Smoothing and Persistence

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In a recent paper, Shapiro (1959) discussed the day-to-day persistence of surface pressure distribution over North America and Europe. Apparently, no comparative compilations have previously been made even though, as Shapiro points out, the recognition of persistence as an important feature of the pressure distribution dates back to the earliest analyses.

Table 1 gives the autocorrelation for lags, τ, of 1, 2 and 3 days. Here $T = 0$ when "instantaneous" synoptic data were used in the correlation and $T = 3$ for three-day means. The increased correlation found by Shapiro when going from $\tau = 2$, $T = 0$ to $\tau = 3$, $T = 3$ must be due, at least in part, to the averaging time, T. It may be worthwhile to make explicit this effect of smoothing on persistence.

Table 1. Shapiro's autocorrelations and corresponding values for a Markov process.

Days		Shapiro		Markov Process	
τ	T	America	Europe	$\gamma = 0.72$	$\gamma = 0.31$
1	0	0.49	0.73	0.49	0.74
2	0	0.25	0.57	0.23	0.54
3	0	not given		0.12	0.39
3	3	0.35	0.61	0.31	0.55

Let $f(t)$ designate any time series and

$$\phi(\tau) = \frac{1}{L}\int_0^L f(t)f(t-\tau)dt = \langle f(t)f(t-\tau)\rangle \quad (1)$$

its autocorrelation (unnormalized), with $\langle\ \rangle$ designating an average[1] over t. In the process of recording and reducing $f(t)$, we usually perform some operation

$$F(t) = \int_{-\infty}^{\infty} f(t-\alpha)w(\alpha)d\alpha. \quad (2)$$

For definiteness, set

$$w(\alpha) = \frac{1}{T} \quad \text{for} \quad -\frac{T}{2} \leqslant \alpha \leqslant \frac{T}{2} \quad (3)$$

and zero otherwise. Then $F(t)$ is a running average, over an interval T, of $f(t)$. This is not a good method of smoothing data, but it conforms to the usual

[1] The reader may be concerned how averages obtained from *integrals* involving $f(t)$ compare with corresponding averages obtained from *summations* of the record read at discrete intervals Δt. It can be shown that

$$\phi(j\Delta t) = \frac{1}{N}\sum_{i=1}^{N} f(i\Delta t)f(i\Delta t - j\Delta t)$$

equals $\phi(\tau)$ as defined in (1) for $\tau = j\Delta t$, provided the spectrum is continuous (as it is for the series discussed by Shapiro). With regard to the convolution (2), the result does not differ *appreciably* from what would be obtained by moving weighted sums at intervals Δt, provided the contribution from frequencies above $(2\Delta t)^{-1}$ is slight (Blackman and Tukey, 1958, *finite Dirac comb*, p. 257). This in turn implies that the record is reasonably smooth from point to point. With these comments in mind, it is easier to discuss the effect of smoothing in terms of integrals rather than summations.

meteorologic practice and will do for the present exercise.

The problem is to relate the autocorrelations of the smoothed and unsmoothed records—

$$\Phi(\tau) = \langle F(t)F(t-\tau)\rangle$$
$$= \int_{-\infty}^{\infty} w(\alpha')d\alpha' \int_{-\infty}^{\infty} f(t-\alpha)f(t-\alpha'-\tau)w(\alpha)d\alpha$$
$$= \int_{-\infty}^{\infty} w(\alpha')d\alpha' \int_{-\infty}^{\infty} \phi(\alpha-\alpha'-\tau)w(\alpha)d\alpha.$$

For the special case (3),

$$\Phi(\tau, T) = \frac{1}{T^2}\int_{-T/2}^{T/2} d\alpha' \int_{-T/2}^{T/2} \phi(\alpha-\alpha'-\tau)d\alpha$$

is the desired relation.

To go further, it is convenient to introduce the power spectrum, $S(f)$, of $f(t)$; accordingly,

$$\phi(x) = \int_0^\infty S(f)\cos 2\pi f x df,$$

$$\Phi(\tau, T) = \frac{1}{T^2}\int_0^\infty S(f)df \int_{-T/2}^{T/2} d\alpha' \times \int_{-T/2}^{T/2} \cos 2\pi f(\alpha-\alpha'-\tau)d\alpha \quad (4)$$
$$= \int_0^\infty S(f)\frac{\sin^2 \pi f T}{(\pi f T)^2}\cos 2\pi f \tau df.$$

Taking cosine transforms of both sides yields the well-known result that the spectrum of the smoothed records equals the spectrum of the unsmoothed record times the power admittance of the smoothing filter.

