PRESSURE-CHANGE THEORY AND THE DAILY BAROMETRIC WAVE

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ABSTRACT

It is proposed that both the semi-diurnal and the diurnal components of the daily pressure wave are the result of horizontal divergence associated with the daily temperature variation. A combined theoretical-empirical approach suggests that the divergence can be attributed to the Brunt-Douglas isallobaric wind if certain modifications are introduced into the original Brunt-Douglas equation. These modifications include the retention of the vertical Coriolis term in the equations of motion, and the assumption that accelerations caused by changes in the thermal gradient are continually and automatically balanced by changes in the pressure gradient resulting from the horizontal divergence. The proposed theory offers a quantitative explanation for the observed latitudinal distribution of the daily barometric wave.

1. Introduction

The classical explanation for the daily atmospheric pressure wave attributes the marked semi-diurnal component of the wave to the phenomenon of resonance. The resonance theory stems from a suggestion by Kelvin that the atmosphere may possess a natural oscillation of period almost exactly 12 hr. As a result of this circumstance, the small semi-diurnal temperature disturbance and also the solar tidal effect may, according to Chapman [1], be amplified many times over. Although the resonance theory has gained wide, if not quite unanimous acceptance, there is no *a priori* reason why it need be considered the only possible explanation for the semi-diurnal pressure wave. An alternative theory developed in the following pages seeks to explain both the diurnal and semi-diurnal components of the wave as a natural consequence of the adjustment of the pressure field to thermal changes.

It will become apparent that the approach followed in the present study is in part theoretical, in part empirical. The study began as an attempt to verify a pressure-change hypothesis which may be readily developed from the Brunt-Douglas [2] isallobaric wind equation and the equation of continuity. Thus the original physical model is essentially that used by Wexler [3], who ascribed the development of polar anticyclones to convergence of the isollabaric wind over a region of pronounced surface cooling. However, the philosophy has been that where theory does not agree with observation, it is the better part of valor to alter the theory. These alterations involved the use of a number of more or less plausible assumptions. While some of the assumptions appeared to be quite justifiable, others, made for purely empirical reasons, were not easily interpreted from a physical point of view and appeared to have disastrous consequences for the original development. The preliminary approach was therefore modified; the revision appears in the form of an appendix to the article proper, its position serving to emphasize the inductive nature of the argument. It is the writer's hope that the appended theory, which proceeds from the fundamental equations of motion and aims at a physical interpretation of the mathematical steps, will present to the reader a reasonable, internally consistent explanation for the daily pressure wave. The physical reasoning owes much to the work of Gilman [4], who in an unpublished doctoral dissertation has discussed a thermal theory of pressure changes, qualitatively, in greater detail than the scope of this article allows.

2. Some general considerations

The diurnal inequalities of temperature, wind and pressure may be visualized as perturbations travelling at constant velocity around the earth, the amplitudes of the waves being dependent on latitude and season alone. This is clearly an oversimplification, since the properties of the atmosphere and of the earth's surface are not everywhere the same. It is, however, an idea sufficiently accurate to justify some of the following conclusions:

1. The spatial scale of the diurnal disturbances is quite large, being comparable with that of the earth; the period is relatively short.
2. The diurnal variation of a property at a fixed point should be, to a close approximation, identical in form with the spatial derivative of the property (along the west-east coordinate axis) evaluated for the same point throughout the day.
3. The horizontal gradients of the several elements may be assumed to be quite small, because of the immense scale of the systems.
4. The diurnal wind velocity is small (of the order of 10 to 10^2 cm/sec except in the upper atmosphere).
5. Because the wind velocity is small and the gradients of the various elements are also small, it is unlikely that horizontal

advection plays an important part in the diurnal changes. Variations in the horizontal plane may be attributed entirely to the local change of the property at a fixed point.

6. It seems probable that the short period of the diurnal temperature wave, in combination with its great lateral extent, imposes a restriction on the air's motion. Humphreys [5] remarks, with reference to diurnal changes in wind direction: "The whole sequence results from the thermal expansion of the atmosphere (progressive from east to west), which causes an increase of pressure and consequently an outward flow at all levels above the surface. The area covered is so vast that the time involved, only a few hours, is insufficient for the completion of the convective circuit, so that even the surface winds are *away* from the heated regions, as stated, and not toward them, as in sea and land breezes, for instance." This conclusion seems to be supported by the fact that diurnal variations in rainfall and cloudiness do not suggest the existence of such closed convective circuits, but are adequately explained by variations in the vertical lapse rate of temperature.

7. The motion of the diurnal temperature wave is so rapid that it is unlikely the diurnal wind ever attains geostrophic equilibrium. Observations in the free air suggest that the accelerational and rotational terms are of about equal magnitude.

These generalizations will be referred to from time to time as justification for several steps in the development to follow.

3. A relation between the diurnal variations of surface temperature and pressure

In the following discussion and throughout the remainder of this article, it is implicity assumed that the divergence of the geostrophic wind is negligible compared with the divergence of the ageostrophic wind. The ageostrophic wind is used here in the Brunt-Douglas [2] sense, and its divergence may be obtained either directly by substitution of the isallobaric wind in the continuity equation or, as Sutcliffe [6] has shown, by making certain assumptions regarding the vorticity and its rate of change. The equation relating the horizontal divergence to the time rate of change of the vorticity may be written as

$$\operatorname{div} \boldsymbol{V} = -\frac{1}{\zeta + f}\frac{d}{dt}(\zeta + f), \qquad (1)$$

where ζ is the relative vorticity, and f is the vorticity due to the earth's rotation. The following assumptions are now made: (1) ζ may be considered small compared with f; (2) the advection of vorticity may be neglected; (3) ζ may be taken as the vorticity of the geostrophic wind; and (4) the latitudinal variation of f may be ignored. Equation (1) then becomes

$$\operatorname{div} \boldsymbol{V} = -\frac{1}{\rho f^2}\frac{\partial}{\partial t}\nabla^2 p, \qquad (2)$$

where $\nabla^2 p$ is the horizontal Laplacian of the pressure, and ρ is the density. This equation expresses the divergence of the Brunt-Douglas isallobaric wind.

It will be evident from generalizations made in the preceding section that assumptions (1) and (2) above are probably justifiable in the present application. The same cannot be said for the third assumption, for it was concluded that the diurnal wind is not quasi-geostrophic in the usual sense. The final assumption is one that is often used, but it is not evident at this point that the latitudinal variation of f may be neglected.

