AN APPROXIMATION TO THE PRODUCT OF DISCRETE FUNCTIONS

George W. Platzman

The University of Chicago[1]

(Manuscript received 25 April 1960)

ABSTRACT

If one regards a discrete function as a vector, the "best" approximation to the product of two discrete functions (defined for the same set of values of the argument) is not necessarily the ordinary scalar product. The "best" approximation is shown to be an approximation-in-the-mean to the product of the trigonometric interpolation polynomials (cardinal functions) which correspond to the given discrete functions. This approximation arises naturally when the product is taken in the spectral domain. However, it can be approached by the ordinary scalar product provided the input functions are smoothed. The smoothing operator is linear and easily computed; it results in the suppression of all harmonics of wave length less than four times the mesh length.

1. Introduction

The first numerical integrations of a meteorological prediction equation—carried out on the ENIAC computer (Aberdeen, Maryland) in the Spring of 1950 by the Princeton group (Charney, Fjørtoft, von Neumann (1950))—were based upon the barotropic vorticity equation applied to initial 500-mb data. One year later, computations were made (on the ENIAC) for a barotropic stability problem, by taking as initial data a sinusoidal disturbance in the streamline field, superimposed upon a non-uniform zonal flow between parallel walls.

The results of the "second" ENIAC computations have remained unpublished.[2] For this reason, and particularly because these integrations did not extend beyond a physical time of about twenty-four hours, the dynamical prediction group at The University of Chicago undertook to repeat this experiment when a high-speed computer facility first became available to it.[3] A report on this work will be published in the next issue of the Journal (Baer, 1961).

One aspect of the ENIAC computations was of particular interest to us in devising a program suitable for numerical integration over an extended time interval—in excess of, say, 30 days. Even for the relatively small number of iterations attained with the ENIAC (about 30), the numerical solution began to show evidence of a characteristic structure termed "noodling" or "spaghetti motion." This structure—which is evident mainly in the vorticity field (rather than the streamline field)—consists in the emergence and persistence of eddies whose length scale is a few grid intervals and which tend to be elongated, or filamented, in shape. A spectral analysis of the kinetic energy was not made at the time but probably would have revealed a slow growth of the "tail" of the spectrum.

A question that immediately arises in discussing "noodling" is whether this phenomenon is a real property of the differential equations or a truncation error (associated only with the corresponding difference equations). Regardless of the answer to this question, any tendency for persistent growth of energy in the short-wave end of the spectrum must be regarded as inimical to the success of the calculation merely because the largest truncation errors will, in general, be made in that part of the spectrum.[4] We concluded, therefore, that an *extended* numerical integration must be predicated upon the control of the growth of the tail of the spectrum. The device that was used for this purpose was presented by the writer (Platzman, 1958) in a report prepared after the integrations had been carried out successfully for an interval of 39 days (936 iterations), thus confirming the validity of the control mechanism. The present paper is a slightly abbreviated version of that report.

Briefly, the considerations which led to the appropriate device are as follows. If the dependent variables of the equations to be integrated are represented by the discrete set of their values on a grid, and if the problem is then posed as a different equa-

[1] This investigation was supported by funds provided by the National Science Foundation (Grant NSF-G 2159, Tech. Rep. 2, April 1958).

[2] In a related study, the writer described a method whereby the initial growth (or decay) rate may be estimated for non-linear barotropic flows posed as initial-value problems (Platzman, 1952).

[3] From July 1956 to December 1958, we had access to the IBM-704 computer facilities at the General Motors Technical Center (Detroit, Michigan), through a generous arrangement for which we are greatly indebted to the Research Staff of the General Motors Corporation.

[4] In regard to the question of the reality of the "noodling" phenomenon, we will remark that a tendency for "noodling" is consistent with the "spectral blocking" theorem given, for example, by Fjørtoft (1953). For two-dimensional nonviscous flow, Fjørtoft showed that there is an upper limit to the amount of energy that may be transferred from low to high wave numbers by inertial exchange.

tion, one may regard each discrete function as the superposition of a finite number of harmonics up to, say, degree N (equal approximately to one-half the number of degrees of freedom in the function). The product of two such discrete functions will yield, therefore, a sum of harmonics up to degree $2N$, but, since this product is itself represented as a discrete function on the same grid, the harmonics of degrees $N+1$ to $2N$ are "ministerpreted" as harmonics of degree N or less—a process that we have termed "smearing." The resulting distortion is compounded as many times as the solution is stepped forward in time and is, according to our hypothesis, primarily responsible for the spurious growth rate mentioned previously.

