the mobile tower. Profiles of mean wind speed and air temperature for these observations have been presented previously [2].

3. Data analysis

Several techniques have been suggested for obtaining turbulent-energy spectra that are defined in terms of the distribution of variance or mean-square displacement as a function of frequency. Very crude estimates may be secured by the use of moving averages [6; 8]; the absence of sharp cut-off properties in the filters is a serious limitation in this procedure. Precise methods of spectral analysis, based on the Fourier cosine transform of the autocorrelation function, have been described by Tukey [12] and others. These methods, however, are practicable only when high-speed computing devices are used, due to the very large number of numerical calculations required. Since this type of equipment was not readily available, a simpler procedure, suggested by J. W. Tukey in private conversation, was adopted; this procedure, which requires only conventional desk calculators and which provides limited resolution, is briefly described below.

The raw data available for spectral analysis consist of sets of N consecutive observations of the eddy-velocity components, u', v' and w'. Each set of raw data forms a finite sequence,

$$x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{N-1}, x_N,$$

with a time interval δt between successive values. According to information theory, the extreme range of frequencies over which spectral estimates may be obtained is $1/(2\,\delta t)$ to $1/(N\,\delta t)$ cycles/sec; the upper limit is fixed by the time interval δt, and the lower limit is set by the length of record available. The first step in the analysis is the formation of sequences of squared differences of the type

$$d_i^{(1)} = (x_{2i} - x_{2i-1})^2, \quad i = 1, 2, \cdots, [N/2],$$

$$d_i^{(2)} = \{(x_{4i} + x_{4i-1}) - (x_{4i-2} + x_{4i-3})\}^2, \quad i = 1, 2, \cdots, [N/4],$$

and

$$d_i^{(3)} = \{(x_{8i} + \cdots + x_{8i-3}) - (x_{8i-4} + \cdots + x_{8i-7})\}^2, \quad i = 1, 2, \cdots, [N/8].$$

The symbol $d_i^{(p)}$ represents the i-th squared difference of order p, and the brackets [ ] indicate that the integral part of the quantity thus enclosed is to be considered. Higher-order differences are formed in a similar manner until the available length of record is exhausted. In the present application, this occurs for $p = 9$, $N = 512$.

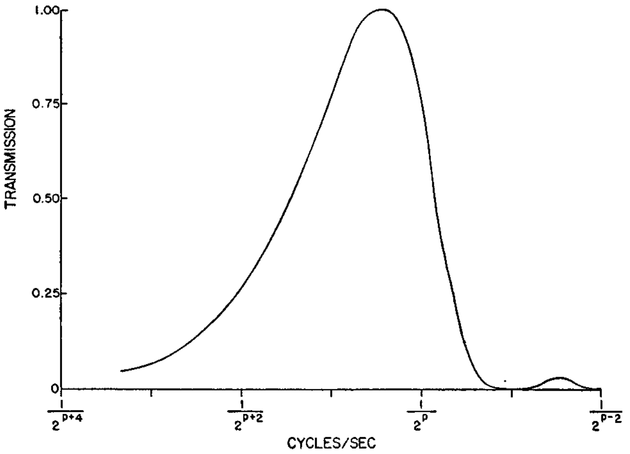

Fig. 1. Transmission curve for quadratic filter.

It is assumed that the individual differences are random samples from a population that includes all frequencies from $1/(2\,\delta t)$ to zero cycles per second. Rough estimates of the mean spectral density averaged over the frequency interval from $1/(2^p\,\delta t)$ to $1/(2^{p+1}\,\delta t)$ cycles/sec are given by the quantity[2]

$$\frac{1}{N}\sum_{i=1}^{N/2^p} d_i^{(p)}.$$

To obtain spectral estimates that represent the contribution to the total variance from this frequency interval, the mean spectral density must be multiplied by a frequency $1/(2^p\,\delta t)$. Thus, when $\delta t = 1$ sec, estimates of the average power within the frequency interval $1/(2^p)$ to $1/(2^{p+1})$ are written as

$$\frac{1}{2^pN}\sum_{i=1}^{N/2^p} d_i^{\prime(p)}.$$

If these spectral estimates are plotted on a logarithmic frequency scale, the area beneath the curve between any two frequencies is proportional to the contribution to the total variance within this frequency interval.

