THE ANNUAL MERIDIONAL TIDE IN THE ATMOSPHERE

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The annual temperature oscillation in the atmosphere induces a pressure oscillation with the same period. In the absence of surface irregularities, the temperature oscillation would have the character of a standing wave whose amplitude depends only on the latitude. The periodicity of the temperature oscillation can be represented approximately by a sine function.

In the analysis of tidal motions, it is convenient to postulate an equilibrium state in which all horizontal gradients and all motion are zero. This state is approached most closely in summer. The meridional gradients and the zonal motion reach their maximum values in winter. The deviations of temperature, pressure and zonal motion (which in the following are considered to be perturbations of small magnitude) from the equilibrium state then do not change sign throughout the year and may be represented by functions of the form $A(\theta)\,(1 + \sin \gamma t)$, where θ is colatitude and γ is the annual frequency. This function has the required properties, *i.e.*, it is zero in summer ($t = 3\pi/2\gamma$) and a maximum in winter. Only one hemisphere is considered.

In a previous article [Spar, 1950], the following perturbation equations were derived:

$$\frac{\partial U}{\partial t} + 2\omega\, V \cos\theta = -\frac{\partial K}{r \sin\theta\, \partial\lambda}, \qquad (1)$$

$$\frac{\partial V}{\partial t} - 2\omega\, U \cos\theta = -\frac{\partial K}{r\, \partial\theta}, \qquad (2)$$

$$\frac{1}{g}\frac{\partial p_s}{\partial t} + \frac{1}{r \sin\theta}\left[\frac{\partial U}{\partial\lambda} + \frac{\partial (V \sin\theta)}{\partial\theta}\right] = 0, \qquad (3)$$

and

$$\frac{\partial p_s}{\partial t} = \frac{1}{h_0}\left(\frac{\partial K}{\partial t} - \frac{\partial Q}{\partial t}\right), \qquad (4)$$

where

$$U = \int_s^\infty \rho_0 u\, dz, \quad V = \int_s^\infty \rho_0 v\, dz, \quad K = \int_s^\infty p\, dz,$$

$$Q = \int_s^\infty \frac{g p_0}{R} \int_s^z \frac{T}{T_0^2}\, dz\, dz, \quad h_0 = \frac{1}{p_{s0}} \int_s^\infty p_0\, dz,$$

and u = zonal velocity, v = meridional velocity, ρ_0 = equilibrium density, p_0 = equilibrium pressure, T_0 = equilibrium temperature, p = pressure perturbation, T = temperature perturbation, R = gas constant, r = mean radius of the earth, ω = angular velocity of the earth's rotation, g = acceleration of gravity, λ = longitude, and the subscript s = surface value.

The integrated temperature perturbation, Q, may be represented by the function

$$Q = \tfrac{1}{2}H(\theta)\,(1 + \sin \gamma t),$$

where $H(\theta)$ is the mid-winter value of Q. We may expect the integrated pressure perturbation, K, and the zonal transport, U, to be of the form

$$K = \tfrac{1}{2}L(\theta)\,(1 + \sin \gamma t)$$

and

$$U = \tfrac{1}{2}M(\theta)\,(1 + \sin \gamma t).$$

The meridional transport, V, on the other hand, must reverse in sign during the year and may be represented by

$$V = \tfrac{1}{2}N(\theta) \cos \gamma t.$$

This function is zero in summer and winter, and reaches its maximum values in the transitional seasons.

Differentiation of the perturbation quantities in (1) and (2) gives

$$\tfrac{1}{2}\gamma M + (2\omega \cos\theta)(\tfrac{1}{2}N) = 0, \qquad (1a)$$

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and

$$\tfrac{1}{2}\gamma N \sin\gamma t + (2\omega\cos\theta)(\tfrac{1}{2}M) + (2\omega\cos\theta)(\tfrac{1}{2}M)\sin\gamma t = +\frac{dL}{2r\,d\theta}\sin\gamma t + \frac{dL}{2r\,d\theta}. \qquad (2b)$$

Since (2b) must be true for all values of $\sin\gamma t$, it follows that

$$\gamma N + (2\omega\cos\theta)M - r^{-1}\,dL/d\theta = 0, \qquad (2c)$$

and

$$(2\omega\cos\theta)M - r^{-1}\,dL/d\theta = 0. \qquad (2d)$$

From (1a) and (2c),

$$N = \frac{\gamma}{r(\gamma^2 - f^2)}\frac{dL}{d\theta}, \qquad (2e)$$

and

$$M = -\frac{f}{r(\gamma^2 - f^2)}\frac{dL}{d\theta}, \qquad (2f)$$

where $f = 2\omega\cos\theta$. But, from (2d),

$$M = \frac{1}{f}\frac{dL}{r\,d\theta}. \qquad (2g)$$

Equations (2f) and (2g) are compatible only if γ^2 can be neglected compared with f^2. In the present case, γ^2 is about five orders of magnitude smaller than f^2 and can, therefore, be ignored.

