WAVE SOLUTIONS OF THE VORTICITY EQUATION FOR THE 2½-DIMENSIONAL MODEL

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ABSTRACT

Wave solutions of the non-linear quasi-geostrophic equations for the so-called "2½-dimensional model" without friction are derived. The solutions describe wave motions which propagate at different speeds in each layer. The amplitudes of the disturbances are either changing periodically with time (stable waves) or increasing exponentially (unstable waves). The critical value of wavelength, at which waves maintain a constant amplitude, is expressed as a function of thermal stability and of vertical wind shear in the basic current.

It is also found that, while the inclination of trough lines to the vertical increases monotonically with time in stable waves, it varies very slowly in the unstable waves and tends to approach a certain limiting value.

1. Introduction

In 1939, a solution of the linearized vorticity equation was first obtained by Rossby for the motion of sinusoidal waves of infinite lateral extent on an earth whose variation of Coriolis parameter with latitude is constant. Later, Haurwitz extended Rossby's method and obtained solutions for waves of finite lateral extent on a horizontal plane (1940a) and on a sphere (1940b).

After that, various solutions of the vorticity equation were obtained without the restriction of small amplitude in the perturbation superimposed on the basic current (Craig, 1945; Neamtan, 1946; Rombakis, 1948; Machta, 1949; Höiland, 1951; Long, 1952, Holmboe, 1953; Arakawa, 1953; Kao, 1954; and Syono, 1955).

These investigations were mainly concerned with non-divergent horizontal motions in which the total amount of kinetic energy does not change. Although the flow patterns described are fairly similar to those observed on weather charts, it is generally accepted that some of the interesting features of large-scale motions in the atmosphere (for example, deepening and filling of cyclones, or the intensification of long waves) are associated with the transformation of potential into kinetic energy. Also, the speed of propagation of a disturbance at a certain level must be related not only to the velocity of the basic current at that level, as is true in the barotropic waves, but also to the wind speed at other levels. Solutions of the equation of motion, including the effect of horizontal divergence, will therefore help our understanding of the physical processes involved in the behavior of long waves.

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It seems obvious that it is difficult mathematically to obtain an exact solution of the complicated three-dimensional equations of motion. However, with the progress of research on numerical weather prediction, many workers have devised systems of prognostic equations which are simple enough to permit computation of the field of motion with a reasonable amount of labor, and at the same time realistic enough to give useful forecasts of contour elevations. Since the effect due to the baroclinicity of the atmosphere is well taken into account in these equations, it seems appropriate to discuss the dynamics of long waves by making use of them. Among these systems of equations, which imply more or less similar dynamical properties, we shall make use here of equations for the so-called "2½-dimensional model" presented by Charney and Phillips (1953).

The purpose of this article is to describe some dynamical properties of the simple baroclinic waves obtained by solving the vorticity equations for the 2-layer model.

2. Equations of motion

With some slight simplications, the quasi-geostrophic equations for the 2-layer model are written (Charney and Phillips, 1953) as

$$\Delta^2 \frac{\partial \phi_1}{\partial t} = \frac{1}{f} J(\nabla^2\phi_1, \phi_1) - \beta \frac{\partial \phi_1}{\partial x} + \pi J(\phi_2, \phi_1) + f\pi \frac{\partial(\phi_1 - \phi_2)}{\partial t}, \quad (1)$$

and

$$\Delta^2 \frac{\partial \phi_2}{\partial t} = \frac{1}{f} J(\nabla^2\phi_2, \phi_2) - \beta \frac{\partial \phi_2}{\partial x} - \pi J(\phi_2, \phi_1) - f\pi \frac{\partial(\phi_1 - \phi_2)}{\partial t}, \quad (2)$$

where π, assumed to be constant, is

$$\pi = \frac{\kappa f}{\Phi_1 - \Phi_2}, \quad \kappa = \frac{\theta_{1\frac{1}{2}}}{\theta_1 - \theta_2},$$

and the subscripts 1, $1\frac{1}{2}$ and 2 refer to quantities measured at the 250-, 500- and 750-mb levels, respectively. Other notations are: $\Phi_1 - \Phi_2$ = a typical (constant) value of the geopotential difference between levels one and two; ϕ = geopotential of the constant-pressure surface; f = Coriolis parameter; $\beta = df/dy$ (assumed to be constant); θ = potential temperature; ∇^2 = Laplace operator in the plane $= \partial^2/\partial x^2 + \partial^2/\partial y^2$; $J(\alpha, \beta)$ = Jacobian of α and β, $= \partial(\alpha, \beta)/\partial(x, y)$; and x and y are Cartesian coordinates pointing eastward and northward, respectively.

