A NOTE ON THE STABILITY OF BAROCLINIC WAVES

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ABSTRACT

The stability characteristics of waves in a thermotropic atmosphere are computed for several velocity profiles. It is shown that the critical wavelength for any wind shear varies considerably with the shape of the velocity profile.

1. Introduction

Stability criteria for the waves in a baroclinic westerly current have been derived for several different atmospheric models. (See, *e.g.*, Charney, 1947; Fjørtoft, 1950; Kuo, 1952; Thompson, 1953.) This article is concerned with the stability properties of the integrated model with invariant thermal wind direction. In this model, first introduced by Sutcliffe (1947), and later referred to by Thompson as "thermotropic" and "equivalent baroclinic," the shape of the vertical wind profile plays an important role. In particular, it affects the stability criterion. The dependence of the stability criterion on the wind profile is computed below.

2. Development

The p-vorticity equation can be written, viscosity and "tipping" of the vorticity vector being neglected, as

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla \zeta + v\beta + \omega \frac{\partial \zeta}{\partial p} + (f + \zeta) \operatorname{div}_p V = 0, \quad (1)$$

where ζ is the relative vorticity, V the horizontal wind vector, ∇ the gradient operator, ω the individual pressure change (dp/dt), f is Coriolis parameter, β the derivative of f with respect to meridional distance, $\operatorname{div}_p V$ the horizontal velocity divergence measured on the constant-pressure map, and v is the meridional wind component.

Equation (1) is integrated with respect to pressure, and the following notation is introduced:

$$\bar{\alpha} = \frac{1}{p_0} \int_0^{p_0} \alpha \, dp, \quad (2)$$

where α is any function of pressure, and p_0 is the surface pressure. It will be assumed that

$$\overline{\omega \frac{\partial \zeta}{\partial p}} = \overline{(f + \zeta) \operatorname{div}_p V} = 0.$$

These assumptions have been discussed frequently in the literature and will not be justified in this article. The integrated vorticity equation then becomes

$$\partial \bar{\zeta}/\partial t + \overline{V \cdot \nabla \zeta} + \beta \bar{v} = 0. \quad (3)$$

As suggested by Sutcliffe (1947), the integrals can be replaced by introducing the wind-shear vector

$$V_T = \bar{V} - V_0,$$

where V_0 is the surface wind. When it is assumed that the direction of the wind shear does not vary with elevation, the variation of wind with pressure can be written as

$$V(p) = \bar{V} + B(p)\, V_T, \quad (4)$$

where $B(p)$ is a scalar function of pressure. Applying the curl operator to (4), and assuming no variation of $B(p)$ in the horizontal plane, one obtains

$$\zeta(p) = \bar{\zeta} + B(p)\, \zeta_T. \quad (5)$$

After substitution of (4) and (5), (3) becomes

$$\partial \bar{\zeta}/\partial t + \bar{V} \cdot \nabla \bar{\zeta} + \beta \bar{v} + \overline{B^2}\, V_T \cdot \nabla \zeta_T = 0. \quad (6)$$

The last term in (6), which is sometimes referred to as the "development" term, is the basis for Sutcliffe's theory of development.

The quantity $\overline{B^2}$ can be calculated from

$$\overline{B^2} = \overline{(V - \bar{V})^2}/V_T^2. \quad (4a)$$

Although it is not a function of pressure, $\overline{B^2}$ does vary in the horizontal and in time. Almost nothing is known of the variations of $\overline{B^2}$ and the effects of these variations on the integration of the thermotropic model. In the present article, it is assumed that $\overline{B^2}$ is a constant.

By introduction of the geopotential height as a stream function through the geostrophic wind equation, (6) may be transformed into

$$\frac{\partial}{\partial t} (\nabla^2 z) + V \cdot \nabla (\nabla^2 z) + \beta \frac{\partial z}{\partial x} + \overline{B^2}\, V_T \cdot \nabla (\nabla^2 h) = 0, \quad (7)$$

where z is the geopotential height of the isobaric surface at which $V = \bar{V}$ [the "level of equivalence" or "equivalent baroclinic level," in Thompson's (1953) terminology], and h is equal to the geopotential

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thickness, $z - z_0$. The assumption that the equivalent baroclinic level is a fixed isobaric surface is implicit in Sutcliffe's theory of cyclogenesis, and is the basis for the expectation that the thermotropic model is capable of forecasting development in that surface.