If the temperature gradient is assumed to be independent of height, (2) may be written with some approximation as

$$\operatorname{div} \boldsymbol{V} = -\frac{p}{\rho p_0 f^2}\frac{\partial}{\partial t}\nabla^2 p_0 - \frac{gz}{T_0 f^2}\frac{\partial}{\partial t}\nabla^2 T_0. \qquad (3)$$

Here the subscripts zero refer to the values at the ground, g is the acceleration of gravity, and T is the temperature.

In section 2, it was noted that the diurnal air motions appear to occur almost exclusively in response to temperature changes. If the speed and lateral extent of the temperature wave are so great that pressure-gradient changes associated with the horizontal divergence play no active role in moving the air toward the heated regions away from the cooled areas, there is justification for neglecting the first term on the right-hand side of (3); for $\nabla^2 p_0$ is different from zero only by virtue of the horizontal divergence, which must initially be caused by the variation of temperature. This admittedly subjective argument is developed more precisely in the appendix. If the term in p_0 is neglected, and density advection is assumed to be unimportant, the pressure change at the ground is given by

$$\frac{\partial p_0}{\partial t} = \frac{g^2}{T_0 f^2}\frac{\partial}{\partial t}\left(\nabla^2 T_0 \int_0^\infty \rho z \, dz\right). \qquad (4)$$

A finite value H' may be chosen as the upper limit of integration, if the atmosphere is considered to be homogeneous. Then, if (4) is integrated with respect to time, and the right-hand side of the equation integrated with respect to height, we have

$$p'_{0,t} = \frac{\rho g^2 H'^2}{2 f^2 T_0} \nabla^2 T_{0,t}. \qquad (5)$$

Here $p'_{0,t}$, interpreted as the deviation of the surface pressure from its daily mean value, at some given time t, is represented as a function of the second space-derivative of the temperature. The variation of T_0 is, in effect, the variation of the mean virtual temperature of the entire atmospheric column.

In section 2 it was concluded that the diurnal variation of temperature should be nearly identical in form with the spatial derivative, taken along the west-east coordinate axis, throughout the course of a day. Thus a preliminary test of the form of (5) may be made by comparing the diurnal variation of pressure at a given point with the second time-derivative of the surface temperature at the same point. Since the diurnal variation of surface temperature is ordinarily well represented by the first two terms of an harmonic series (the correlation between fitted and observed curves being usually in the neighborhood of 0.98), such a comparison is readily made. It is to be expected that the observed pressure curves will show a lag with respect to the curves of $\partial^2 T_0/\partial t^2$, since the mean virtual temperature of the entire atmospheric column may be presumed to follow by several hours the temperature variation observed near the ground.

Comparisons of the sort described above are illustrated in fig. 1. The selection of stations was governed by the availability of data and by the desire to include points covering a wide range of latitudes, as well as island, coastal and continental localities. The amplitude of $\partial^2 T_0/\partial t^2$ is arbitrary in each case, the main preoccupation at this phase of the study being to establish the form of a possible relationship. Temperature and pressure data for fig. 1 were obtained from Hann [7; 8; 9; 10] and Shaw [11].

The similarity between the sets of curves shown in fig. 1 was accepted as encouraging evidence of the essential validity of (5), in spite of the doubtfulness of certain assumptions made in the derivation. It was therefore decided to try first to establish the quantitative relationship implied by (5) and, if this proved feasible, to attempt theoretical justification later. A preliminary study disclosed that the latitudinal distribution of the daily pressure wave, given any reasonable expression for the temperature wave, could be obtained only by replacing the Coriolis parameter, $f = 2\omega \sin\phi$, in (5) by the constant 2ω; thus,

$$p'_{0,t} = \frac{\rho g^2 H'^2}{8\omega^2 T_0} \nabla^2 T_{0,t}. \tag{6}$$

Since (6) is the result of a combined theoretical and empirical approach, the reader may prefer to regard this relationship between the diurnal changes of temperature and pressure as essentially empirical in nature. The details of this relationship are the subject of the following two sections; in the appendix, the physical implications of (6) are discussed in similar detail.

4. Diurnal temperature wave at the earth's surface and in the free atmosphere

Any convenient solution of (6) necessitates the choice of an appropriate analytical expression for T_0. Since the variation of T_0 is in effect the variation of the mean temperature of the entire atmospheric

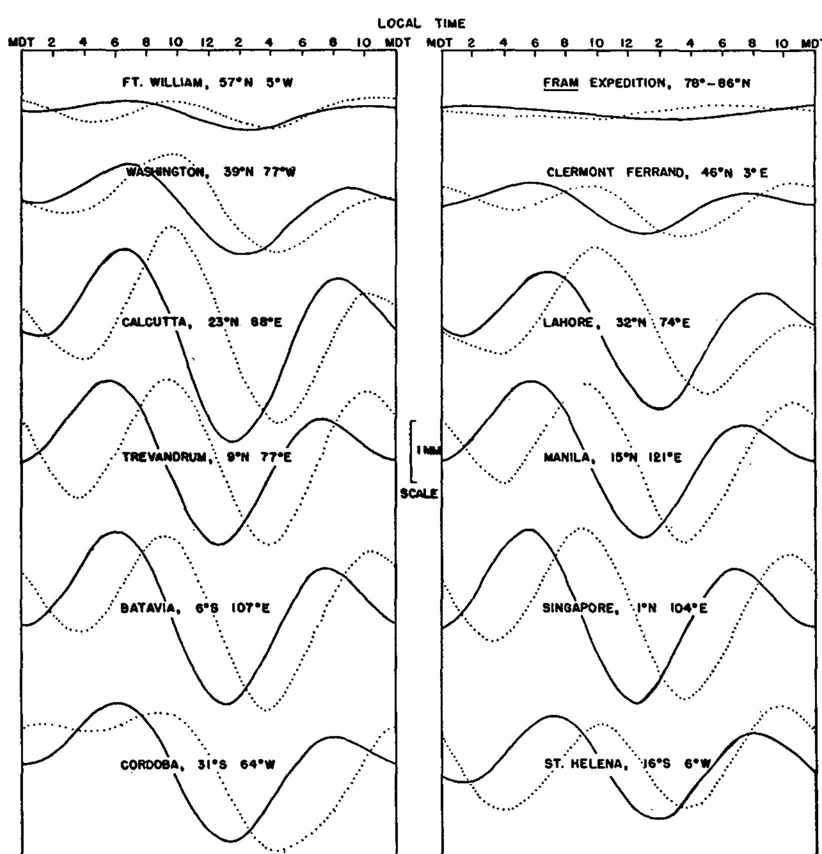

FIG. 1. Dotted curves: observed daily pressure oscillations, scale in millimeters. Solid curves: second time-derivative of surface temperature, arbitrary scale. Annual values.