It is evident that this distortion depends essentially upon the nonlinearity of the basic equations (in particular, owing to inertial and other transport terms, the hydrodynamical equations are quadratic). To prevent the "smearing" of high-order onto low-order harmonics, each product of discrete functions is evaluated only after the multiplier and multiplicand functions are smoothed by suppressing all harmonics of wave length less than four times the basic grid interval (that is, of order less than $\frac{1}{2}N$). The product of two functions which are smoothed in this way will contain harmonics up to order N only (rather than $2N$), and, since these are resolvable, no "smearing" occurs.

Recently, Phillips (1959) has given a very simple but effective example which proves that the distortion in question here may have an exponential growth rate and has substantiated this finding by means of a controlled numerical experiment. Phillips' illuminating example was presented after our work had been completed. We were, indeed, unaware that the distortion being controlled could lead to true computational instability, because we did not perform the null experiment that would have revealed instability in the absence of the control mechanism. The discussion presented below represents our attempt to establish an interpretation and a rationale for this type of "harmonic" smoothing, within the frame-work of interpolation and approximation theory and apart from the question of computational stability.

2. Trigonometric functions and polynomials

Let $a_l(x)$ designate a set of functions orthogonal and normalized with respect to a non-negative real weight function $\rho(x)$, in the interval $0 \leq x \leq L$; that is,

$$\int_0^L \rho(x) a_l^*(x) a_{l'}(x) dx = \begin{Bmatrix} 1 & \text{if} & l' = l \\ 0 & \text{if} & l' \neq l \end{Bmatrix} \quad (2.1)$$

where $\int_0^L \rho(x) dx = 1$, and a^* is the complex conjugate of a. It is well-known that, if we wish to approximate a function $f(x)$ by a linear combination of a finite number of the $a_l(x)$,

$$g(x) \equiv \textstyle\sum_l g_l a_l(x),$$

then the "best" approximation to $f(x)$, in the sense that the mean square deviation

$$\int_0^L \rho(x) |f(x) - g(x)|^2 dx$$

is a minimum, is provided by choosing g_l equal to

$$f_l \equiv \int_0^L \rho(x) a_l^*(x) f(x) dx, \quad (2.2)$$

the corresponding *expansion coefficient* of $f(x)$ with respect to $a_l(x)$. The function $g(x)$ thus determined is called the *approximation-in-the-mean* to $f(x)$. The validity of the foregoing statement does not depend upon the number of terms in $g(x)$, or upon the specific forms of a_l or ρ; further, the same result holds for discrete functions.

The set of trigonometric functions

$$a_l(x) \equiv \exp\left(\sqrt{-1}\,\frac{2\pi}{L} lx\right) \quad (2.3)$$

($l = 0, \pm 1, \pm 2, \cdots$) satisfies (2.1) with $\rho = 1/L$. Further, if $f(x)$ is a continuous periodic function (of period L), the Fourier series

$$\sum_{l=-\infty}^{\infty} f_l a_l(x),$$

where f_l is the expansion coefficient (2.2), converges uniformly to $f(x)$.

We now examine the corresponding properties of discrete periodic functions. We divide the interval $0 \leq x \leq L$ into n equal subintervals each of length $\Delta x = L/n$; the end points of these subintervals are then $x = i\Delta x$ where $i = 0, 1, 2, \cdots, n$. We denote by f^i the discrete function $f(i\Delta x)$; in particular, we have the set of discrete periodic functions

$$a_l^i \equiv \exp\left(\sqrt{-1}\,\frac{2\pi}{n} li\right). \quad (2.4)$$

These are of course very similar to the corresponding continuous functions (2.3), but they differ in this important respect: whereas the elements of the set (2.3) are all linearly independent, only a finite number (in fact, exactly n) of the members of (2.4) are independent; more specifically,

$$a^i_{l+kn} = a_l^i \qquad (k = 0, \pm 1, \pm 2, \cdots). \quad (2.5)$$

As a consequence of (2.5), the status of orthogonality among the a_l^i may be expressed in the identity

$$\frac{1}{n}\sum_{i=0}^{n-1} a_l^{i*}a_{l'}^{\,i} = \begin{cases} 1 & \text{if} \quad l' = l \bmod n \\ 0 & \text{if} \quad l' \neq l \bmod n \end{cases} \tag{2.6}$$

which is the analogue[5] of (2.1).