To restrict the problem to one dimension it will be assumed that

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = 0,$$

where y is the meridional coordinate. Thus, meridional shear of the zonal wind is ignored here, and no consideration is given to dynamic (shearing) instability.

The undisturbed zonal velocity is considered to be a function only of height (or pressure). The wind components at the level of equivalence are

$$\bar{u} = U + u' \quad \text{and} \quad \bar{v} = v', \tag{8}$$

while the thermal wind components are

$$u_T = U_T = U - U_0 \quad \text{and} \quad v_T = 0. \tag{9}$$

U is the undisturbed zonal velocity at the level of equivalence, primes denote perturbation quantities, and U_T is the constant vertical shear of the undisturbed zonal wind. The latter is proportional to the meridional gradient of thickness. Substituting (8) and (9) in (7), and neglecting quadratic perturbation terms, we obtain the linearized vorticity equation,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial^2 z}{\partial x^2} + \beta\frac{\partial z}{\partial x} + \overline{B^2}\, U_T \frac{\partial}{\partial x}\left(\frac{\partial^2 h}{\partial x^2}\right) = 0. \tag{10}$$

The solution of (10) may be written

$$\begin{Bmatrix} z \\ h \end{Bmatrix} = \begin{Bmatrix} Z \\ H \end{Bmatrix} e^{ik(x-ct)}, \tag{11}$$

where $k(=2\pi/L)$ is the wave number (L is the wavelength), c the phase velocity, and Z and H are the amplitudes of the z and h perturbations. The frequency equation, obtained by substituting (11) in (10), is

$$c - U + \frac{\beta}{k^2} - \overline{B^2}\, U_T \frac{H}{Z} = 0, \tag{12}$$

which is identical with the Rossby equation except for the baroclinic term containing U_T.

The ratio of H to Z in (12) can be determined from the adiabatic equation for horizontal flow,

$$\partial T/\partial t + V\cdot\nabla T = 0, \tag{13}$$

where T is the virtual air temperature. Equation (13) also applies to an atmosphere in neutral equilibrium (*i.e.*, with an adiabatic lapse rate). We now introduce a new integral,

$$\alpha^* = \frac{1}{\ln(p_0/p_e)}\int_{p_e}^{p_0} \alpha\, d(\ln p), \tag{14}$$

in which α^* represents the mean value of any quantity α with respect to logarithm of pressure in the layer between the surface (p_0) and the level of equivalence (p_e). The following definition is also convenient:

$$\alpha'' = \alpha - \alpha^*.$$

Thus,

$$\partial T^*/\partial t + (V''\cdot\nabla T'')^* + V^*\cdot\nabla T^* = 0. \tag{15}$$

From the hypsometric formula,

$$h \sim T^*.$$

Thus, if we assume that

$$(V''\cdot\nabla T'')^* = 0,$$

it follows that

$$\partial h/\partial t + V^*\cdot\nabla h = 0. \tag{16}$$

Now, from (4),

$$V^* = \bar{V} + B^* V_T,$$

from which it follows that

$$V^*\cdot\nabla h = \bar{V}\cdot\nabla h + B^* V_T\cdot\nabla h.$$

But, on the geostrophic assumption,

$$V_T\cdot\nabla h = 0.$$

Thus,

$$\partial h/\partial t + \bar{V}\cdot\nabla h = 0. \tag{17}$$

Equation (17) is linearized to obtain

$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + \bar{v}\frac{\partial h}{\partial y} = 0. \tag{18}$$