One of the particular properties of (1) and (2) is that the second equation is derived from the first by interchanging the subscripts 1 and 2. Consequently, we shall hereafter write down only the equation referring to the upper level, except when the explicit expression for the lower level is needed.

We shall now try to find the solutions of (1) and (2) in the particular form:

$$\left.\begin{aligned} \phi_1(x, y, t) &= -fU_1 y + \phi_1'(x, y, t) \\ \phi_2(x, y, t) &= -fU_2 y + \phi_2'(x, y, t) \end{aligned}\right\}, \tag{3}$$

where ϕ_1' and ϕ_2' denote the disturbances superimposed on the zonal basic current and are assumed to be given by

$$\phi_1' = \varphi_1(t)\, e^{i(kx+my)}, \quad \text{and} \quad \phi_2' = \varphi_2(t)\, e^{i(kx+my)}.$$

Then we can see that the non-linear terms in (1) and (2) drop out, and the equation which determines $\varphi_1(t)$ becomes

$$(\gamma + 2)\frac{\partial \varphi_1}{\partial t} = -ik\varphi_1[(\gamma+1)(U_1 - B) + U_2] + ik\varphi_2(V + B), \tag{4}$$

where $\gamma = (k^2 + m^2)/f\pi$, $B = \beta(k^2 + m^2)^{-1}$, and $V = U_1 - U_2$.

3. The phase velocity and instantaneous amplifying factor

Before we solve (4), it is of interest to note some results which are derived directly from it. By putting

$$\varphi_1(t) = \alpha_1(t) \exp[i\,\delta_1(t)],$$

and

$$\varphi_2(t) = \alpha_2(t) \exp[i\,\delta_2(t)], \tag{5}$$

and by separating the real and imaginary parts of (4), we get the differential equations which express the variation in amplitude and phase angle, respectively, of the long wave with respect to time:

$$\frac{d\alpha_1}{dt} = -\frac{\alpha_2 k(V + B)}{(\gamma + 2)}\sin(\delta_2 - \delta_1), \tag{6}$$

and

$$\frac{d\delta_1}{dt} = -kU_1 + Bk + \frac{k}{(\gamma+2)}\left[V\left\{1 + \frac{\alpha_2}{\alpha_1}\cos(\delta_2 - \delta_1)\right\} - B\left\{1 - \frac{\alpha_2}{\alpha_1}\cos(\delta_2 - \delta_1)\right\}\right]. \tag{7}$$

These equations are essentially similar in form to those obtained by Manabe (1955), who used slightly different methods.

If we define the speed of propagation of the disturbance, at time t, by the equation

$$c_1 = -\frac{1}{k}\frac{d\delta_1}{dt}, \tag{8}$$

c_1 is given by

$$c_1 = U_1 - \frac{\beta}{k^2 + m^2} - \frac{1}{(\gamma+2)}\left[V\left\{1 + \frac{\alpha_2}{\alpha_1}\cos(\delta_2 - \delta_1)\right\} - B\left\{1 - \frac{\alpha_2}{\alpha_1}\cos(\delta_2 - \delta_1)\right\}\right]. \tag{9}$$

Particularly, when $\gamma \to \infty$, we obtain

$$\frac{d\alpha_1}{dt} = 0, \quad \text{and} \quad c_1 = U_1 - \frac{\beta}{k^2 + m^2}.$$

In other words, the disturbance propagates without change in amplitude at the speed first given by Rossby and collaborators (1939) for the barotropic atmosphere. In the baroclinic case, however, the propagation of the disturbances in the upper level is affected by the velocity of the basic current in the lower level. The third term on the right-hand side of (9) thus demonstrates the coupling effect or mutual interaction between the waves at the upper and lower levels.

Equation (6) shows that the difference in the phase angles between upper and lower levels is of predominant importance in the development of the long waves, in the sense that disturbances are amplified when the disturbance at the upper level lags behind that of the lower level and *vice versa*.