It is convenient to select from the a_l^i a set of linearly independent members, which we do by restricting l to the interval

$$-N \leqslant l \leqslant N \tag{2.7}$$

$$N \equiv \begin{cases} \frac{1}{2}(n-1) & \text{if} \quad n \text{ is odd} \\ \frac{1}{2}n & \text{if} \quad n \text{ is even} \end{cases}.$$

When n is even, (2.7) admits a single redundancy by including $l = +\frac{1}{2}$ and $l = -\frac{1}{2}$, which give the same a_l^i. This leads to no real difficulty and is advantageous because it preserves symmetry in the range of l. Henceforth, N will have the meaning defined in (2.7).

Now let f^i designate an arbitrary discrete periodic function ($f^{i\pm n} = f^i$), and consider the expansion

$$f^i = \sum_{l=-N}^{N} f_l a_l^i. \tag{2.8}$$

Multiply by a_l^{i*} and sum over i; because of (2.6), we get

$$\frac{1}{n}\sum_{i=0}^{n-1} a_l^{i*}f^i = \begin{cases} f_l & \text{if} \quad |l| \neq \frac{1}{2}n \\ f_l + f_{-l} & \text{if} \quad |l| = \frac{1}{2}n \end{cases}.$$

When $l = \pm\frac{1}{2}n$, only the sum $f_l + f_{-l}$ (and not f_l or f_{-l} separately) enters in the representation (2.8); hence, there is no loss of generality in taking $f_l = f_{-l}$ in which case

$$f_l = \frac{c_l}{n}\sum_{i=0}^{n-1} a_l^{i*}f^i \tag{2.9}$$

$$c_l \equiv \begin{cases} 1 & \text{if} \quad |l| < \frac{1}{2}n \\ \frac{1}{2} & \text{if} \quad |l| = \frac{1}{2}n \\ 0 & \text{if} \quad |l| > \frac{1}{2}n \end{cases}.$$

We may show now that the functions a_l^i are "complete," in the sense that an arbitrary periodic function f^i can be represented by the expansion (2.8), the coefficients of which are given by (2.9).

To prove this, it suffices to demonstrate that (2.8) is satisfied identically by the f_l defined in (2.9). Now, from (2.9),

$$\sum_{l=-N}^{N} f_l a_l^i = \frac{1}{n}\sum_{i'=0}^{n-1} c^{i'-i}f^{i'}$$
$$c^i \equiv \sum_{l=-N}^{N} c_l a_l^i. \tag{2.10}$$

With the help of the identity

$$\sum_{l=-p}^{p} a_l^i = \frac{\sin\left[(2p+1)\dfrac{\pi}{n}i\right]}{\sin\left(\dfrac{\pi}{n}i\right)}, \tag{2.11}$$

we find that, in (2.10),

$$c^i/n = \begin{cases} 1 & \text{if} \quad i = 0 \bmod n \\ 0 & \text{if} \quad i \neq 0 \bmod n \end{cases}. \tag{2.12}$$

Consequently, (2.10) reduces to f^i. The function c^i given in (2.12) is a periodic pulse (of intensity n), the expansion coefficients of which are the c_l of (2.9).

It should be remarked that, if f^i is a more general discrete function with unequal end-point values ($f^0 \neq f^n$), the expansion coefficients f_l in (2.9) produce the arithmetic mean $\frac{1}{2}(f^0 + f^n)$ at the end points. Throughout this discussion, we assume that the function to be represented has no jump ($f^0 = f^n$). If a jump occurs, it can be removed by subtracting a suitable linear function.