If we substitute the geostrophic equations,

$$\partial h/\partial y = -(f/g)U_T \tag{19}$$

and

$$\bar{v} = \frac{g}{f}\frac{\partial z}{\partial x} \tag{20}$$

in (18), it follows that

$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} - U_T\frac{\partial z}{\partial x} = 0. \tag{21}$$

Substitution of (11) in (21) yields

$$\frac{H}{Z} = \frac{U_T}{U - c}. \tag{22}$$

From (12) and (22), we obtain the frequency equation

$$c = U - \frac{\beta}{2k^2} \pm \frac{1}{2}\left(\frac{\beta^2}{k^4} - 4\overline{B^2}\, U_T^2\right)^{\frac{1}{2}}, \tag{23}$$

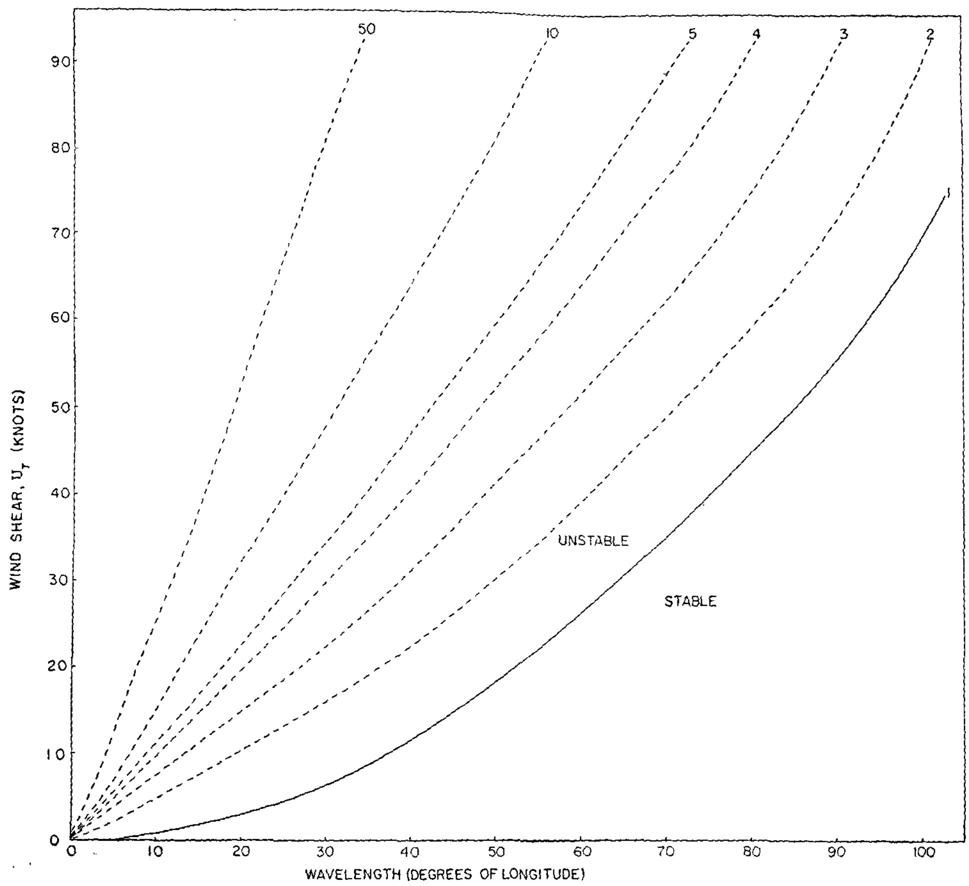

FIG. 1. Amplification of unstable baroclinic waves after one day at latitude 40 deg for $\overline{B^2}$ equal to 0.2. Solid curve represents critical (neutral) wave.

