NUMERICAL METHOD FOR ESTIMATING THE STRATOSPHERIC TEMPERATURE DISTRIBUTION FROM SATELLITE MEASUREMENTS IN THE CO_2 BAND [1]

Giichi Yamamoto

U. S. Weather Bureau

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ABSTRACT

Based on Kaplan's idea that the temperature structure of the upper atmosphere may be inferred from satellite measurements, new methods of estimating the temperature distribution are presented and sample calculations are made assuming observations in four intervals of the 15-micron CO_2 band.

1. Introduction

Kaplan [1] has proposed an interesting idea that the temperature distribution in the atmosphere can be inferred from satellite measurements. He proposed to measure outgoing radiation in various band regions of the 15 micron CO_2 band and showed a method of estimating the temperature distribution [2].

Later King [3] showed the possibility of estimating the temperature profile by measuring radiation coming to a satellite from varying directions in a single spectral region, for which he proposed the 9.6 micron O_3 band. Practically, however, it seems that Kaplan's method will lead us to a better estimation, because the amount and distribution of CO_2 in the atmosphere are better known and simpler than those of O_3, and to use the limb darkening effect one must assume horizontal homogeneity of the temperature and ozone amount, which is not always so in the atmosphere.

Recently Wark [4] proposed a different method of solving the Kaplan problem. In this paper a third alternative for solving the Kaplan problem will be shown.

2. Outline of the problem

First we assume the atmosphere to be free from water vapor and ozone, and the mixing ratio of carbon dioxide to be constant throughout the atmosphere. Then the transmission function for an air column from the top of the atmosphere to a layer of pressure p is a function of p and the temperature distribution in the air column. Initially we do not know the temperature distribution, so that as a first approximation we will assume the temperature distribution of the standard atmosphere, so far as it is concerned with the temperature correction for the transmission function. Then the transmission function is expressed as a function of p alone.

The intensity of the vertically outgoing radiation coming to a satellite is then given by

$$J_i = -\int_0^{p_s} B_i(p)\frac{d\tau_i(p)}{dp}dp, \quad (i = 1, \cdots, m), \quad (1)$$

where J_i is the intensity of radiation of the i-th band region, m is the number of band regions to be used, $B_i(p)$ is the Planck function of the i-th region, taking p as the independent variable instead of the temperature T of the layer, $\tau_i(p)$ is the transmission function, and p_s is the surface pressure. As the band regions which will concern us are very opaque, p_s can safely be replaced by infinity. Also, since they are all located near the center of the 15 micron CO_2 band, we can assume

$$B_i(p) = \alpha_i B(p), \quad (i = 1, \cdots, m), \quad (2)$$

where α_i's are constants and $B(p)$ is the Planck function of a given wave number near the center of the band. In the later calculation we take 680 cm^{-1} as this wave number. Then (1) is given by

$$I_i = -\int_0^{\infty} B(p)\frac{d\tau_i(p)}{dp}dp, \quad (i = 1, \cdots, m), \quad (3)$$

where

$$I_i = J_i/\alpha_i, \quad (i = 1, \cdots, m). \quad (4)$$

Our problem is therefore, given I_i and $\tau_i(p)$, to evaluate B as a function of p by solving the simultaneous eq (3). It is essentially the problem of quadrature with unknown parameters in the integral. Kaplan and Wark divided the integral into several parts, say, from 0 to p_1, p_1 to p_2, etc., and obtained B or its slope in each part. An alternative method of solving eq (3)

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is to expand $B(p)$ into polynomials and determine the coefficients by solving the resulting simultaneous equations. If the available band regions are few, *i.e.*, m is small, both methods encounter difficulty. In the former method (Kaplan's and Wark's) a unique solution is unobtainable, because the division of the integral in (3) is arbitrary. In the latter method it is questionable whether or not $B(p)$ may be expressed by polynomials of so limited a number of terms. To overcome these difficulties it will be necessary, in the former method, to find dividing points $p_1, p_2, \cdots, p_{m-1}$, which give us a best estimation, perhaps by statistical investigation of known temperature distributions. In the latter method one must find polynomials which can express the actual $B(p)$ using a small number of terms, also by statistical investigation of known profiles. The main advantage of the latter method is that it makes possible an analytical approach to the problem, as will be shown later.