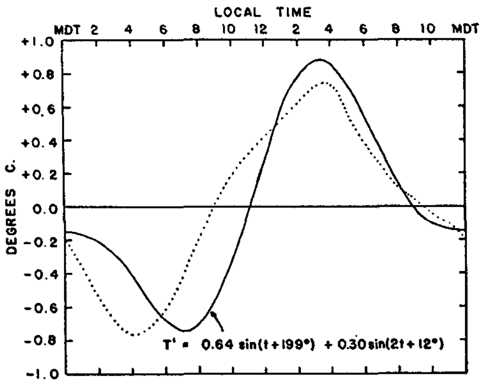

Fig. 2. Dotted curve: diurnal variation of temperature at Lindenberg, surface to 3000 m. Solid curve: diurnal variation of temperature in assumed vertical column between 300-m level (Clermont Ferrand) and 3100-m level (Sönnblick) over central Europe. Annual values.

column, this expression must be descriptive of the diurnal temperature changes in the free air. Observations of the diurnal changes to any great height are quite limited and perhaps not altogether reliable. It is, however, well established that the times of minimum and maximum temperatures are delayed by about two hours only a few hundred meters above the ground. Observations in the United States [12] and at Lindenberg [13] exhibit this lag to a height of 1500 m, after which the observed variation is small, irregular, and in all likelihood not representative. Radiosonde measurements at San Juan, and Antigua, reported by Riehl [14], show a mean diurnal range below 16 km of 2.34C. In the latter study, the occurrence of the maximum temperature near noon at even the extremely high levels suggests that the error introduced by direct insolation on the radiosonde instruments may not have been entirely removed.

In view of the instrumental difficulties involved in direct measurements, it seems probable that the most reliable estimate of the diurnal temperature variation within a vertical column is obtained by observing the hourly deviations from the mean pressure difference

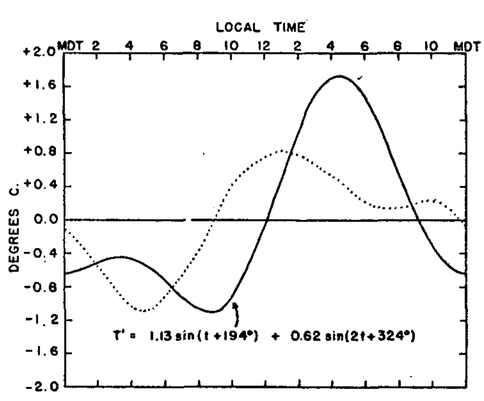

Fig. 3. Dotted curve: diurnal variation of temperature at Batavia, surface to 2000 m. Solid curve: diurnal variation of temperature in assumed vertical column between sea level (Madras) and 2600-m level (Dodabetta Peak) near equator. Annual values.

between a mountain-peak station and another station near sea level. In fig. 2, the diurnal temperature variation of an assumed vertical column between Sönnblick, in the Alps, and Clermont Ferrand, a station at approximately the same latitude, has been computed in this fashion. The data represent averages throughout the year and were obtained from Hann's work on the daily oscillation of pressure [8]. The observed temperature variation in the free air above Lindenberg [13] is shown in the same figure, the temperature being that of the column between the ground and the 3-km level.

Fig. 3 presents similar data for stations near the equator. The free-air temperature measurements are those reported by Bemmelen [15] for the first 2 km over Batavia, and the computed temperature variation was obtained from pressure observations at Dodabetta Peak, 2634 m, and Madras, near sea level. The latter observations were made for 77 corresponding days [7]. In figs. 2 and 3, the harmonic formulae representing the temperature variation obtained from the observed pressure differences are shown on the diagram. The ratios of the amplitude of the first to the second harmonic term are 1.82 at Madras and 2.13 over central Europe. This suggests that the amplitude of the semi-diurnal temperature wave in the free atmosphere decreases much more rapidly with increasing latitude than does the amplitude of the diurnal wave.

A similar conclusion with respect to the temperature near the ground is suggested in figs. 4 and 5. Here the amplitudes of the two terms are separately plotted, to show the latitudinal variation of the surface-temperature wave; the data are from [9; 10; 11] and from hourly temperature records in the United States. Annual means were used where these could be obtained; observations for the United States stations are averages for the months of June and December. The effect of continentality on the distribution of amplitudes (see also Wexler [16] and Terada [17]) is readily apparent, the observations having been arbitrarily divided into three groups denoting distance from the nearest large body of water. Because these groups are unevenly distributed according to latitude, the data were averaged by 10-deg latitude zones, and a weighted mean was obtained by treating each group of stations as a separate observation. The mean value is plotted at the center of each zone.

It might be expected that the dominant term in the daily temperature wave, the diurnal term, would bear some resemblance in its latitudinal distribution to the intensity of solar radiation. It is not surprising, therefore, that a simple cosine function, shown by the dotted curve in fig. 4, represents fairly well the amplitude of the diurnal temperature term. The amplitude of the semi-diurnal wave appears to be adequately