3. Trigonometric interpolation; the cardinal function

Let $\hat{f}(x)$ and $\hat{g}(x)$ denote *continuous* periodic functions, and form the discrete functions

$$f^i \equiv \hat{f}(i\Delta x), \qquad g^i \equiv \hat{g}(i\Delta x).$$

If $f^i = g^i$ for all integral i, we may say that $\hat{f}$ and $\hat{g}$ are *cotabular* (E. T. Whittaker, 1915). Evidently, there is an infinite number of periodic functions $\hat{g}(x)$ cotabular with a given periodic function $\hat{f}(x)$.

An important example of a set of cotabular functions is the set of trigonometric functions

$$a_{l+kn}(x) \qquad (k = 0, \pm 1, \pm 2, \cdots), \tag{3.1}$$

for a fixed value of l. Clearly, there are n distinct sets (3.1). The fact that all functions in each set are cotabular follows directly from (2.5). It is also evident that in each set the function of *lowest order* is the one for which $|l + kn| \leq \frac{1}{2}n$. To put this another way: in each set, the function of lowest order is the one corresponding to $k = 0$ in (3.1) if l is restricted as in (2.7) (except that for $l = \pm\frac{1}{2}n$ there are two functions of lowest order, corresponding to $k = 0$ and ∓ 1).

The preceding remarks enable us to answer the following questions. If $\hat{f}(x)$ is an arbitrary continuous periodic function and $\hat{g}(x)$ is any other continuous periodic function cotabular with $\hat{f}(x)$, we ask (i) what

[5] A strict analogy with (2.1) would require the inclusion of the end point $i = n$ in (2.6). This could be done by inserting under the sum in (2.6) a weight function equal to $\frac{1}{2}$ at the end points ($i = 0$ and $i = n$) and equal to 1 at all other points. The simpler form given in (2.6) suffices because the summand has equal values at the end points.

is the connection between the coefficients of the respective Fourier series for $\hat{f}(x)$ and $\hat{g}(x)$ and (ii) what is the function of *lowest-order* cotabular with $\hat{f}(x)$? Since $g^i = f^i$, we have

$$\sum_{l'=-\infty}^{\infty} \hat{g}_{l'} a_{l'}{}^i = \sum_{l'=-\infty}^{\infty} \hat{f}_{l'} a_{l'}{}^i.$$

Multiply this equation by $a_l{}^{i*}$, and sum over i; by making use of (2.6), we find

$$\sum_{k=-\infty}^{\infty} \hat{g}_{l+kn} = \sum_{k=-\infty}^{\infty} \hat{f}_{l+kn}. \tag{3.2}$$

Eq (3.2) answers the first question and shows that the sum of all coefficients belonging to cotabular harmonics must have the same value for all cotabular functions. The answer to the second question stems directly from this result, for it is clear from (3.2) that the function, $f(x)$ say, of lowest-order cotabular with $\hat{f}(x)$ is obtained by setting $f_{l+kn} = 0$ for all $|l + kn| > \frac{1}{2}n$. This means that (3.2) is satisfied in the form

$$\sum_{k=-\infty}^{\infty} \hat{f}_{l+kn} = \begin{Bmatrix} f_l & \text{if} \quad |l| < \frac{1}{2}n \\ f_l + f_{-l} & \text{if} \quad |l| = \frac{1}{2}n \end{Bmatrix}.$$

If we agree to take $f_l = f_{-l}$ when $|l| = \frac{1}{2}n$, then the desired function is

$$\begin{aligned} f(x) &= \sum_{l=-N}^{N} f_l a_l(x) \\ f_l &\equiv c_l \sum_{k=-\infty}^{\infty} \hat{f}_{l+kn}, \end{aligned} \tag{3.3}$$

with c_l as in (2.9).

The function $f(x)$ defined by (3.3) can be determined for every periodic function $\hat{f}(x)$ and is the same for each function cotabular with $\hat{f}(x)$; thus, a function of the type $f(x)$ is uniquely associated with each cotabular set. This function is called the *cardinal function* of the set (E. T. Whittaker, 1915).

We will show now that the coefficients f_l of the cardinal function are equal, respectively, to the expansion coefficients in the trigonometric series representation of the discrete function f^i. In (3.3), replace x by $i\Delta x$:

$$f^i = \sum_{l=-N}^{N} f_l a_l{}^i; \tag{3.4}$$

now multiply by $a_l{}^{i*}$ and sum over i. By applying (2.6), we see that f_l has exactly the form (2.9). This result makes it clear that the simplest construction of the cardinal function consists in evaluating the expansion coefficients for f^i and then forming the cardinal function (3.3) directly from these coefficients.