The transmission characteristics for a quadratic filter of the type described above are shown in fig. 1. These data were obtained by applying the filtering operation to sequences derived from sine waves of unit amplitude and random phasing. The curve presented in the figure is strictly correct only for $p > 5$; minor deviations occur at low values of p. To compute the transmission, the $d_i^{(p)}$ are expressed in terms of sample ordinates of a sine wave. For example, the i-th term for the third-order sequence of squared differences

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[2] Provided that N is an integral power of 2.

may be written as

$$d_i^{(3)} = \left\{\left[\cos\left(\theta+\frac{7\pi}{4}\frac{k_1}{k_2}\right)+\cdots+\cos\left(\theta+\pi\frac{k_1}{k_2}\right)\right] - \left[\cos\left(\theta+\frac{3\pi}{4}\frac{k_1}{k_2}\right)+\cdots+\cos\theta\right]\right\}^2,$$

where k_1 is the frequency of the sine wave to be filtered, k_2 the frequency of the filter given by $1/(2^p\,\delta t)$, and θ is the phase angle. If it is assumed that the filter samples the sinusoidal fluctuations for an infinitely long time, the average value of the $d_i^{(p)}$ at a particular frequency k_1 is

$$\overline{d_i^{(p)}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d_i^{(p)}(\theta)\,d\theta, \quad k_1 \le k_2.$$

When $k_1 > k_2$, the expression on the right must be multiplied by the factor k_2/k_1. The transmission is defined as the ratio of the $\overline{d_i^{(p)}}$ at a particular frequency to the maximum $\overline{d_i^{(p)}}$. The frequency interval over which the filter provides resolution is approximately delineated by transmission values in excess of 0.7. Thus, the transmission curve of fig. 1 supports the statement made earlier that the filter provides spectral estimates for the frequency band from $1/(2^p\,\delta t)$ to $1/(2^{p+1}\,\delta t)$. It is evident that the cutoff at low frequencies is not sharply defined, and the filter therefore might be described as a band-pass type of limited resolving power. There is some advantage to the use of a filter with this characteristic when relatively short pieces of record (preferably from a continuous spectrum that does not vary rapidly with frequency) are available for analysis. The filter in this case tends to smooth the results, and the power spectrum thus obtained is more satisfactory than that resulting from the use of a sharply tuned filter.

A measure of the statistical reliability of the spectral estimates is available from the distribution of the ratio of chi square to the number of degrees of freedom. The maximum number of degrees of freedom for a given estimate is equal to the number of squared differences included in the summation. As indicated by Tukey [12], the number of effective degrees of freedom is generally less than this and depends upon the type of filter as well as the form of spectrum being studied. The factors by which the results of this study should be multiplied to give approximate 90 per cent confidence limits are presented in table 1. These minimum limits become very wide at low frequencies, due to the small number of degrees of freedom obtainable. In view of this lack of stability, the lowest frequency interval considered in the analysis is from 1/128 to 1/256 cycles/sec. The confidence limits are interpreted as defining the range within which the population variances will be found approximately 90 per cent of the time under identical conditions of mean wind speed and direction, surface roughness, thermal stability, *etc.*

TABLE 1. Factors by which spectral estimates are to be multiplied to obtain approximate 90 per cent confidence limits; table based on $N = 512$.

Deg. of freedom (max)	Frequency interval (cycles/sec)	5 per cent	95 per cent
256	1/2–1/4	0.87	1.14
128	1/4–1/8	0.80	1.22
64	1/8–1/16	0.72	1.32
32	1/16–1/32	0.63	1.44
16	1/32–1/64	0.50	1.64
8	1/64–1/128	0.34	1.94
4	1/128–1/256	0.18	2.37

4. Results

Rough estimates of the contributions to the total variance for seven frequency intervals were obtained from the basic data by the above procedure. A complete summary of these results is available elsewhere [3] and will not be reproduced here. The general nature of the results is indicated in figs. 2 to 7, where the component spectra at 2.3 and 11.9 m for six observation periods are presented. Individual variances have been entered at the midpoints of the appropriate frequency intervals, the *v*- and *w*-values being slightly offset to avoid confusion. The data in figs. 2 to 4 are considered representative of unstable flow over a rough land surface; the general similarity of the spectral distributions for the three sets of measurements presumably reflects the existence of rather uniform meteorological conditions during the observations. Approximate equipartition of turbulent energy among the three orthogonal components is indicated near $\frac{1}{2}$ cycles/sec at 2.3 m, and near $\frac{1}{8}$ cycles/sec at 11.9 m. The relatively large contribution by the vertical component in the $\frac{1}{2}$ to $\frac{1}{4}$ cycles/sec interval at 11.9 m suggests a high frequency interval in which the turbulent energy is principally directed along the vertical coordinate. As pointed out below, this feature of the results may not be real but may be produced by the "aliasing" of high-frequency fluctuations. Panofsky [8] has cited some controversial evidence concerning the partition of eddy energy at high frequencies.