4. Stable and unstable waves

Under the initial conditions $\varphi_1 = \varphi_1(0)$ and $\varphi_2 = \varphi_2(0)$ at $t = 0$, the solutions of (4) are given by

$$\varphi_1(t)=\frac{e^{i\omega_1 t}}{2D}[\varphi_1(0)\,(V\gamma+D)-2\varphi_2(0)\,(V+B)]$$
$$+\frac{e^{i\omega_2 t}}{2D}[\varphi_1(0)\,(D-V\gamma)+2\varphi_2(0)\,(V+B)], \quad (10)$$

where

$$\omega_1=\frac{k}{2(2+\gamma)}[-(U_1+U_2)(\gamma+2)+2B(\gamma+1)-D],$$
$$(11)$$
$$\omega_2=\frac{k}{2(2+\gamma)}[-(U_1+U_2)(\gamma+2)+2B(\gamma+1)+D],$$

and

$$D=[V^2(\gamma^2-4)+4B^2]^{\frac{1}{2}}. \quad (12)$$

We see that, for a given value of k and m, it is possible that ω_1 and ω_2 may be complex. A complex value of ω_1 and ω_2 means that the disturbances amplify exponentially with time. Fig. 1 shows the critical values of the vertical wind shear in the basic current against wavelength which satisfy the relation

$$V^2(\gamma^2-4)+4B^2=0, \quad (13)$$

where

$$V=\frac{\Phi_1-\Phi_2}{g}\left(\frac{dU}{dZ}\right),$$

and numerical values of parameters are $f=9.37\times10^{-5}$, $\beta=1.75\times10^{-13}$, $\pi=8.609\times10^{-13}$ ($\kappa=9$), $m=0$, and $\Phi_1-\Phi_2=9.81\times8.16\times10^3$ m²/sec². This illustrates more or less similar critical values to those obtained by Fjørtoft (1950), Thompson (1953) and Kuo (1953). And (13) is, of course, exactly the same as the equation obtained by Phillips (1954) by means of linearized perturbation techniques.

We shall now discuss in somewhat more detail the stable and unstable cases separately. When the initial conditions are given by

$$\phi_1(0)=-fU_1y+\alpha_1(0)\cos[kx+my+\delta_1(0)], \quad (14)$$

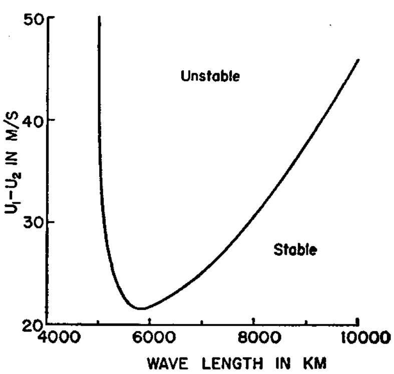

FIG. 1. Critical values of vertical wind shear in basic current against wavelength, for which perturbations maintain constant amplitude.

and

$$\phi_2(0)=-fU_2y+\alpha_2(0)\cos[kx+my+\delta_2(0)], \quad (15)$$

the solution for ϕ_1 in the stable case is given by

$$\phi_1(t)=-fU_1y+\alpha_1(t)\times\cos[kx+my+\tfrac{1}{2}(\omega_1+\omega_2)t+\delta_1(t)], \quad (16)$$

where

$$\begin{aligned}\alpha_1^2(t)=\;&\alpha_1^2(0)\cos^2\omega_i t+D^{-2}\sin^2\omega_i t\\&\times[\alpha_1^2(0)\,V^2\gamma^2+4\alpha_2^2(0)\,(V+B)^2\\&-4\alpha_1(0)\,\alpha_2(0)\,V\gamma(V+B)\\&\times\cos\{\delta_2(0)-\delta_1(0)\}]\\&-2\alpha_1(0)\,\alpha_2(0)\,D^{-1}(V+B)\\&\quad\times\sin2\omega_i t\sin[\delta_2(0)-\delta_1(0)], \quad (17)\end{aligned}$$

$$\begin{aligned}\tan\delta_1(t)=\;&[\alpha_1(0)\sin\delta_1(0)\cos\omega_i t\\&-D^{-1}\sin\omega_i t\{\alpha_1(0)\,V\gamma\cos\delta_1(0)\\&-2\alpha_2(0)\,(V+B)\cos\delta_2(0)\}]\\&\times[\alpha_1(0)\cos\delta_1(0)\cos\omega_i t\\&+D^{-1}\sin\omega_i t\{\alpha_1(0)\,V\gamma\sin\delta_1(0)\\&\quad-2\alpha_2(0)\,(V+B)\sin\delta_2(0)\}]^{-1}, \quad (18)\end{aligned}$$

and $\omega_i=(\omega_2-\omega_1)/2$.