The cardinal function may be regarded as an expression for interpolation between the values f^i, or it may be regarded as an approximation to any member of the set of associated cotabular functions. In the latter sense, it is important to note that in view of (3.3), the cardinal function is not necessarily an approximation-in-the-mean to any of the cotabular functions.

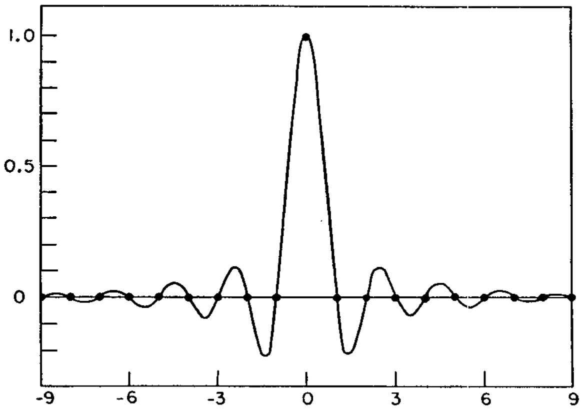

FIG. 1. The cardinal function $c(x)/n$ for $n = 18$ (see eq (3.6)); the abscissa is marked in units of $x/\Delta x$.

Since the coefficients f_l in (3.3) are given by (2.11), we may express $f(x)$ explicitly as a linear combination of the f_l:

$$\begin{aligned} f(x) &= \frac{1}{n} \sum_{i=0}^{n-1} c(x - i\Delta x) f^i \\ c(x) &\equiv \sum_{l=-N}^{N} c_l a_l(x). \end{aligned} \tag{3.5}$$

By comparing (3.5) with (2.10), we see that $c(x)$ is the cardinal function of the periodic pulse c^i (of intensity n). By carrying out the l-sum in (3.5), we find that

$$c(x) = \left(\sin \frac{\pi x}{\Delta x}\right) \div \begin{Bmatrix} \sin \dfrac{\pi x}{n\Delta x} & (n \text{ odd}) \\ \tan \dfrac{\pi x}{n\Delta x} & (n \text{ even}) \end{Bmatrix}. \tag{3.6}$$

This function for $n = 18$ is portrayed in fig. 1, together with the associated pulse.

Throughout this discussion, our attention has been confined to periodic functions. By keeping Δx fixed and taking the limit $n \to \infty$, one may obtain without difficulty the corresponding results for the infinite interval. The cardinal function in this more general form has many interesting properties and has been studied extensively (for a summary, see J. M. Whittaker, 1935). For the infinite interval, the cardinal function of a *unit* pulse, analogous to (3.6) is equal to $\theta^{-1} \sin \theta$, where $\theta \equiv \pi x/\Delta x$. The Fourier transform of this function is equal to $\Delta x/2\pi$ when $|k| < \pi/\Delta x$ and equal to zero when $|k| > \pi/\Delta x$ (where k is the wave number), which is similar to the frequency-response function for a low-pass filter with sharp cut-off at frequency $\pi/\Delta x$ (period $2\Delta x$). The cardinal function may, in fact, be interpreted as the output resulting from the transmission of a unit pulse through an ideal low-pass filter. Similarly, the more general cardinal function analogous to the $f(x)$ of (3.5) may be regarded as the output resulting from the transmission through an ideal low-pass filter, of a sequence of pulses of intensity f^i spaced apart twice the cut-off period of the filter. This interpretation arises naturally in the field of communication theory (see Shannon, 1949 and Hall, 1950).

4. Approximation for the product of two functions

The typical nonlinear operation in problems of integrating the hydrodynamical equations consists in the multiplication of two functions. Suppose that the integration is performed numerically and that f^i and g^i are two discrete functions which at some stage of the numerical process are to be multiplied. We ask, simply, is f^ig^i the "best" approximation to the product function?