Comparison of the spectral distributions for the horizontal components in figs. 2 to 4 shows larger increases with decreasing frequency at 11.9 m than at 2.3 m. These indicated variations with height are not statistically significant for a single set of observations. To obtain a more reliable estimate of this factor, the data for all three observation periods have been combined. This was accomplished by first normalizing the spectral estimates at each of the four levels with respect to the variance E_0 in the $\frac{1}{2}$ to $\frac{1}{4}$ cycles/sec interval; then, the normalized data at each level were fitted, by the method of least squares, to power laws

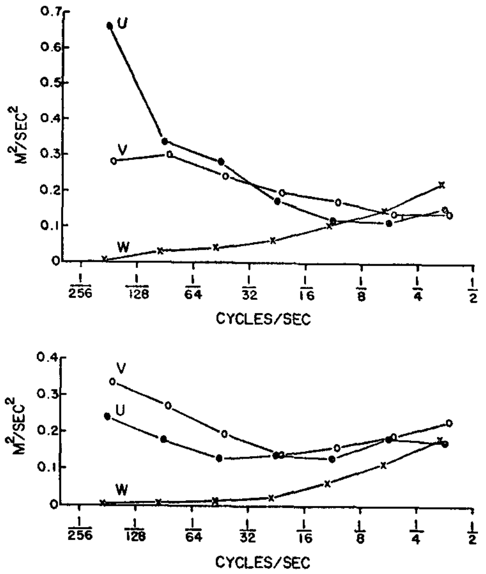

FIG. 2. Power spectra at 11.9 m (top) and 2.3 m (bottom), 0955 to 1006 EST 11 June 1952.

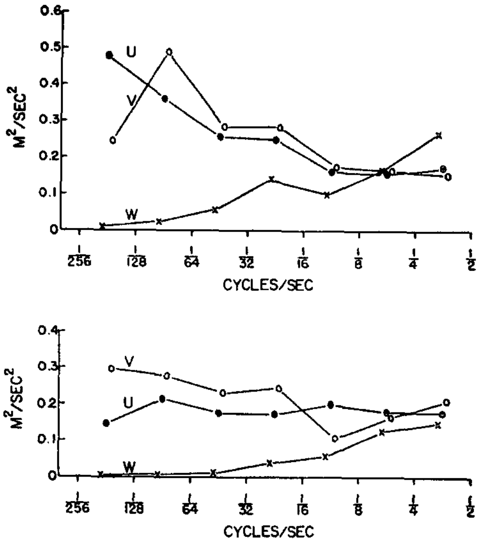

FIG. 3. Power spectra at 11.9 m (top) and 2.3 m (bottom), 1310 to 1322 EST 11 June 1952.

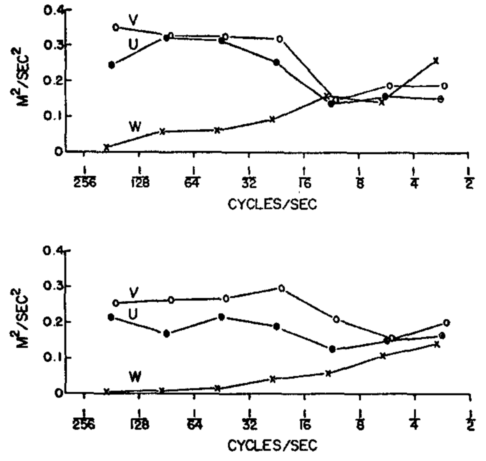

FIG. 4. Power spectra at 11.9 m (top) and 2.3 m (bottom), 1524 to 1534 EST 11 June 1952.

TABLE 2. Computed values of exponent β for combined daytime data of 11 June 1952.