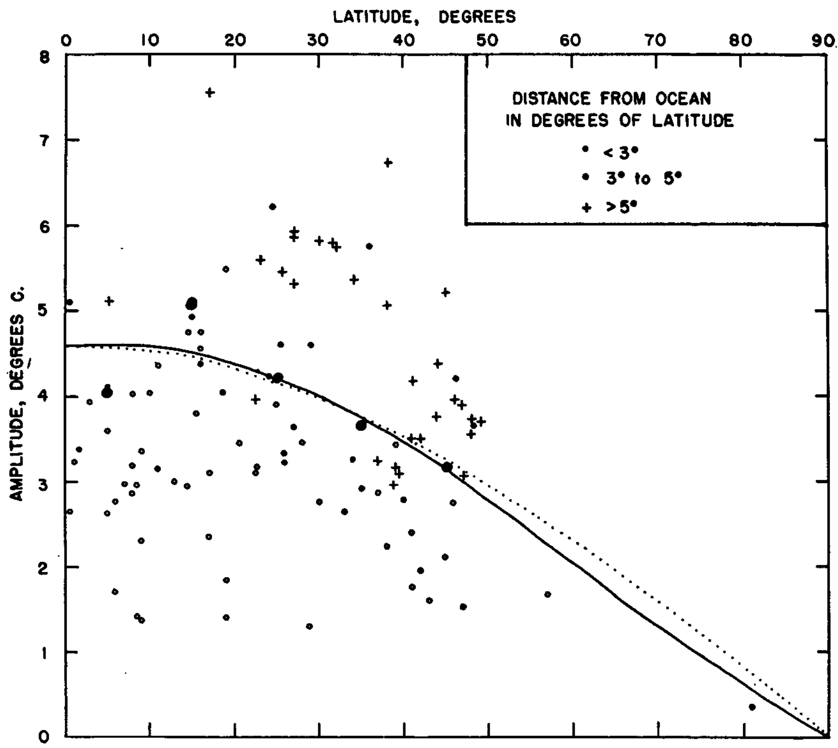

FIG. 4. Amplitude of diurnal temperature wave as function of latitude and continentality, annual values. Larger filled-in circles are weighted means of observations within each 10-deg latitudinal zone. Dotted curve: 4.6 cos ϕ; solid curve: 4.6 (cos ϕ + 0.06 cos 3ϕ).

represented by a power of the cosine of the latitude, as shown by the curve in fig. 5. The latter function decreases much more rapidly with increasing latitude than does either of the curves used to delineate the diurnal amplitude in fig. 4. At the equator, the ratio of the 24-hr to the 12-hr temperature term is 3.47. It was previously noted, from the observations at Dodabetta Peak and Madras, that near the equator in the free atmosphere the ratio appears to be about 1.82. Thus it would seem that the importance of the semi-diurnal temperature term is much more marked in the free air than at the surface of the earth.

From the foregoing, it appears that an idealized representation of the temperature wave at some fixed time t' should resemble the function

$$T' = A_1 \cos\phi \sin(\theta + \alpha_1) + A_2 \cos^n \phi \sin(2\theta + \alpha_2), \quad (7)$$

where A_1 and A_2 are the amplitudes of the diurnal and semi-diurnal temperature terms, respectively, ϕ is the latitude, θ the longitude measured eastward, and $-\alpha_1$ and $-\alpha_2$ are the longitudes at which the curves cross the axis upwards. If t' is chosen as midnight Greenwich time, θ_0 is the longitude of Greenwich.

It has been implicity assumed that the phase of the temperature wave is independent of latitude. Since the phase is probably largely determined by the times of sunrise and sunset, this assumption should be useful in working with the annual means. Similarly it will be convenient to assume that the temperature wave is independent of height, to avoid integration through numerous layers; the temperature data now available do not warrant any additional refinement.

In the following section, that function will be de-

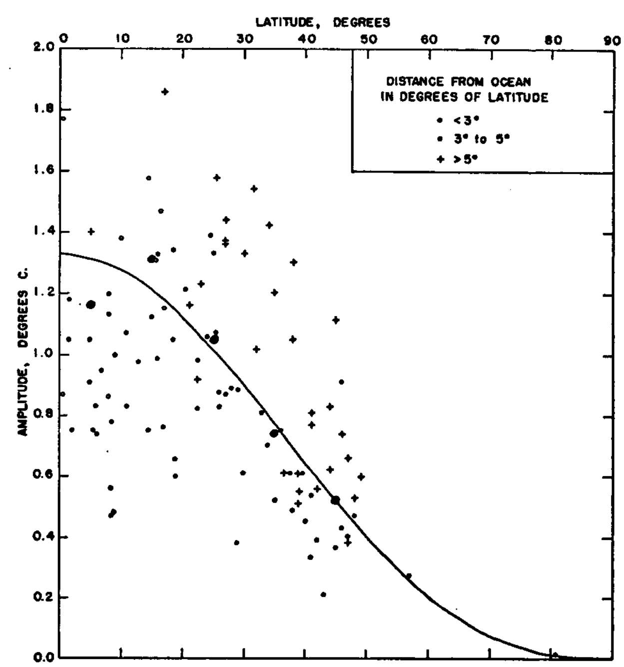

FIG. 5. Amplitude of semi-diurnal temperature wave as function of latitude and continentality, annual means. Larger filled-in circles are weighted means of observations within each 10-deg latitudinal zone. Curve: 1.327 $\cos^{2.3} \phi$.

termined which, having the general form of (7), best explains the observed daily oscillation of pressure. It will then be shown that this temperature function closely fits the observed temperature wave so far as the latter is known.

5. Daily pressure oscillation as a function of the temperature wave

If the physical model is to be realistic, the use of a homogeneous atmosphere should be avoided and the equations should apply to a spherical earth. Equation (6) may then be written as

$$p'_{0,t} = 0.308 \left(\frac{1}{\cos^2 \phi} \frac{\partial^2 T_0}{\partial \theta^2} + \frac{\partial^2 T_0}{\partial \phi^2} - \tan \phi \frac{\partial T_0}{\partial \phi} \right)_t . \quad (8)$$

The derivation of this equation is given in the appendix. It is sufficient, at this point, to remark that the model is based on a two-layer atmosphere in which the U. S. National Advisory Committee for Aeronautics constants have been used. Equation (8), in which a function of the type given by (7) is to be substituted for T_0, expresses the deviation from the mean pressure in millibars.

The work of Hann [7; 8] makes it possible to determine quite readily the average amplitude and phase of the daily pressure oscillation at each latitude. The data for all stations in the northern hemisphere, a total of 84, for which Hann obtained observations, were averaged vectorially to estimate the mean amplitude and phase of both the diurnal and semi-diurnal pressure waves within eight latitudinal zones. The resulting means are plotted on the polar diagrams shown in figs. 7 and 8. The individual values for each zone would show a wide scatter about the average position. In fig. 6, empirical functions used to represent the variation with latitude of the diurnal and semi-diurnal amplitudes are indicated by the circles plotted at every tenth degree of latitude. These empirical expressions are attributed by Hann [7] to Kämtz and Forbes. The amplitudes of the two terms