To clarify this question, we remark that, in an analytic (as distinguished from numerical) solution, continuous functions $\hat{f}(x)$ and $\hat{g}(x)$ exist for which f^i and g^i are the discrete representations. The question is then: will f^ig^i give the "best" discrete representation of $\hat{f}(x)\cdot\hat{g}(x)$? To discuss this question, assume that at the particular stage being considered there are no truncation errors present, so that

$$f^i = \hat{f}(i\Delta x), \qquad g^i = \hat{g}(i\Delta x).$$

As approximations to the unknown functions $\hat{f}(x)$ and $\hat{g}(x)$, take the corresponding cardinal functions $f(x)$ and $g(x)$; and, as an approximation of the product $\hat{f}(x)\hat{g}(x)$, take $\hat{h}(x) \equiv f(x)g(x)$.

Since $f(x)$ and $g(x)$ are given by the expansions

$$f(x) = \sum_{l=-N}^{N} f_l a_l(x); \qquad g(x) = \sum_{l=-N}^{N} g_l a_l(x),$$

the product is

$$\hat{h}(x) = \sum_{l=-2N}^{2N} \hat{h}_l a_l(x), \tag{4.1}$$

$$\hat{h}_l \equiv \sum f_{l'} g_{l''} \begin{pmatrix} l' + l'' = l, & |l| \leqslant 2N \\ |l'| \leqslant N, & |l''| \leqslant N \end{pmatrix}.$$

It is important to note that the product $\hat{h}(x)$ contains harmonics of order twice the maximum order contained in the factors. In accordance with (3.3), the cardinal function $h(x)$ associated with $\hat{h}(x)$ has expansion coefficients

$$h_{-\frac{1}{2}n} = \hat{h}_{-\frac{1}{2}n} + \tfrac{1}{2}(\hat{h}_{\frac{1}{2}n} - \hat{h}_{-\frac{1}{2}n}) \quad (n \text{ even})$$
$$h_l = \hat{h}_l + \hat{h}_{l+n} \quad (-\tfrac{1}{2}n < l \leqslant -1)$$

$$h_0 = \begin{cases} \hat{h}_0 & (n \text{ odd}) \\ \hat{h}_0 + \hat{h}_{-n} + \hat{h}_n & (n \text{ even}) \end{cases}$$
$$h_l = \hat{h}_l + \hat{h}_{l-n} \quad (1 \leqslant l < \tfrac{1}{2}n)$$
$$h_{\frac{1}{2}n} = \hat{h}_{\frac{1}{2}n} + \tfrac{1}{2}(\hat{h}_{-\frac{1}{2}n} - \hat{h}_{\frac{1}{2}n}) \quad (n \text{ even}).$$

These relations show that each expansion coefficient of the cardinal function $h(x)$ associated with the product $f(x)g(x)$ is equal to the corresponding expansion coefficient of the product, plus (except for $l = 0$ or $\pm\frac{1}{2}n$) one other coefficient of the product, the latter corresponding to a higher-order harmonic generated by the product operation but incapable of being resolved with the n degrees of freedom to which the representation of the product function is restricted. Since $h(x)$ is cotabular with $\hat{h}(x)$ and, therefore, $h(i\Delta x) = f^ig^i$, we may say, from the preceding remarks, that the discrete function $h^i \equiv f^ig^i$ fails to be the "best" approximation to $\hat{h}(x) \equiv f(x)g(x)$ in the sense that the cardinal function associated with h^i is not an approximation-in-the-mean to $\hat{h}(x)$.

If we ask for a function $\bar{h}(x)$ with n degrees of freedom, which is an approximation-in-the-mean to $\hat{h}(x)$, we know that the answer is given by taking $\bar{h}_l = \hat{h}_l$, where $\bar{h}_l$ are the expansion coefficients of $\bar{h}(x)$. In particular,

$$\bar{h}(x) = \sum_{l=-N}^{N} \hat{h}_l a_l(x).$$

We now assert that $\bar{h}^i \equiv \bar{h}(i\Delta x)$ is a "better" approximation to $\hat{h}(x) \equiv f(x)\cdot g(x)$ than is $h^i \equiv f^ig^i$ ($\neq \bar{h}^i$ in general), in the sense that the trigonometric interpolation polynomial (cardinal function) which corresponds to $\bar{h}^i$ is an approximation-in-the-mean to $\hat{h}(x)$, while that which corresponds to h^i is not.