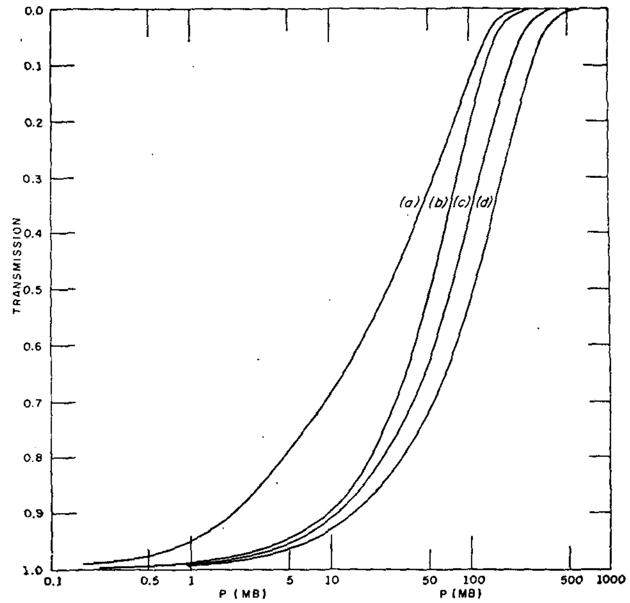

FIG. 1. Transmission curves, corresponding to the U. S. extension of the ICAO standard atmosphere. The band intervals are (a) 665–670 cm^{-1} (b) 675–680 cm^{-1} (c) 686–691 cm^{-1} and (d) 692–697 cm^{-1}.

3. Transmission values

In this investigation the author will use four band regions: 665–670, 675–680, 686–691 and 692–697 cm^{-1}. It should be mentioned that Kaplan uses seven band regions for the entire atmosphere and Wark uses three for the stratosphere, and that the accuracy of estimation depends highly upon the number of band regions used. The first region used by the author involves the Q branch of the 15-micron fundamental. Neither Kaplan nor Wark used this region. However the author believes that the use of it is favorable for a better understanding of the temperature structure of the upper stratosphere, as it is the most opaque region in the 15-micron CO_2 band even if we take a 5 cm^{-1} interval around the Q branch.

The transmission values were calculated by the method given by Yamamoto and Sassamori [5], which will not be duplicated here. In fig. 1 are shown the transmission values, corresponding to the U. S. extension of the ICAO standard atmosphere, as a function of pressure.

4. Methods of estimating the temperature profile

(*a*) *Using numerical values of the transmission functions.* As already stated in section 2, one of the key points of this research is to determine a variable by which the Planck function corresponding to the atmospheric temperature distribution can, in general, be represented by a polynomial with a small number of terms—in our case four. After several trials it was found that the power series of the new variable, $t = p^m$, where m is between $\frac{1}{3}$ and $\frac{1}{5}$, is suitable for the purpose. In the present investigation we will use a new variable given by

$$t = p^{1/5} + 0.2752. \qquad (5)$$

Here the numerical constant is added for convenience on calculation and it has no special physical significance.

Now, owing to the small amount of carbon dioxide in the upper atmosphere, we want to set the upper limit above which our estimation of the temperature profile cannot be applied. We will assume this limit to be $p = 0.2$ mb, or $t = 1$, because the transmission of even the most intense interval, 665–670 cm^{-1}, is 0.99 at this limit. Then eq (3) becomes

$$I_i = -\int_1^\infty B(t)\frac{d\tau_i(t)}{dt}dt, \quad (i = 1, 2, 3, 4). \qquad (6)$$

We now assume

$$B(t) = B(1) + a_1(t-1) + a_2(t-1)^2 + a_3(t-1)^3, \qquad (7)$$

where $B(1)$ is the Planck function at $t = 1$, and a_1, a_2, and a_3 are unknown constants to be determined.

Integrating (6) by parts and applying (7) we have

$$I_i = B(1) + a_1\int_1^\infty \tau_i(t)dt + 2a_2\int_1^\infty (t-1)\tau_i(t)dt + 3a_3\int_1^\infty (t-1)^2\tau_i(t)dt, \quad (i = 1, 2, 3, 4). \qquad (8)$$

Each integral in (8) can be calculated from the tabulated values of τ_i, so that, given the values of I_i, we can determine the values of $B(1)$, a_1, a_2, and a_3 from (8).