Z (m)	u-component	v-component	w-component
11.9	-0.29	-0.20	0.67
6.4	-0.21	-0.20	0.77
3.7	-0.12	-0.14	0.75
2.3	-0.04	-0.11	0.83

of the form

$$(E_k/E_0) \propto k^{\beta},$$

where k is the frequency, and β is a constant. The values thus obtained for the exponent β are shown in table 2. Similar quantities for the w-spectra are also listed. While the numerical results are to be regarded merely as very rough approximations, the indicated variations of the component spectra with height and frequency appear quite reasonable.

This segment of the turbulence spectrum has apparently not been subjected to power-law analysis by other investigators. As might be anticipated, quite different values of the exponent apply to the region of isotropic turbulence. MacCready [4], for example, finds approximate agreement between the power spectra for all three components and the theoretical $k^{-5/3}$ law in the frequency interval from 2.5 to 40.0 cycles/sec at a height of 15 cm. In their analysis of component spectra at 300 ft, McCormick and Singer [7] report a somewhat larger negative exponent for frequencies greater than 90 cycles/hr where approximate equipartition of turbulent energy is indicated. Power spectra for the vertical component at 300 ft have been studied by Panofsky [8; 9] and a function has been suggested describing the variation of the vertical-velocity spectrum with frequency and height. The w-spectra values of table 2 seem to be in qualitative agreement with this expression in the low-frequency region. No attempt has been made, however, to fit the observed data to the proposed law, since the frequency range investigated does not appear to be sufficiently broad to justify this procedure.

Component spectra for the nighttime observations of 11 June are presented in fig. 5. At 2.3 m, the general behavior is quite similar to that of the daytime cases previously discussed; as might be anticipated, the magnitude of the estimates is considerably lower at night than during the day. At 11.9 m, there is a very definite tendency for the v-component to follow the w-component so that both distributions show minimum contributions at low frequencies. It is not known whether or not this is typical of nighttime data. The behavior of the u-component, except for a surprising peak near 1/64 cycles/sec that is probably due to sampling variations, is similar to that observed in the daytime cases mentioned previously.

The results for the water-trajectory observation of 20 August appear in fig. 6; it will be noted that the relationship between the spectra follows the general pattern for land trajectories described earlier. This result is in agreement with the findings of McCormick and Singer [7]. The magnitude of the individual estimates is, on the average, only about one-tenth that of comparable daytime values for land trajectories and about one-half that obtained for the nighttime observations. When the fluctuations become this small, the resolving power of the instrumentation may not be sufficient to permit a very precise determination of the spectra.

Discussion of the data for 10 June has purposely been delayed, because the results are in a sense intermediate between those obtained during the day over land and those of 20 August for a water trajectory. The spectral distributions for all three components (see fig. 7) decline with decreasing frequency, the estimates at 2.3 m showing a very marked decrease. At this level, the variances for the highest frequency interval are approximately the same magnitude as those of the 11 June daytime data; the variances at low frequencies are comparable in magnitude with the estimates of 20 August. Throughout the range of frequencies investigated, contributions from the u-component are relatively larger than those from the other components; the spectral estimates for the v- and w-components are approximately equal. It seems reasonable to infer that, during the short land trajectory, there has been an injection of energy at high frequencies with the largest injection occurring at 2.3 m.

With the exception of the data for 10 June, which are considered a special case in view of the complex trajectory, the general relationship between the spectral distribution of the eddy-velocity components indicated by the results may be summarized as follows. At the higher frequencies studied, the variances for all three components are approximately equal; this signifies equipartition of turbulent energy. At lower frequencies, the u- and v-spectra either remain constant (lower levels) or increase (upper levels). The w-spectra, on the other hand, decline sharply with decreasing frequency; contributions to the vertically-directed energy are considered negligible in the lowest frequency intervals studied. The tendency for low-frequency turbulent energy to be directed predominantly along the horizontal coordinates has previously been noted by McCormick [6] from the analysis of observations at 91 m. Approximate equipartition of turbulent energy at high frequencies has been reported by Panofsky [8], McCormick [6], MacCready [4] and others. The frequency at which turbulence becomes isotropic varies with height; Taylor [11] has suggested that a minimum value is given by $\bar{u}/z$, where $\bar{u}$ is the mean wind speed and z is the height.