In order to compute $\bar{h}^i$ from f^i and g^i, it is necessary, practically, first to compute f_l and g_l; then, from (4.1), $\bar{h}_l$ ($= \hat{h}_l$) may be computed. It is impossible to express $\bar{h}^i$ as a product of a linear combination of the f^i and a linear combination of g^i. This point may be clarified by reference to the diagram in fig. 2. The axes here are l', l'' so that each point may be imagined to correspond to an harmonic interaction generated in the product $f(x)g(x)$ and represented by $f_{l'}g_{l''}$. From (4.1), we see that the coefficient $\hat{h}_l$ is the sum of the products taken along the diagonal segments $l' + l'' = l$ within the interaction square and that the function $\hat{h}(x)$ is composed of contributions covering the outer square in fig. 2. The function $\bar{h}(x)$, however, covers only the region enclosed by the solid heavy lines in the figure and excludes the triangles in the lower left and upper right corners of the large square ($|l| > N$). If $\bar{f}^i$ and $\bar{g}^i$ denote any linear transformations of f^i and g^i, respectively, then the product $\bar{f}^i\bar{g}^i$ would necessarily cover a square (or possibly a rectangle) in the interaction diagram, and we see, therefore, that $\bar{h}^i$ cannot be represented as such a product.

There is an alternative procedure, in which a somewhat weaker approximation to the product function can be represented correctly as the product of two discrete functions which are linear transformations of f^i and g^i. Instead of approximating $\hat{f}(x)$ and $\hat{g}(x)$ by their respective cardinal functions, we take the following approximations-in-the-mean to the cardinal functions

$$\bar{f}(x) = \sum_{l=-p}^{p} \bar{f}_l a_l(x); \qquad \bar{g}(x) = \sum_{l=-p}^{p} \bar{g}_l a_l(x) \quad (4.2)$$

$$p \equiv \left\{ \begin{array}{ll} \frac{1}{2}(N-1) & (N \text{ odd}) \\ \frac{1}{2}N & (N \text{ even}) \end{array} \right\}.$$

Here the order of the highest harmonic is, at most, half of that needed to represent the data exactly. Consequently, in forming the product $\bar{f}(x)\bar{g}(x)$, the highest harmonic generated is of order N, at most. It follows that the cardinal function of this product is equal to the product itself; in other words, the expansion coefficients of the discrete function $\bar{f}^i\bar{g}^i = \bar{f}(i\Delta x)\bar{g}(i\Delta x)$ are equal to the expansion coefficients of the continuous function $\bar{f}(x)\bar{g}(x)$. In terms of the interaction diagram, this product covers the square indicated by the broken lines in fig. 2.

The functions $\bar{f}(x)$ and $\bar{g}(x)$ defined in (4.2) can be represented explicitly as linear combinations of f^i and g^i. To do this, we note that the coefficients $\bar{f}_l$ of $\bar{f}(x)$ can be expressed as follows:

$$\bar{f}_l = (c_l)^{-1} w_l{}^* f_l$$

$$w_l \equiv \left\{ \begin{array}{ll} 1 & \text{if} \quad l \leqslant p \\ 0 & \text{if} \quad l > p \end{array} \right\}. \quad (4.3)$$

With the aid of the identity given in (2.11), one may easily determine the "filter function" w^i which corresponds to the response w_l defined in (4.3); the result is

$$w^i = \frac{\sin\left[(2p+1)\dfrac{\pi}{n} i\right]}{\sin\left(\dfrac{\pi}{n} i\right)}. \quad (4.4)$$

Here p is defined as in (4.2), so we see that the application of the filter (4.4) produces a smoothed function $\bar{f}^i$ in which the highest wave number (frequency) is not greater than $\frac{1}{2}N$ or the smallest wave length (period) not less than $4\Delta x$. Fig. 3 illustrates the form of w^i.

The iterative process of integrating the hydrodynamical equations requires that each function formed as a product must itself be used subsequently in a product. For this reason, the filter (4.4) must be applied also to the product $\bar{f}^i\bar{g}^i$. The effect of this operation is to remove the lower left and upper right corners of the inner interaction square in fig. 2, leaving the region marked out by the heavy, broken lines.