(*b*) *Expressing the transmission functions by Legendre polynomials.* By our method of expressing the Planck function by polynomials, we can make an analytical approach to the problem by expressing the transmission values too by polynomials. This will reduce numerical calculation to some extent especially when the the number of band intervals is increased.

First we will use Legendre polynomials, and in order to do so we will use a new variable

$$x = 0.6970p^{1/5} - 1.5050, \qquad (9)$$

which makes $x = -1$ at $p = 0.2$ mb and $x = +1$ at $p = 600$ mb. The transmission values of each interval used in this paper are all zero at $p = 600$ mb, so the use of the variable given by (9) may be admitted. Eq (3) now becomes

$$I_i = -\int_{-1}^{+1} B(x)\frac{d\tau_i(x)}{dx}dx, \quad (i = 1, 2, 3, 4). \qquad (10)$$

Now we will write

$$\frac{d\tau_i(x)}{dx} = \sum_{n=0}^{m} C_{in}P_n(x), \quad (i = 1, 2, 3, 4), \qquad (11)$$

where $p_n(x)$ are Legendre polynomials and C_{in}'s are constants which can be determined from the tabulated values of $\tau_i(x)$ by using the orthogonal property of the Legendre polynomials as follows:

$$C_{in} = \frac{(2n+1)}{2} \times \left[\left| \tau_i(x)P_n(x) \right|_{-1}^{+1} - \int_{-1}^{+1} \tau_i(x)P_n'(x)dx \right],$$
$$(i = 1, 2, 3, 4), \quad (n \quad 0, \cdots, m). \qquad (12)$$

Using Gauss's quadrature formula we have

$$\int_{-1}^{+1} \tau_i(x)P_n'(x)dx = \sum_{j=1}^{m} \alpha_j \tau_i(x_j)P_n'(x_j), \qquad (13)$$

where $x_1, \cdots, x_m$ are the zeros of $P_m(x)$ and

$$\alpha_j = \frac{1}{P_m'(x_j)} \int_{-1}^{+1} \frac{P_m(x)}{x - x_j}dx. \qquad (14)$$

The values of x_j's and α_j's are given by Lowan, Davids and Levenson [6]. Now we assume

$$B(x) = \sum_{n=0}^{3} a_n'P_n(x). \qquad (15)$$

Inserting (11) and (15) into (10) we have

$$I_i = -2\{C_{i0}a_0' + (\tfrac{1}{3})C_{i1}a_1' + (\tfrac{1}{5})C_{i2}a_2' + (1/7)C_{i3}a_3'\}, \quad (i = 1, 2, 3, 4). \qquad (16)$$

Thus by the orthogonal property of the Legendre polynomials we only need the values of $C_{i0}, \cdots, C_{i3}$ to calculate the unknown constants $a_0', \cdots, a_3'$. The values of $C_{i0}, \cdots, C_{i3}$ for our transmission functions are listed in table 1.

TABLE 1. Coefficients of eq (11).

	C_{i0}	C_{i1}	C_{i2}	C_{i3}
665–670 ($i = 1$)	−0.49500	+0.28109	+0.62328	−0.08274
675–680 ($i = 2$)	−0.50000	+0.00786	+0.91893	+0.26016
686–691 ($i = 3$)	−0.50000	−0.14694	+0.78455	+0.53207
692–697 ($i = 4$)	−0.50000	−0.32042	+0.55985	+0.64852

(*c*) *Using Chebyshev polynomials.* It is well known that Chebyshev polynomials have excellent properties in curve-fitting, interpolation and similar investigations. The designation "Chebyshev polynomials" had been used with several different meanings. In this paper we follow the definitions and notations given in "Tables of Chebyshev polynomials, National Bureau of Standards, Applied Mathematics Series 9." We again use the variable given by (9). We expand the transmission functions by Chebyshev polynomials

$$\tau_i(x) = \sum_{m=0}^{l} C_{im}'T_m(x), \quad (i = 1, 2, 3, 4), \qquad (17)$$

where $T_m(x)$ is defined by

$$T_m(x) = \cos(m \arccos x). \qquad (18)$$

Then C_{im}''s are given by

$$C_{im}' = \frac{2}{l}\sum_{\alpha=0}^{l-1} \tau_i(x_\alpha) \cos\frac{m\pi}{2l}(2\alpha + 1),$$
$$(i = 1, 2, 3, 4), \quad (m = 0, \cdots, l), \qquad (19)$$

where

$$x_\alpha = \cos\frac{\pi}{2l}(2\alpha + 1). \qquad (20)$$

The values of C_{im}''s for our transmission functions, which were calculated by taking nine terms in (17), *i.e.* $l = 8$, are listed in table 2. The derivatives of $\tau_i(x)$ are then given by

$$\frac{d\tau_i(x)}{dx} = \sum_{m=1}^{l-1} mC_{im}'U_{m-1}(x), \quad (i = 1, 2, 3, 4), \qquad (21)$$

TABLE 2. Coefficients of eq (17).