Equations (3) and (4) lead, upon differentiation, to

$$\frac{\gamma}{gh_0}(L - H) + \frac{1}{r\sin\theta}\left(\sin\theta\frac{dN}{d\theta} + N\cos\theta\right) = 0. \qquad (3a)$$

Upon substituting (2e) and (3a), and neglecting γ^2 compared with f^2, we obtain, after some simplification,

$$\frac{d^2L}{d\theta^2} + \left(\frac{\cos\theta}{\sin\theta} + 2\frac{\sin\theta}{\cos\theta}\right)\frac{dL}{d\theta} - \left(\frac{4r^2\omega^2}{gh_0}\cos^2\theta\right)L = -\left(\frac{4r^2\omega^2}{gh_0}\cos^2\theta\right)H. \qquad (5)$$

It will be assumed that the mid-winter distribution of Q may be represented by

$$H = C\cos\theta. \qquad (6)$$

The solution for the mid-winter distribution of K will be represented by the infinite series

$$L = C\sum_n \alpha_n\cos^n\theta, \quad (n = 0, 1, 2, 3, \cdots). \qquad (7)$$

Substitution of (6) and (7) in (5) yields

$$\sum_n\left[-\frac{4r^2\omega^2}{gh_0}\alpha_n\cos^{n+2}\theta + n(1-n)\alpha_n\cos^n\theta + n(n-3)\alpha_n\cos^{n-2}\theta\right] = -\frac{4r^2\omega^2}{gh_0}\cos^3\theta. \qquad (8)$$

Upon equating the coefficients of like powers of $\cos\theta$ on both sides of (8), we obtain

$$\alpha_1 = \alpha_2 = 0,$$
$$(4r^2\omega^2/gh_0)\alpha_0 - 4\alpha_4 = 0,$$
$$2\alpha_4 - 3\alpha_6 = 0,$$
$$(4r^2\omega^2/gh_0)\alpha_{m-2} + m(m-1)\alpha_m - (m+2)(m-1)\alpha_{m+2} = 0, \quad (m = 6, 8, \cdots),$$
$$6\alpha_3 - 10\,\alpha_5 = 4r^2\omega^2/gh_0,$$

and

$$(4r^2\omega^2/gh_0)\alpha_{m-2} + m(m-1)\alpha_m - (m+2)(m-1)\alpha_{m+2} = 0, \quad (m = 5, 7, 9, \cdots).$$

For the perturbation to vanish at the equator, α_0 must be zero. If we impose this boundary condition, all the even coefficients (α_4, α_6, *etc.*) vanish, and only the odd coefficients need be evaluated from the last two equations.

The last equation can be transformed into a continued fraction by dividing by α_m. Thus,

$$\frac{\alpha_m}{\alpha_{m-2}} = \frac{\dfrac{4r^2\omega^2}{gh_0(m-1)}}{-m + (m+2)\dfrac{\alpha_{m+2}}{\alpha_m}}. \qquad (8a)$$

Repeated application of (8a) yields the formula

$$\frac{\alpha_m}{\alpha_{m-2}} = \frac{\dfrac{4r^2\omega^2}{gh_0(m-1)(m)}}{-1 + \left\{\dfrac{\dfrac{4r^2\omega^2}{gh_0(m+1)(m)}}{-1 + \left\{\dfrac{\dfrac{4r^2\omega^2}{gh_0(m+3)(m+2)}}{-1 + \cdots\cdots}\right\}}\right\}}.$$

The continued fraction converges rapidly and may be used, with (8a), to obtain the coefficients of (7). The

TABLE 1. Coefficients of series for integrated pressure perturbation, K.

h_0 (km)	α_3	α_5	α_7	α_9	α_{11}	α_{13}
6.5	+1.235	−0.607	+0.161	−0.0266	+0.00298	−0.000240
7.5	+1.128	−0.494	+0.117	−0.0159	+0.00156	−0.000113
8.5	+1.031	−0.411	+0.0866	−0.0112	+0.000974	−0.0000610

α_n for $n = 3, 5, 7, 9, 11$ and 13 were obtained for $h_0 = 6.6, 7.5$ and 8.5 km, and are shown in table 1.

To find the surface pressure-wave, we make use of (4). The mid-winter surface pressure-distribution, $P(\theta)$, is then given by

$$P(\theta) = (C/h_0)(-\cos\theta + \sum_n \alpha_n \cos^n \theta), \qquad (n = 3, 4, 7 \cdots),$$

or

$$P(\theta) = (C/h_0)\, G(\theta).$$

The function $G(\theta)$ is tabulated in table 2 for the three values of h_0 and for every 10 deg lat between the equator and the North Pole.

From the results of a previous study[2] [Spar, 1950], it appears that a reasonable value for C is about 4×10^{10} dy/cm, while h_0 is about 7.5 km. Thus, C/h_0 is equal to about 53 mb. The computed pressure variation is thus seen to be of reasonable magnitude. Except at the equator, the pressure is higher in winter than in summer at all latitudes, since the pressure and temperature perturbations are seen to be 180 deg out of phase.

[2] The article referred to contains a mathematical error which invalidates the results contained therein. In the present note, the form of the temperature disturbance has been simplified in order to arrive at a solution.

TABLE 2. Pressure-perturbation function, $G(\theta)$.

θ	$h_0 = 6.5$ (km)	$h_0 = 7.5$	$h_0 = 8.5$
0	−0.235	−0.263	−0.304
10	−0.244	−0.273	−0.312
20	−0.270	−0.298	−0.335
30	−0.307	−0.335	−0.368
40	−0.347	−0.371	−0.399
50	−0.374	−0.392	−0.411
60	−0.364	−0.373	−0.383
70	−0.296	−0.299	−0.303
80	−0.168	−0.168	−0.169
90	0	0	0

According to the computations, the pressure in winter reaches a maximum value in the vicinity of colatitude 50 deg (latitude 40°N) and has two minima, one at the pole and one at the equator. This result agrees qualitatively with the observed mean meridional pressure profile in winter. However, the results should be interpreted with caution in view of the highly simplified model employed and the assumption that the pressure is uniform in the summer hemisphere.

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REFERENCE

Spar, J., 1950: On the theory of annual pressure variations. *J. Meteor.*, 7, 167–180.