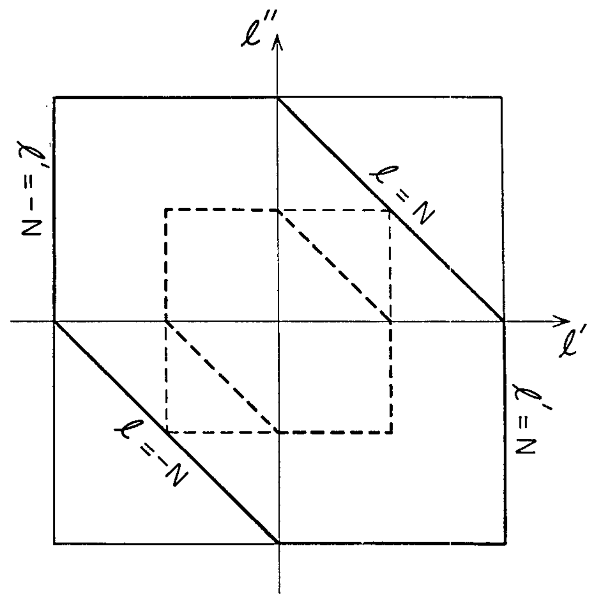

FIG. 2. Interaction diagram.

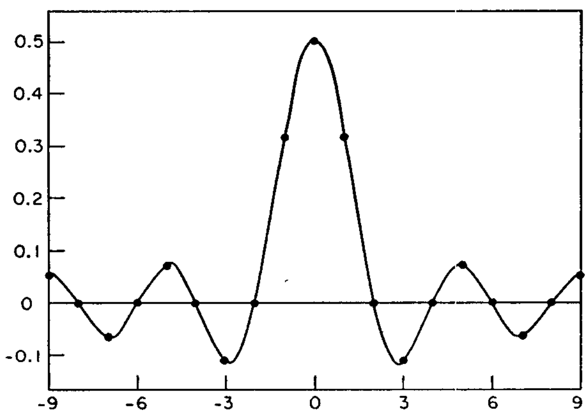

FIG. 3. The filter function w^i/n for $n = 18$ (see eq. (4.4)); the abscissa is marked in units of $i = x/\Delta x$.

We may summarize these conclusions as follows. If f^i and g^i are suitable discrete representations of $\hat{f}(x)$ and $\hat{g}(x)$, the "best" approximation to the product $\hat{f}(x)\hat{g}(x)$ is not $f^i g^i$ but rather the discrete representation of an approximation-in-the-mean to the product $f(x)g(x)$ of the cardinal functions associated with f^i and g^i. However, the latter approximation can be obtained only by operations with the expansion coefficients of f^i and g^i, and, unless the integration program is formulated explicitly in terms of the expansion coefficients, this is not an efficient scheme. An alternative method—which yields an approximation-in-the-mean to the "best" approximation—consists in first smoothing f^i and g^i and then forming the ordinary product of the smoothed functions. The smoothing consists in suppressing all harmonics of wave length less than four times the basic grid interval.

REFERENCES

Baer, F., 1961: The extended numerical integration of a simple barotropic model. *J. Meteor.*, **17**, (to be publ.).

Fjørtoft, R., 1953: On the changes in the spectral distribution of kinetic energy in two-dimensional, non-divergent flow. *Tellus*, **5**, 225–230.

Hall, W. F., 1950: Communication theory applied to meteorological measurements. *J. Meteor.*, **7**, 121–129.

Jackson, D., 1930: *The theory of approximation.* Amer. Math. Soc. Colloquium Publs., Vol. XI.

Phillips, N. A., 1959: *An example of non-linear computational instability.* The Rossby Memorial Volume, Rockefeller Inst. Press, New York.

Platzman, G. W., 1952: The increase or decrease of mean-flow energy in large-scale horizontal flow in the atmosphere. *J. Meteor.*, **8**, 347–358.

Shannon, C. E., 1948: Communication in the presence of noise. *Proc. Inst. radio Eng.*, **37**, 10–12.

Whittaker, E. T., 1915: On the functions represented by the expansions of the interpolation theory. *Proc. r. Soc. Edinburgh*, **35**, 181–194.

Whittaker, J. M., 1935: *Interpolatory function theory.* Cambridge Tracts in Math. and Math. Phys. No. 33, Cambridge Univ. Press.