The solution (16) describes a type of wave motion in which both the amplitude and phase velocity, defined by

$$c_i=-\frac{1}{k}\left(\frac{\omega_1+\omega_2}{2}+\frac{d\delta_i}{dt}\right), \quad i=1 \text{ or } 2,$$

vary periodically with time. Since the phase velocities have different values for each level, the phase difference $\delta_2-\delta_1$ is also a function of time. Table 1 shows some numerical examples of periods of fluctuation in amplitude. In this table, $2\pi/(\omega_2-\omega_1)$ is presented as a function of the wavelength and U_1-U_2.

Replacing the sine and cosine in (16) by the hyperbolic sine and cosine, respectively, we obtain solutions for ϕ_1 in the unstable case. Figs. 2 and 3 illustrate the change of the amplitude and phase difference $\delta_2-\delta_1$ respectively. Numerical values of the parameters used are as follows: $U_1-U_2=30$ m/s, $\alpha_1(0)/\alpha_2(0)=2$, $\delta_2(0)-\delta_1(0)=-\pi/2$, $m=0$, $L=2\pi/k$, and the constant values of f and β are evaluated at 40 deg lat.

It is obvious that the solutions obtained above in the unstable case do not hold true beyond a certain limit of time, because of the assumptions used in

TABLE 1. Examples of periods of fluctuation in amplitude.

L \ U_1-U_2:	10 m/s	30 m/s	50 m/s
2000 km	2.6 day	0.8 day	0.6 day
3000	5.0	1.7	1.0
4000	8.5	3.3	2.0
5000	10.4	11.5	—
6000	8.2	—	—
7000	6.1	—	—
8000	4.6	20.5	—
9000	3.9	6.2	—
10,000	3.3	4.3	—

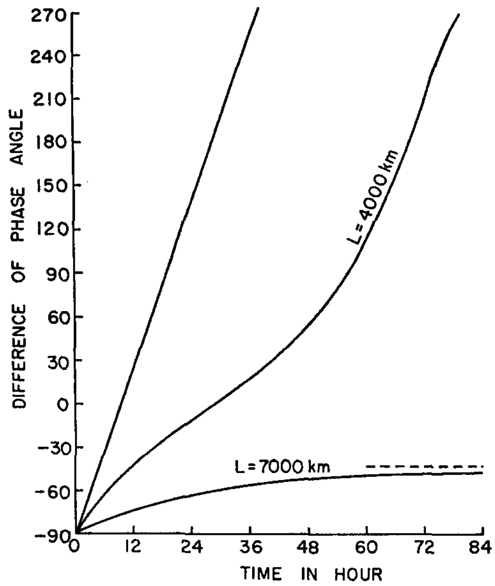

FIG. 2. Variation of difference in phase angle between perturbations at upper and lower levels. As wavelength of perturbation approaches zero, curves approach solid straight line.

deriving (1) and (2). Nevertheless, fig. 2 illustrates an interesting tendency of the baroclinic waves. Owing to the difference in the velocities of basic flow, the disturbances at each level propagate with different speed; in other words, the inclination angle of the trough line in the vertical plane varies with time. While this difference in phase angles increases monotonically with time in the stable wave, its change is very slow in the unstable wave; and as t tends to

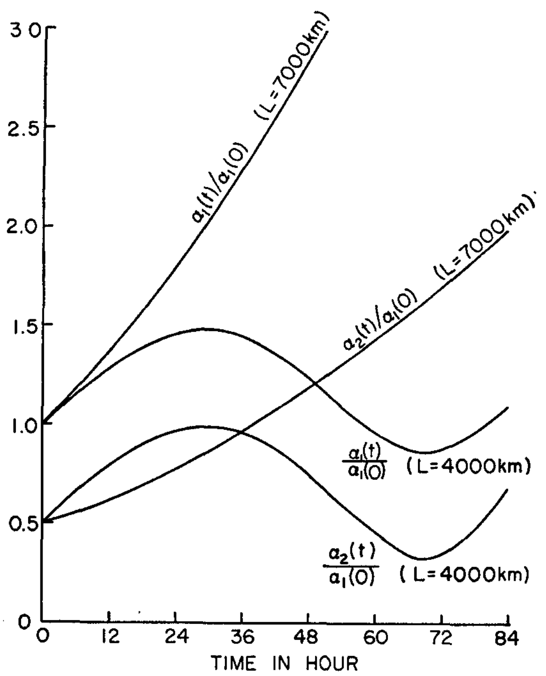

FIG. 3. Variations of amplitudes with time. Subscripts 1 and 2 refer to quantities observed at upper and lower levels, respectively.