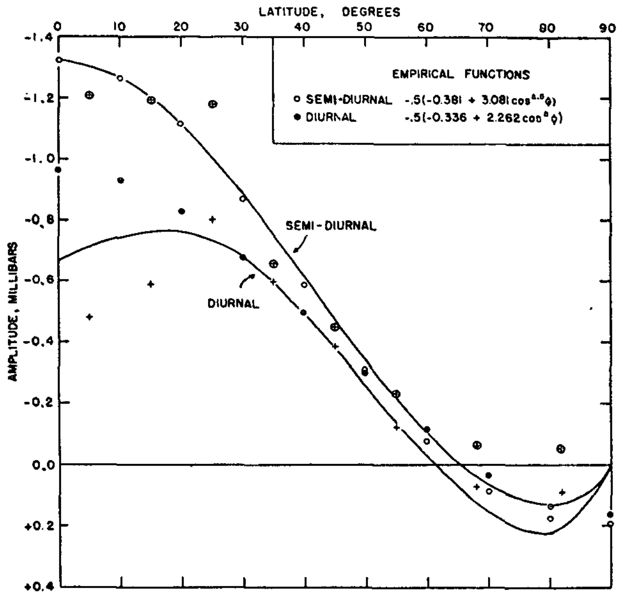

FIG. 6. Amplitude of diurnal and semi-diurnal pressure oscillations: filled-in circles, diurnal oscillation according to Kämtz; open circles, semi-diurnal oscillation after Forbes; crosses, observed amplitude of diurnal oscillation; circumscribed crosses, observed amplitude of semi-diurnal oscillation. Curves represent diurnal and semi-diurnal amplitude computed from (10). Annual values.

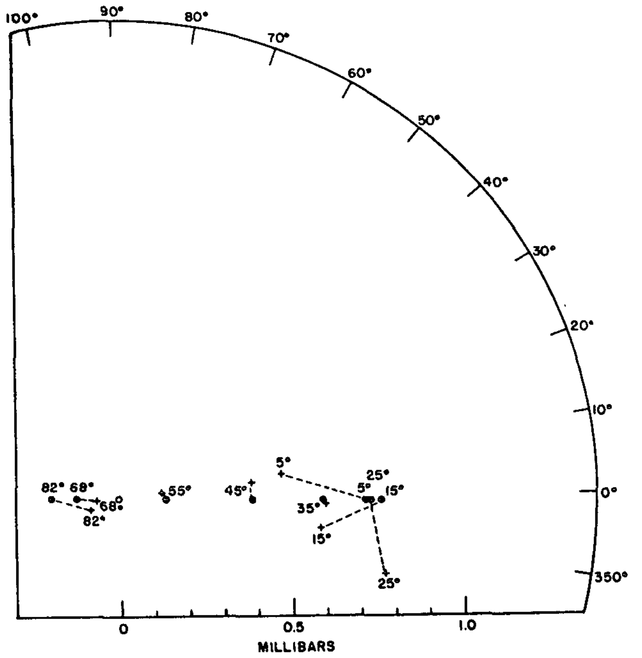

FIG. 7. Amplitude and phase of diurnal pressure oscillation at latitudes indicated. Filled-in circles, observed values; crosses, computed values. Annual values.

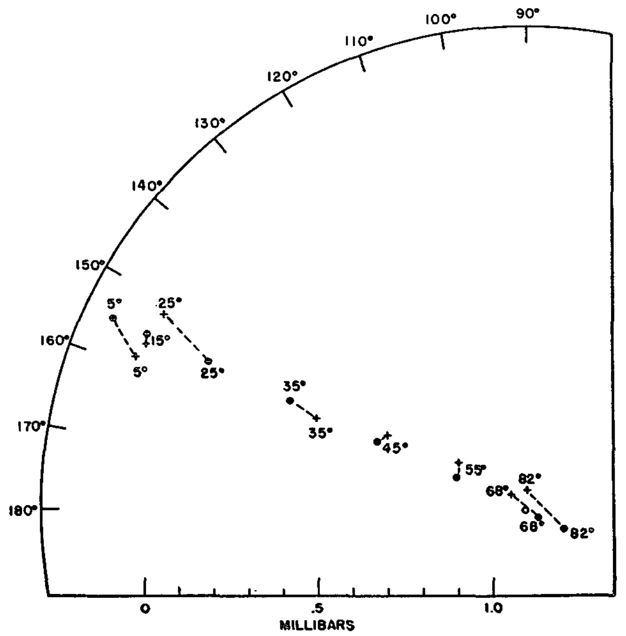

FIG. 8. Amplitude and phase of semi-diurnal pressure oscillation at latitudes indicated. Filled-in circles, observed values; crosses, computed values. Annual averages.

obtained from the actual observations are also shown in fig. 6, at the midpoints of the zones selected. The amplitudes have been arbitrarily prefixed by a negative sign where the observed phase of the pressure wave is approximately 180 deg different from the phase of the temperature wave; the positive amplitudes indicate an in-phase relationship, found by inspection, between the pressure and temperature waves.

If the following expression for the temperature wave is substituted in (8),

$$T' = A_1 (\cos\phi + 0.06\,\phi\cos 3\phi)\sin(\theta + \alpha_1) + A_2\cos^{2.3}\phi\,\sin(2\theta + \alpha_2), \quad (9)$$

and the indicated differentiation is carried out, the result is

$$\begin{aligned} p'_{0,t} = {} & 0.308\,A_1 [-\cos\phi + \sin\phi\tan\phi \\ & - \sec\phi + 0.06\,(-9\,\phi\cos 3\phi - 6\sin 3\phi \\ & + 3\,\phi\sin 3\phi\tan\phi - \cos 3\phi\tan\phi \\ & - \phi\cos 3\phi\sec^2\phi]\,[\sin(\theta + \alpha_1)]_t \\ & + 0.308\,A_2 [-2.3\cos^{0.3}\phi\,(\cos^2\phi \\ & - 2.3\sin^2\phi) - 4\cos^{0.3}\phi] \\ & [\sin(2\theta + \alpha_2)]_t. \quad (10) \end{aligned}$$

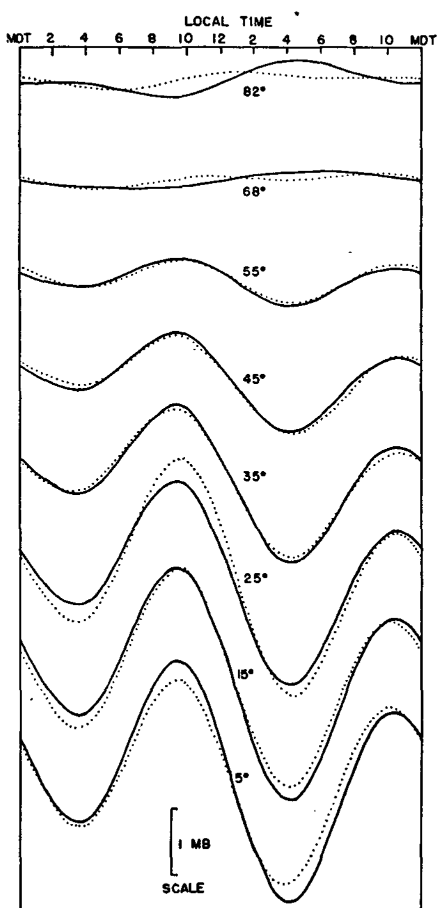

FIG. 9. Daily pressure variation at latitudes indicated. Dotted curves, observed variation; solid curves, computed variation. Annual values.