For a Markov process, the Fourier pairs are

$$\phi(\tau) = e^{-\gamma|\tau|}, \quad \text{and} \quad S(f) = \frac{4\gamma}{\gamma^2 + (2\pi f)^2}.$$

At high frequencies, $S(f) \sim f^{-2}$, thus giving a cut-off by 6 db per octave so representative of many geophysical processes. According to (4),

$$\Phi(\tau, T) = \frac{2}{\gamma^2 T^2}[\gamma(T-\tau) + e^{-\gamma T}\cosh \gamma\tau - e^{-\gamma\tau}], \quad 0 \leqslant \tau \leqslant T, \quad (5)$$
$$= \frac{4}{\gamma^2 T^2}\sinh^2\left(\frac{\gamma T}{2}\right)e^{-\gamma\tau}, \quad T \geqslant \tau,$$

which could have been obtained directly from (3), but at the expense of heavier algebra. Fig. 1 shows $\Phi(\tau, T)$ for various degrees of smoothing. The case $\gamma T = 0$ yields $\Phi(\tau, T) = \phi(\tau)$, which is the expected asymptotic behavior for no smoothing. The normalized correlation of the smoothed function

$$\Psi(\tau, T) = \Phi(\tau, T)/\Phi(0, T)$$

always lies above $\phi(\tau)$: the persistence of a Markov process is increased by smoothing.

By an appropriate choice of γ, Shapiro's correlations for the daily means can be reproduced (table 1). An extrapolation to $\tau = 3$ days, $T = 0$ leads to very low correlations. These are materially raised by setting $T = 3$ days but not quite enough to bring them to the observed levels. Evidently, large-scale pressure patterns show somewhat more persistence over 3 days than a Markov process. In all events, the effect of smoothing on persistence cannot be ignored.

An important special case is $T = \tau$. This refers to the day-to-day persistence of daily means, week-to-week persistence of weekly means, *etc.* The resulting function

$$\Psi(\tau, \tau) = \frac{2\sinh^2(\frac{1}{2}\gamma\tau)e^{-\gamma\tau}}{\gamma\tau + e^{-\gamma\tau} - 1}$$

is portrayed by the dashed curve in fig. 1. $\Psi(\tau, \tau)$ equals $1 - \frac{2}{3}\gamma\tau \cdots$ for small $\gamma\tau$, and $\frac{1}{2}(\gamma\tau)^{-1}$ for large

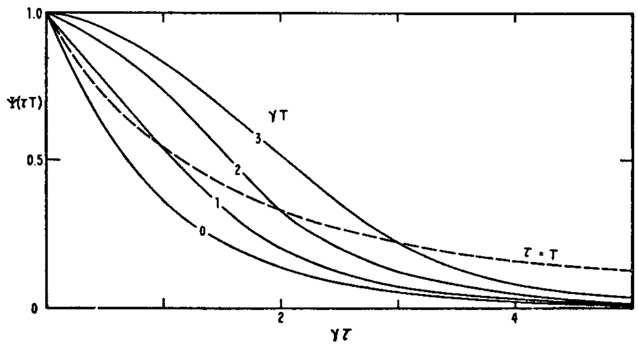

FIG. 1. The normalized autocorrelation, $\Psi(\tau, T)$, of a Markov process for various degrees of smoothing ($\gamma T = 0, 1, \cdots$). The dashed curve refers to special case $\tau = T$ for which the lag time and average time are the same.

$\gamma\tau$. Thus, $\Psi(\tau, \tau)$ diminishes monotonically for increasing lags, though far less rapidly than $\Psi(\tau, T)$.

This condition does not hold in general. For a "white spectrum," $S(f) =$ constant, and $\Psi(\tau, T) = 1 - \tau/T$ for $\tau \leqslant T$, and zero otherwise. Thus, $\Psi(\tau, \tau)$ vanishes identically. For spectra peaked at some frequency $f_0 \neq 0$, $\Psi(\tau, \tau)$ is peaked near $\tau = f_0^{-1}$ as well as at $\tau = 0$. The atmospheric spectrum is peaked around 2 cy per wk, and the excess of the observed to the Markov correlation for $\tau = 3$ days may perhaps be explained in this way.

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REFERENCES

1. Blackman, R. B., and J. W. Tukey, 1958: The measurement of power spectra from the point of view of communications engineering. *Bell System Tech. J.*, **37**, 185–282.
2. Shapiro, R., 1959: Some observations of the persistence of the surface pressure distribution. *J. Meteor.*, **15**, 435–439.