infinity, it approaches a limiting value δ_c. For the case $\alpha_2(0) = \delta_2(0) = 0$, we have $\delta_c = \tan^{-1}(iD/\gamma V)$.

Referring to (6), we note that this value of δ_c is such that the amplitude of the disturbance increases indefinitely. The table below shows some values of δ_c as a function of k, for the particular case where $m = 0$ and $U_1 - U_2 = 40$ m/s:

L:	6000 km	7000 km	8000 km	9000 km
δ_c:	$-42°$	$-53°$	$-58°$	$-46°$.

It is interesting to note that (10) gives

$$\frac{\partial}{\partial t}\int_0^{2\pi/k}\frac{1}{2f^2}(\nabla\phi_1')^2\,dx = \frac{\partial}{\partial t}\int_0^{2\pi/k}\frac{1}{2f^2}(\nabla\phi_2')^2\,dx.$$

That is, the change of the kinetic energy per unit time at the upper level is exactly the same as that at the lower level for any values of $\alpha_i(0)$ and $\delta_i(0)$. This would suggest that growth of disturbances occurs simultaneously and at about the same rate throughout a large part of the troposphere. Needless to say, the change of the total kinetic energy for our waves is easily proved to be related to the change of potential energy in the manner demonstrated by Phillips (1954) in a general form.

So far we have discussed baroclinic waves by making use of the equations for the 2-layer model, because these equations are simple enough to permit physical interpretation; at the same time, the essential features of the baroclinic effect are taken well into account in this model. The results obtained above can easily be generalized to a n-layer model. Instead of doing so, we shall make here brief remarks on the mutual interaction between the waves at the upper and lower levels for the 3-layer model.

Following the procedure in section 3, we obtain, from the equations of vorticity for the 3-layer model, the expressions for change in amplitude of the disturbance at the lower level and for the phase velocity defined in section 3:

$$\frac{d\alpha_3}{dt} = E\sin(\delta_1 - \delta_3) + F\sin(\delta_2 - \delta_3), \quad (19)$$

and

$$c_3 = -(k\alpha_3)^{-1}[G + E\cos(\delta_3 - \delta_1) + F\cos(\delta_2 - \delta_3)], \quad (20)$$

where

$$E = \alpha_1 kD^{-1}(U_1 - U_2 - B),$$
$$F = \alpha_2 kD^{-1}[\gamma_1(U_1 - U_2 - B) + U_1 + U_2 + 2U_3 + 4\gamma_1\gamma_2^{-1} - B],$$
$$G = \alpha_3 k(U_3 - B) + \alpha_3 kD^{-1}[\gamma_1(U_2 - U_3 + B) - 2(U_3 + U_2 + U_2\gamma_1\gamma_2^{-1} - B)],$$
$$D = \gamma_1\gamma_2 + 2(\gamma_1 + \gamma_2) + 3,$$
$$\gamma_1 = (k^2 + m^2)/f\pi_1,$$
$$\gamma_2 = (k^2 + m^2)/f\pi_2,$$

and the subscripts 1, 2 and 3 refer to quantities measured at the upper, middle and lower levels, respectively.

When $k^2 + m^2 \gg f\pi_1 \approx f\pi_2$, (20) is reduced to

$$c_3 = U_3 - B + \gamma_1 D^{-1}\left[(U_2 - U_3)\left\{1 + \frac{\alpha_2}{\alpha_3}\cos(\delta_2 - \delta_3)\right\} + B\left\{1 - \frac{\alpha_2}{\alpha_3}\cos(\delta_2 - \delta_3)\right\}\right].$$

Since this equation does not include any parameters referred to the upper level except γ_1, we can say that the upper-level disturbance has no influence on the motion of the air at the lower level. On the other hand, we can also see that the effect of the upper and middle levels is of the same order of magnitude when $k^2 + m^2 \ll f\pi_1$.

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