	(665–670) $i = 1$	(675–680) $i = 2$	(686–691) $i = 3$	(692–697) $i = 4$
C_{i0}'	+0.856444	+0.993999	+1.064888	+1.146221
C_{i0}'	−0.569351	−0.606472	−0.593015	−0.569773
C_{i2}'	+0.085162	−0.003252	−0.058941	−0.108975
C_{i3}'	+0.087758	+0.140682	+0.109505	+0.069475
C_{i4}'	−0.013917	+0.014871	+0.039442	+0.043530
C_{i5}'	−0.018355	−0.044659	−0.014910	+0.006049
$C_{i}^{6}{}'$	−0.010444	−0.010444	−0.015222	−0.005889
$C_{i}^{7}{}'$	+0.007305	+0.012075	−0.003639	−0.007130
$C_{i}^{8}{}'$	+0.009413	+0.010543	+0.001951	−0.003505

where

$$U_{m-1}(x) = (1 - x^2)^{-\frac{1}{2}} \sin (m \arc\cos x). \quad (22)$$

Next we assume

$$B(x) = \sum_{n=0}^{3} a_n'' T_n(x). \quad (23)$$

Inserting (21) and (22) into (10) and applying the following relations,

$$\begin{aligned} T_n(x) U_m(x) &= (1/2)\{U_{m+n}(x) + U_{m-n}(x)\} \quad \text{for } m \geqslant n, \\ &= (1/2) U_{m+n}(x) \quad \text{for m} = n - 1, \\ &= (1/2)\{U_{m+n}(x) - U_{m-n-2}(x)\} \quad \text{for } m \leqslant n - 2, \quad (24) \end{aligned}$$

and

$$\int_{-1}^{+1} U_n(x) dx = \frac{1}{n+1}\{T_{n+1}(+1) - T_{n+1}(-1)\}, \quad (25)$$

we have

$$\begin{aligned} I_i = & - 2(C_{i1}' + C_{i3}' + C_{i5}' + C_{i7}')a_0'' \\ & - \{(8/3)C_{i2}' + (16/15)C_{i4}' + (72/35)C_{i6}' + (128/63)C_{i8}'\}a_1'' \\ & - \{ - (2/3)C_{i1}' + (18/5)C_{i3}' + (50/21)C_{i5}' + (98/45)C_{i7}'\}a_2'' \\ & - \{ - (8/5)C_{i2}' + (32/7)C_{i4}' + (72/27)C_{i6}' + (128/55)C_{i8}'\}a_3'', \\ & (i = 1, 2, 3, 4). \quad (26) \end{aligned}$$

5. Result of estimation and discussion

For a known temperature profile we can easily calculate I_i, and then by solving (8) or (16), or (26), we can know the accuracy of our estimation. The estimation was made for several radiosonde observations which were selected, so far as possible, to represent the cases occurring in arctic, temperate and tropical regions, and for the profile of the standard atmosphere. The given profiles and estimated profiles are shown in fig. 2. As can be seen from the figure the general trend of the profiles is fairly well reproduced by the estimated ones between $x = -0.8$ and $+0.8$, or $p = 1$ mb and 400 mb. Beyond this interval errors of estimation increase owing to the decreasing slopes of the transmission functions. Even in this interval the fine structure, including the position of the tropopause, is not well reproduced by the present estimation, which was made by using four band regions. If the available number of band regions is increased, it is certain that we can make better estimations. It is interesting to observe that the estimated profiles by three methods are different, although the differences will decrease if the number of band regions are increased. For a small number of band regions, as in the case of our calculation, the differences of estimation are caused by using different polynomials for expressing the Planck function. Here we come to the statistical problem; what polynomials will best represent the Planck function corresponding to the atmospheric temperature distribution? A satisfactory answer to this question is not yet obtained, as we have only calculated for a small number of cases. However, from the standpoint of ease of calculation the use of Legendre polynomials is preferred.

