SPECTRUM MODIFICATION DUE TO THE USE OF FINITE DIFFERENCES

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ABSTRACT

The modification of the vorticity spectrum associated with the use of finite differences instead of derivatives is considered for an isotropic turbulence field. The results permit a rough estimate of the dependence of the magnitude of vorticity on the grid spacing.

1. Introduction

In the statistical studies of a turbulent flow, we often meet statistical quantities which involve derivatives of a certain physical quantity with respect to space coordinates or time. Vorticities and wind velocities, computed under the geostrophic approximation, are examples of these quantities. In computing these properties, it is common procedure to replace the derivatives by finite differences. The purpose of this article is to demonstrate the modification of spectral distribution of vorticity induced by the finite-difference technique. Although we will restrict our considerations to a purely two-dimensional motion, the results obtained here will be easily extended to n-dimensional motions and to other quantities which involve derivatives.

2. The modification of the vorticity spectrum due to the use of finite differences

When the α-component of velocity is assumed to be a steady, random function of space, it can be represented as a stochastic Fourier integral,

$$u_\alpha(\boldsymbol{x}) = \int_\Lambda e^{i(k_1x_1+k_2x_2)}\, dh_\alpha(\boldsymbol{k}), \tag{1}$$

where $h_\alpha(\boldsymbol{k})$ is a random function of k_1 and k_2, and the double integral extends over the entire wave-number space Λ. For $h_\alpha(\boldsymbol{k})$, we have the statistical relationships

$$\overline{dh_\alpha^*(\boldsymbol{k})\, dh_\beta(\boldsymbol{k}')} = 0 \quad \text{if} \quad \boldsymbol{k} \neq \boldsymbol{k}',$$

and

$$\overline{dh_\alpha^*(\boldsymbol{k})\, dh_\beta(\boldsymbol{k})} = \Phi_{\alpha\beta}(\boldsymbol{k})\, dk,$$

where the asterisk indicates that the quantity is a complex conjugate, and $\Phi_{\alpha\beta}(\boldsymbol{k})$ is the so-called spectrum tensor of the velocity.

From (1), the vorticity is given by

$$\zeta = \frac{\partial u_\alpha}{\partial x_\beta} - \frac{\partial u_\beta}{\partial x_\alpha} = i \int_\Lambda e^{i(k_1x_1+k_2x_2)}(k_\beta\, dh_\alpha - k_\alpha\, dh_\beta). \tag{2}$$

Therefore, we have

$$\zeta^2 = \int_{\Lambda\times\Lambda} e^{i[(k_1'-k_1)x_1+(k_2'-k_2)x_2]} \times [k_\beta\, dh_\alpha^*(\boldsymbol{k}) - k_\alpha\, dh_\beta^*(\boldsymbol{k})] \times [k_\beta'\, dh_\alpha(\boldsymbol{k}') - k_\alpha'\, dh_\beta(\boldsymbol{k}')].$$

The average is obtained by an operation under the integral sign. The result is zero, except when the points $\boldsymbol{k}$ and $\boldsymbol{k}'$ coincide in the wave-number space Λ. Thus,

$$\overline{\zeta^2} = \int_\Lambda [k_\beta^2\, \Phi_{\alpha\alpha}(\boldsymbol{k}) + k_\alpha^2\, \Phi_{\beta\beta}(\boldsymbol{k}) - k_\alpha k_\beta\{\Phi_{\alpha\beta}(\boldsymbol{k}) + \Phi_{\beta\alpha}(\boldsymbol{k})\}]\, dk. \tag{3}$$

For isotropic and purely two-dimensional flow, $\Phi_{\alpha\beta}(\boldsymbol{k})$ is represented by

$$\Phi_{\alpha\beta}(\boldsymbol{k}) = \frac{E(k)}{\pi k}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right), \tag{4}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. $E(k)$ denotes the amount of kinetic energy within the interval k and $k + dk$, and $k^2 = k_1^2 + k_2^2$. By substituting (4) into (3), we have

$$\overline{\zeta^2} = 2\int_0^\infty k^2\, E(k)\, dk. \tag{5}$$

Now, in finite-difference form, the vorticity is computed by

$$\zeta_f = \frac{u_1(x_2+a) - u_1(x_2-a)}{2a} - \frac{u_2(x_1+a) - u_2(x_1-a)}{2a}, \tag{6}$$

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