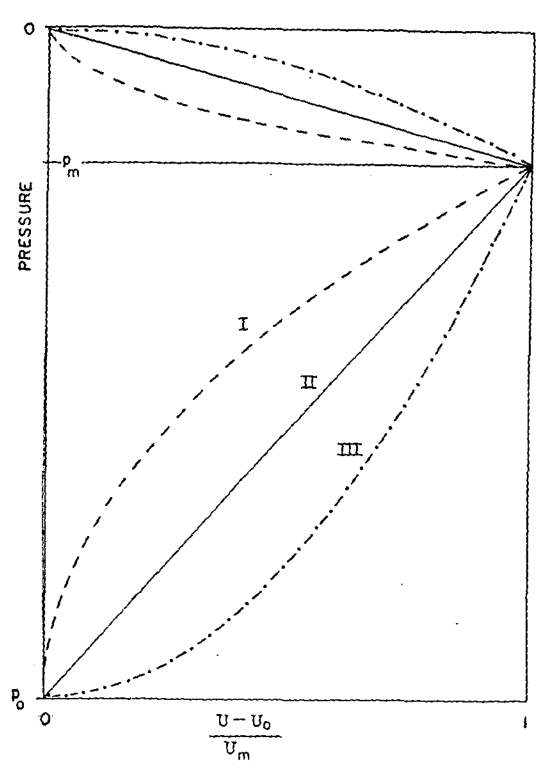

FIG. 2. Examples of undisturbed zonal velocity profiles. Vertical coordinate is pressure on linear scale. Velocity is plotted as dimensionless ratio, $(U - U_0)/U_m$.

which reduces to the Rossby equation for non-divergent or equivalent barotropic waves if U_T equals zero. Since a complex value of the perturbation phase-velocity corresponds to unstable waves, the stability criterion is

$$S = \left[\frac{\beta^2}{k^4} - 4\overline{B^2}\, U_T^{\,2} \right] \left. \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \left\{ \begin{matrix} \text{stable} \\ \text{neutral} \\ \text{unstable} \end{matrix} \right. . \qquad (24)$$

3. Implications

The stability criterion (24) is similar to those derived by Kuo (1952) and Thompson (1953), except for the fact that the effect of static (vertical) stability, which is important for the shorter wavelengths, has not been included here. The omission of the static stability is a consequence of the assumptions used in the advection equation (13), *i.e.*, that either the mean flow in the layer below $p(\bar{V})$ is horizontal or the lapse rate is adiabatic.

The amplitude of the unstable perturbation, according to (11), increases with time by a factor

$$a = \exp \tfrac{1}{2}k(-S)^{\frac{1}{2}}t, \qquad (25)$$

where a is the ratio of the wave amplitude after time t to its initial value. The variation of the amplification with wavelength and wind shear is shown in fig. 1 for latitude 40 deg and t equal to one day. $\overline{B^2}$ is assumed to be equal to 0.2. Except for the omission of the static stability factor, fig. 1 is similar to those given by Kuo and Thompson. The static stability factor in the analyses of Kuo and Thompson tends to stabilize the short-wavelength disturbances, yielding a wavelength of maximum amplification ("most unstable wave") for any given wind shear.

It is of some interest to calculate how variations in the model parameters affect the stability characteristics of the system. The wind-profile parameter ($\overline{B^2}$) and the latitude completely determine the stability of the waves in the model above, since the static stability is not included. (We are only interested in the critical wavelength corresponding to the "just unstable" state. Therefore the static stability is of little importance, for it has its main effect at the sub-critical wavelengths.)

The three velocity profiles for which stability criteria will be computed are illustrated in fig. 2. The velocity in fig. 2 is plotted as the dimensionless ratio $(U - U_0)/U_m$, where U_0 is the velocity at the surface, and U_m is the maximum velocity. The pressure is plotted on a linear scale.

Profile I is sharply cusped and resembles a well developed jet-stream wind profile. In profile II, the variation of wind with pressure is linear with a single maximum. Profile III is characterized by a flat maximum with small shear in the vicinity of the maximum, unlike the jet-stream profile.

The wind profiles may be represented by the equations

$$(U - U_0)/U_m = [(p_0 - p)/(p_0 - p_m)]^n, \quad \text{for} \quad p > p_m, \tag{26}$$

$$(U - U_0)/U_m = (p/p_m)^n, \quad \text{for} \quad p < p_m,$$

where p_m is the pressure at which U is a maximum (U_m). For simplicity, the velocity at the top of the atmosphere is also taken to be U_0. The exponent n determines the shape of the profile. For profile II, n equals one. For profile I, n is greater than one; and for profile III, n is less than one.