It is evident that A_1 and A_2 determine the amplitudes of a 24-hr and a 12-hr pressure oscillation, respectively. When $A_1 = 1.08$C and $A_2 = 0.68$C, the amplitudes are represented by the curves drawn in fig. 6. These resulting curves agree about as well as could be expected with the observed amplitudes of the pressure wave.

When in addition the phase angle α_1 is 180 deg, the "computed" amplitude and phase (corresponding to each mean latitude for which the observed values were obtained) are shown by the crosses on the polar diagram of fig. 7. Similarly, when α_2 is 335 deg, the computed amplitudes and phase angles of the semidiurnal pressure oscillation are those indicated in fig. 8. The agreement between observed and computed values is fairly close.

The computed and observed curves showing the daily pressure variation corresponding to the points on the polar diagrams are represented in fig. 9.

Thus, a temperature wave determining the observed latitudinal distribution of the daily barometric wave can be represented by the expression

$$T' = 1.08\,(\cos\phi + 0.06\,\phi\cos 3\phi)\sin(\theta + 180°) + 0.68\cos^{2.3}\phi\,\sin(2\,\theta + 335°). \quad (11)$$

The amplitude of this temperature wave has the same latitudinal variation as the amplitude of the surface-temperature wave represented by the solid curves of figs. 4 and 5. At the equator, the temperature wave is

$$T'_{\phi 0} = 1.08\sin(\theta + 180°) + 0.68\sin(2\,\theta + 335°), \quad (12)$$

and the ratio $A_1{:}A_2$, 1.59, may be compared with the observed value (1.82) obtained for Madras. The phase angles of the idealized wave also show reasonable agreement with the observed values for α_1, 194 deg, and α_2, 324 deg. That the agreement is probably within the limits of observational accuracy is sug-

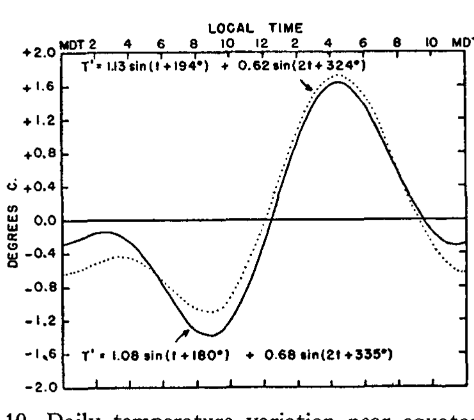

FIG. 10. Daily temperature variation near equator. Dotted curve, observed variation in assumed vertical column between sea level and 2600-m level; solid curve, assumed variation throughout troposphere and stratosphere, determined by (12). Annual values.

gested by fig. 10, where the two curves, observed and hypothetical, are plotted on the same diagram. At latitude 40 deg, the ratio $A_1:A_2$ is 2.18, and this value agrees closely with the observed ratio (2.13) obtained from the observations over central Europe.

The solid curve of fig. 10 supposedly describes the temperature variation at the equator of a column of air between sea level and the maximum height of the stratosphere. The mean daily range of temperature throughout the column is thus assumed to be about 3C, and this is nearly the same as the observed range between sea level and 2600 m. Observations at middle latitudes [12; 13] suggest that the diurnal variation becomes negligible in the vicinity of 5 km. Although Riehl's data [14] for the West Indies show the dirunal range between sea level and 16 km to be 2.34C, it has already been pointed out that the reliability of the latter data seems open to question. It does not appear to be possible to determine from data now available whether or not a variation of 3C for the entire atmospheric column is approximately correct. Johnson [18], using direct observations of ozone concentrations and of the ultraviolet spectral intensity of sunlight made during a rocket flight over New Mexico on 14 June 1949, has computed the diurnal temperature changes in the ozonosphere. The maximum variation, 5.3C, occurred at a height of 48 km, with minimum and maximum temperatures about an hour and a half after sunrise and before sunset, respectively. Since the diurnal temperature variation in the ozonosphere is thus roughly comparable in amplitude and phase to that observed in the lower troposphere, it seems at least possible that the upper troposphere and the stratosphere may exhibit a similar if somewhat smaller variation.

Tentatively it may be concluded that a temperature function determined by (8) and the observed daily oscillation of pressure is in agreement with the observed temperature wave in the free atmosphere, so far as the latter is known from measurements now available.

6. Concluding remarks

A theory has been presented which attempts to explain the daily pressure oscillation as a natural consequence of the daily variation of temperature. The phenomenon of the semi-diurnal pressure wave is attributed not to resonance amplification but to the divergence of the isallobaric wind. The resonance theory and the theory presented here may be regarded as alternative explanations for the semi-diurnal wave. In a sense, neither explanation can claim to be more than a rational hypothesis; but while each must exist on its own merits, there would appear to be a certain practical advantage in pursuing that hypothesis offering the wider application. That the theory developed above, and elaborated in the appendix to follow, traces both the semi-diurnal and the diurnal wave to the same cause may possibly be taken as a point in its favor and as an invitation to further investigation.

APPENDIX

Although it is a meteorological truism that the actual wind, above the layer of surface friction, closely approximates the geostrophic value, the probable reason for this rather remarkable phenomenon is rarely discussed at any great length. The usual explanation, that the accelerational term is small, is obviously incomplete, since it merely poses another question: why is the accelerational term small? The problem is rationalized in the familiar concept of "a mutual adjustment between wind and pressure gradient". Shaw expresses this fundamental rationalization in the following way [19]:

> "Apart from the influence of penetrative convection the gradient of pressure at any given level is controlled by the earth's rotation acting through the uncompensated component of horizontal velocity.
>
> "The force $B-2\omega V\rho \sin\phi$ is always in operation at any level for reestablishment at that level of the condition $B-2\omega V\rho\sin\phi=0$, and air is transported up or down the gradient to secure the adjustment."