There is an additional feature of these data that deserves comment. Due to the resonance character-

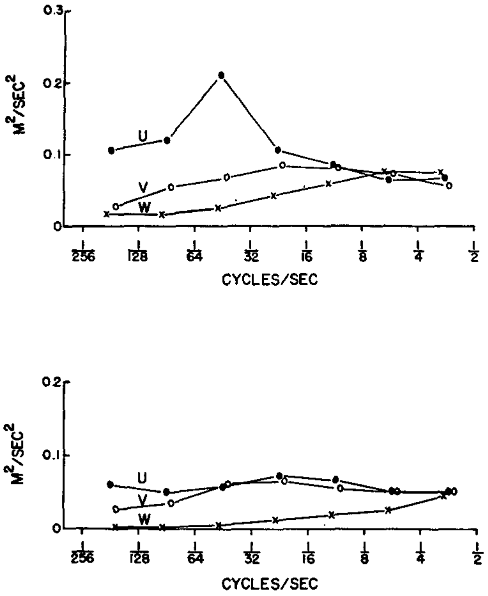

FIG. 5. Power spectra at 11.9 m (top) and 2.3 m (bottom), 2119 to 2130 EST 11 June 1952.

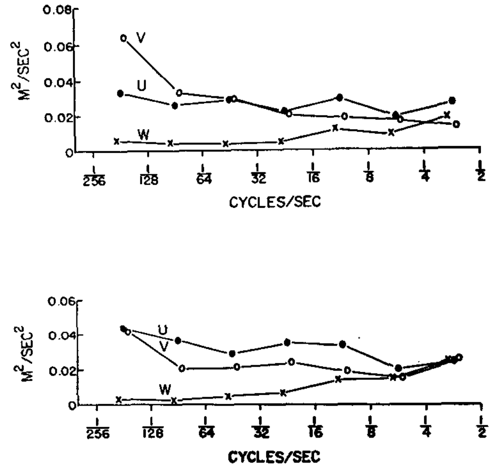

FIG. 6. Power spectra at 11.9 m (top) and 2.3 m (bottom), 1557 to 1609 EST 20 August 1952.

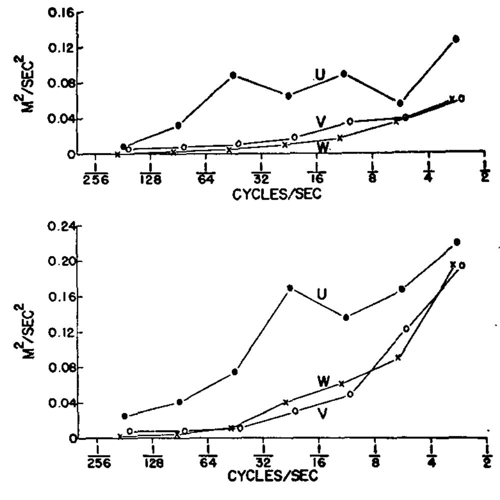

FIG. 7. Power spectra at 11.9 m (top) and 2.3 m (bottom), 1500 to 1512 EST 10 June 1952.

istics of the bivane response, allowance should be made for the possible "aliasing" of high frequency fluctuations in the *v*- and *w*-spectra; estimates for the *u*-component are probably not significantly affected. "Aliasing" refers to the fact that fluctuations with frequencies greater than $1/(2\ \delta t)$ cycles/sec add energy to the computed spectrum at lower frequencies. In particular, the greatest contribution would appear in the $\frac{1}{2}$ to $\frac{1}{4}$ cycles/sec interval of the present investigation. While a precise estimate of this effect is not possible, it appears that the computed spectral estimates for the *w*-component may be 20 to 25 per cent too large in this frequency interval; estimates for the *v*-component would presumably also be too large. If the results were arbitrarily corrected in this rough fashion, the principal effect would be to displace the region of equipartition of turbulent energy towards somewhat higher frequencies.

5. Conclusion

It is believed that the above results indicate the gross features of the power spectra of the eddy-velocity components within the 2- to 12-m layer over an important range of frequencies and under various conditions of mean wind speed, thermal stability, and surface roughness. This conclusion is supported by an independent analysis of approximately one-half of the basic data used in the investigation by a moving-average technique; although this is admittedly a less precise method, the characteristics of the spectral distributions thus obtained are in very close agreement with those discussed above.

Acknowledgments.—The writers are indebted to Dr. J. W. Tukey and Dr. A. Fleisher for many helpful discussions of spectral analysis, to Dr. E. W. Hewson for his interest and encouragement, and to Mr. George Fontes for assistance with the computations and for completing the figures.

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