$\overline{B^2}$ can be computed as a function of n by substituting (26) in (4a) and integrating. It is easily shown that

$$\overline{B^2} = \frac{n^2}{(2n+1)}, \tag{27}$$

for the profiles described by (26). Values of $\overline{B^2}$ are given in table 1 for several values of n. (At n equal to zero, which corresponds to zero wind shear, $\overline{B^2}$ is strictly indeterminate. However, $\overline{B^2}$ approaches the limit zero as n approaches zero.) It is interesting to

Table 1. $\overline{B^2}$ as a function of n.

	n	$\overline{B^2}$
	$\frac{1}{4}$	0.041
(Profile III)	$\frac{1}{2}$	0.125
(Profile II)	1	0.333
(Profile I)	2	0.800
	3	1.29

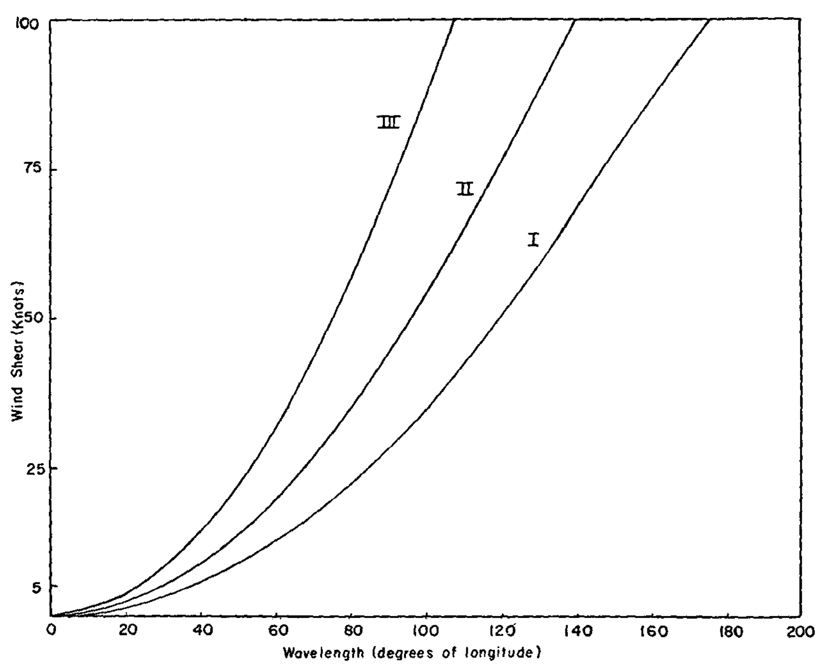

Fig. 3. Critical wavelength at latitude 40 deg as function of wind shear for velocity profiles shown in fig. 2. Wavelengths shorter than critical wavelength are unstable.

note that (27) does not depend on the values of U_m, p_m or U_0, but only on the shape of the profile.

In fig. 2, profiles I, II and III are constructed for n equal to 2, 1 and $\frac{1}{2}$, respectively. The amplification factors shown in fig. 1 are computed for a velocity profile intermediate between II and III, *i.e.*, for $\overline{B^2}$ equal to 0.2. The critical wavelengths for the three profiles at latitude 40 deg were computed from (24) and are shown in fig. 3.

The dependence of the stability criterion on the shape of the velocity profile is shown clearly in fig. 3. The jet-stream type of velocity profile is the most unstable, with the longest critical wavelength for any given wind shear. For example, with a wind shear of 50 kn (between p_0 and p_e), the critical wavelength is about 120 deg long for profile I, 96 deg for profile II, and 75 deg for profile III. A wavelength of 80 deg long is unstable if the wind shear exceeds 22 kn in profile I, 35 kn in profile II, and 57 kn in profile III. The differences between the curves diminish as the wind shear and wavelength decrease.

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