The symbol B is used to denote barometric gradient. Since the change in the pressure gradient obviously depends on the transport of air at all levels above and below the level in question, Shaw's statement is perhaps not sufficiently explicit. However, the simplicity of the precept renders it useful for mathematical discussion. If the vertical Coriolis term is retained, the statement may be rewritten as

$$B_x + 2\omega\, w \cos\phi - 2\omega\, v \sin\phi = 0, \qquad (13)$$

with B_x representing the pressure-gradient force acting along the x-axis. If B_x changes by an amount ΔB_x, the initial acceleration across the gradient is

$$du/dt|_{t_0} = \Delta B_x. \qquad (14)$$

The pressure gradient is altered, Shaw's argument implies, as a result of horizontal divergence accompanying the cross-isobar flow; and the gradient is changed in such a way as always to oppose the initial accleration; i.e., $du/dt|_{t_0}$ approaches zero. It may be assumed, therefore, that after a short interval of time $(t' - t_0)$ equilibrium is reestablished, so that

$$B'_x + 2\omega w' \cos\phi - 2\omega v' \sin\phi = 0. \qquad (15)$$

It appears that a realistic theory of pressure changes must account for the change in the pressure gradient force $B'_x - B_x$ in the interval of readjustment to equilibrium flow. It may also be noted that $v' - v$ represents the change in the geostrophic wind over a time interval during which an ageostrophic component develops and diminishes to zero. According to the Brunt-Douglas derivation, the change in the geostrophic wind is a measure of the ageostrophic or isallobaric wind. The development which follows differs slightly from the original, in that the vertical component of the earth's deflecting force is retained in the primitive equations.

For convenience of representation, an initial state is assumed in which the horizontal pressure and temperature gradients are zero. For B_x, the pressure gradient force may be substituted in terms of the pressure gradient at the ground and the temperature gradient throughout the intervening layer,

$$B_x = -\frac{T_0 - \beta z}{\rho_0 T_0}\frac{\partial p_0}{\partial x} - \frac{gz}{T_0}\frac{\partial T_0}{\partial x}. \tag{16}$$

Here the temperature gradient is assumed to be independent of height, and β is the vertical lapse rate of temperature. Then if the temperature gradient, which at t_0 is zero, takes some other value, the initial equation of motion is

$$\left.\frac{du}{dt}\right|_{t_0} = -\frac{gz}{T_0}\frac{\partial T_0}{\partial x}. \tag{17}$$

As soon as motion occurs, the particles are deflected from a straight path across the temperature gradient. The equation of motion is then

$$\frac{du}{dt} + 2\omega w \cos\phi - 2\omega v \sin\phi = -\frac{T_0 - \beta z}{\rho_0 T_0}\frac{\partial p_0}{\partial x} - \frac{gz}{T_0}\frac{\partial T_0}{\partial x}. \tag{18}$$

According to Shaw's precept, it may be inferred that

$$\left.\frac{du}{dt}\right|_{t_0} + \left.\frac{T_0 - \beta z}{\rho_0 T_0}\frac{\partial p_0}{\partial x}\right|_{t_0+\Delta t} = 0, \tag{19}$$

since the acceleration is shortly balanced by a change in the pressure gradient caused by the horizontal divergence accompanying cross-gradient flow. At time $t_0 + \Delta t$, equilibrium will have been reestablished, and velocity components v and w will have been developed; these components are implicity a measure of the cross-gradient velocity component, u. This may be seen by differentiating (18) with respect to time, following the particle in its path,

$$\frac{d^2u}{dt^2} + 2\omega\cos\phi\frac{dw}{dt} - 2\omega\sin\phi\frac{dv}{dt} = -\frac{d}{dt}\left(\frac{T_0 - \beta z}{\rho_0 T_0}\frac{\partial p_0}{\partial x}\right) - \frac{d}{dt}\left(\frac{gz}{T_0}\frac{\partial T_0}{\partial x}\right), \tag{20}$$

and substituting for dw/dt and dv/dt from the equations defining the acceleration on a moving particle due to the earth's rotation,

$$du/dt = -2\omega w\cos\phi + 2\omega v\sin\phi,$$
$$dv/dt = -2\omega u\sin\phi, \tag{21}$$

and

$$dw/dt = 2\omega u\cos\phi.$$

It is well to remember that the Coriolis deflection is independent of any real force and occurs only as a result of motion on a rotating earth. With this substitution and a slight rearrangement of terms, (20) becomes

$$\frac{d}{dt}\left(\frac{du}{dt} + \frac{T_0 - \beta z}{\rho_0 T_0}\frac{\partial p_0}{\partial x}\right) + 4\omega^2 u = -\frac{d}{dt}\left(\frac{gz}{T_0}\frac{\partial T_0}{\partial x}\right). \tag{22}$$

But if Shaw's statement is valid, the relationship implied by (19) may be used as justification for the neglect of the first term in (22); for if the time interval required for readjustment to equilibrium flow is sufficiently short, this term is, essentially, of higher order than the remaining terms considered separately. The equation for the isallobaric wind may then be written as a function of the time-rate-of-change of the temperature gradient alone,

$$u' = -\frac{gz}{4\omega^2}\frac{d}{dt}\left(\frac{1}{T_0}\frac{\partial T_0}{\partial x}\right) \tag{23}$$

and

$$v' = -\frac{gz}{4\omega^2}\frac{d}{dt}\left(\frac{1}{T_0}\frac{\partial T_0}{\partial y}\right),$$

the primes in this case being used to indicate the ageostrophic wind.

Equations (23) may be written to represent the motion at a fixed point,

$$u' = -\frac{gz}{4\omega^2}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\left(\frac{1}{T_0}\frac{\partial T_0}{\partial x}\right),$$

and (24)

$$v' = -\frac{gz}{4\omega^2}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\left(\frac{1}{T_0}\frac{\partial T_0}{\partial y}\right),$$

or, since it was concluded that advection is of no significance for the diurnal changes,

$$u' = -\frac{gz}{4\omega^2}\frac{\partial}{\partial t}\left(\frac{1}{T_0}\frac{\partial T_0}{\partial x}\right),$$

and (25)

$$v' = -\frac{gz}{4\omega^2}\frac{\partial}{\partial t}\left(\frac{1}{T_0}\frac{\partial T_0}{\partial y}\right).$$

In other applications, such as those discussed by Gilman [4] and Appleby [20], it might be preferable to neglect the local change of temperature gradient and to consider instead the advective terms responsible for "differential advection."

It may be easily verified that substitution of u' and v' from (25) in the tendency equation yields, after integration,

$$p'_{0,t} = (\rho g^2 H'^2/8\omega^2 T_0)\, \nabla^2 T_{0,t}, \tag{26}$$

when a homogeneous atmosphere is assumed. This is identical with (6), which essentially satisfies a relationship between the diurnal variations of temperature and pressure. The assumption of a homogeneous atmosphere may be avoided by writing the tendency equation in its exact form,

$$\frac{\partial p_0}{\partial t} = -g \int_0^\infty \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \right] dz, \tag{27}$$

and substituting u' and v' from (25) for u and v, respectively:

$$\begin{aligned}\frac{\partial p_0}{\partial t} = \frac{g^2}{4\omega^2 R}\frac{\partial}{\partial t}\int_0^\infty z\Bigg\{ &\frac{p}{T_0(T_0-\beta z)}\left[\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right] \\ &+ \frac{1}{T_0(T_0-\beta z)}\left[\frac{\partial T_0}{\partial x}\frac{\partial p}{\partial x}+\frac{\partial T_0}{\partial y}\frac{\partial p}{\partial y}\right] \\ &- \left[\frac{p}{T_0(T_0-\beta z)^2}+\frac{p}{T_0^2(T_0-\beta z)}\right] \\ &\times\left[\left(\frac{\partial T_0}{\partial x}\right)^2+\left(\frac{\partial T_0}{\partial y}\right)^2\right]\Bigg\} dz. \end{aligned} \tag{28}$$

Here R is the gas constant for dry air. The last two terms under the integral sign of (28) are assumed to be small and will be neglected. Since the temperature changes are taken to be independent of height, this equation becomes, after integration with respect to time,

$$p'_{0,t} = \left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right)_t \frac{g}{4\omega^2 R T_0}\int_0^\infty \frac{pz}{T_0-\beta z}\,dz. \tag{29}$$

For a two-layer atmosphere, with $\beta \neq 0$ in the layer $z = H_0$ to $z = H_1$, and $\beta = 0$ in the layer $z = H_1$ to $z = H_2$, (29) becomes

$$\begin{aligned} p'_{0,t} = \left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right)\Bigg\{ &\frac{g^2 p_0}{4\omega^2 R T_0^{n+1}} \times \int_{H_0}^{H_1} z(T_0-\beta z)^{n-1}\,dz \\ &+ \frac{g^2 p_0 H_1 (T_0-\beta H_1)^{n-1}}{4\omega^2 R T_0^{n+1}} \times \int_{H_1}^{H_2} \exp\frac{-g(z-H_1)}{R(T_0-\beta H_1)}\,dz \\ &+ \frac{g^2 p_0 (T_0-\beta H_1)^{n-2}}{4\omega^2 R T_0^{n}} \times \int_{H_1}^{H_2} (z-H_1)\exp\frac{-g(z-H_1)}{R(T_0-\beta H_1)}\,dz \Bigg\}, \end{aligned} \tag{30}$$

where $n = g/R\beta$. The two integrations through the layer $H_2 - H_1$ arise from the necessity of considering the thermal pressure change in the stratosphere as the sum of two components: a component due to the variation of temperature in the troposphere, the stratosphere temperature constant; and a component due to the temperature change in the stratosphere, the pressure at the tropopause held constant.

For the U. S. Standard Atmosphere, the following constants are appropriate:

$$\begin{aligned} T_0 &= 288\text{A}, \\ p_0 &= 1013.25 \text{ mb}, \\ H_1 &= 1.0769 \times 10^6 \text{ cm}, \\ H_2 &= 3.2 \times 10^6 \text{ cm}, \\ \beta &= 6.5 \times 10^{-5} \text{ deg/cm}, \; H_0 \leq z \leq H_1, \end{aligned}$$

and

$$\beta = 0,\; H_1 \leq z \leq H_2.$$

Values for the remaining constants were chosen as follows: $g = 980$ cm/sec.^{2}, $R = 2.87 \times 10^6$ cm^2 sec^{-2} deg^{-1}, and $\omega = 7.29 \times 10^{-5}$ sec^{-1}. The deviation of the pressure from its daily mean value, in millibars, is then

$$p'_{0,t} = 1.25 \times 10^{17}\left(\frac{\partial^2 T_0}{\partial x^2}+\frac{\partial^2 T_0}{\partial y^2}\right)_t, \tag{31}$$

which may be written, in polar spherical coordinates, as

$$p'_{0,t} = \frac{1.25 \times 10^{17}}{a^2} \times \left(\frac{1}{\cos^2\phi}\frac{\partial^2 T_0}{\partial\theta^2}+\frac{\partial^2 T_0}{\partial\phi^2}-\tan\phi\frac{\partial T_0}{\partial\phi}\right)_t. \tag{32}$$

Here a is the radius of the earth, and a suitable value is 6.37×10^8 cm. The coefficient of the right-hand side of (32) is thus 0.308, making this equation identical with (8).

It will perhaps be helpful at this point to recall those assumptions which, in the preliminary approach, were justified only on empirical grounds. The investigation of these assumptions has, of course, been the principal subject of this appendix.

1. The assumption that the diurnal wind is quasi-geostrophic proved to be unnecessary. The quasi-geostrophic nature of the large-scale wind currents is offered as evidence for the existence of forces tending to bring about a mutual adjustment between wind and pressure gradient. These forces may be presumed to operate also in the case of the diurnal inequalities of wind and pressure gradient; it is not necessary that the diurnal wind approach an equilibrium state to the degree attained by the large-scale air currents.

2. The variation of the pressure gradient at the ground, attributable to the horizontal divergence, was neglected in the preliminary study. It has been seen that the horizontal divergence, through its partial control of the pressure gradient, acts as a continual brake upon the acceleration, or as a force tending to restore an equilibrium state never quite attained. If the interval required for the restoring force to act is short, the time-rate-of-change of the two opposed forces may be assumed to be essentially

a higher-order term than the remaining forces considered separately.

3. The substitution of the term 2ω for $f = 2\omega \sin \phi$ has been justified by a consideration of the vertical as well as the horizontal components of the deflecting force. It follows that the latitudinal variation of the Coriolis parameter is of no significance in the determination of the isallobaric divergence.

The physical model examined in the foregoing pages represents an attempt to explain a specific phenomenon, namely the relationship between the diurnal variations of temperature and pressure. The proposed theory attributes ageostrophic motion to thermal changes alone; it seems unlikely, therefore, that it can account for the commonly observed fact that the residual divergence represents the difference between convergence throughout one or more layers and divergence throughout other layers. The theory does not allow for any appreciable vertical motion, although paradoxically it may serve to indicate where vertical motion is taking place. So far as interdiurnal pressure changes are concerned, the theory may perhaps reveal the pattern but not the magnitude of the actual changes.

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