The transmission functions hitherto used were those corresponding to the standard atmosphere. After estimating the temperature profile we can calculate the transmission functions corresponding to the known profile and using these transmission functions we can make the second estimation and so on, iteratively.

In this investigation we neglected water vapor and ozone absorptions. As both absorptions in the 15 micron region are weak, the correction due to these absorptions will be made by assuming the amounts and distributions of these gases, even without knowing observed values of them. Assuming the total transmission τ to be

$$\tau = \tau_{CO_2} \cdot \tau_{H_2O} \cdot \tau_{O_3} \quad (27)$$

we have

$$\begin{aligned} I = & - \int B \frac{d\tau_{CO_2}}{dp} \cdot \tau_{H_2O} \cdot \tau_{O_3} dp \\ & - \int B \frac{d\tau_{H_2O}}{dp} \tau_{CO_2} \tau_{O_3} dp \\ & - \int B \frac{d\tau_{O_3}}{dp} \tau_{CO_2} \tau_{H_2O} dp. \quad (28) \end{aligned}$$

The second and third terms of the right side of (28) are correction terms in which B can safely be replaced by that of the standard atmosphere, as a first approximation. In this case, too, the iterative calculation will improve the estimation.

All these second order approximations are neglected in this investigation, the main purpose of which is to show the method of calculation. Also, the author believes that in these estimations using only four band regions, neither will the former correction using the transmission functions corresponding to the calculated profiles improve the estimation much, nor will the latter correction due to water vapor and ozone significantly decrease the accuracy of it. Both corrections are necessary in actual calculation, however.

The accuracy of estimation depends mostly upon how many band regions we can use. A desirable way

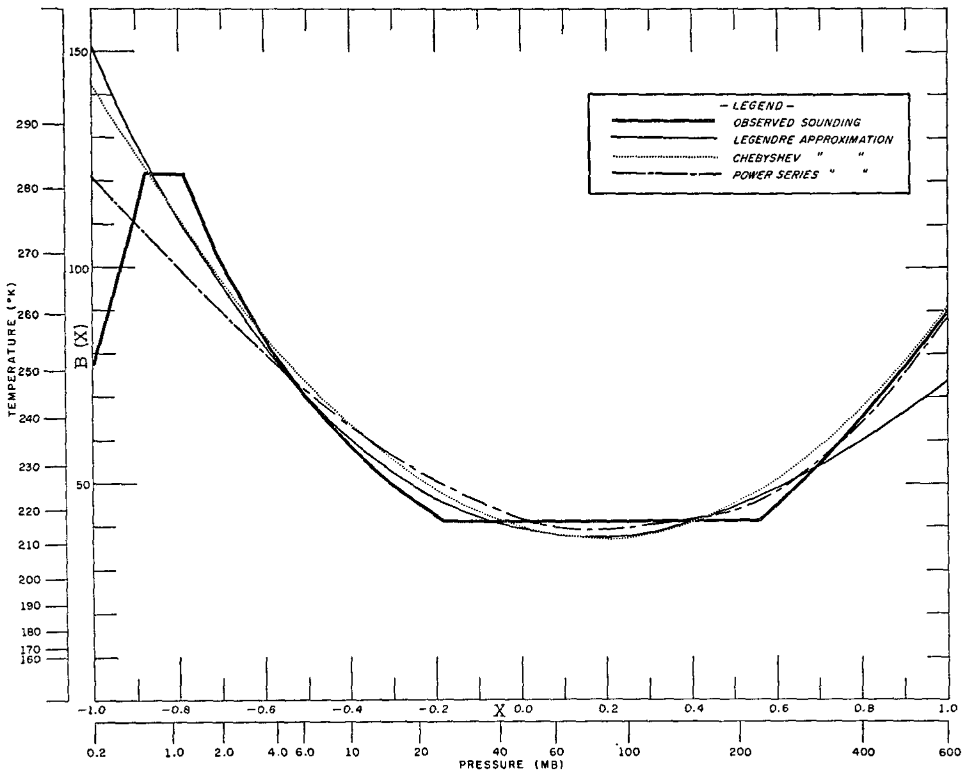

FIG. 2a. Estimated temperature profiles for the standard atmosphere. The ICAO standard atmosphere is indicated as the observed sounding.

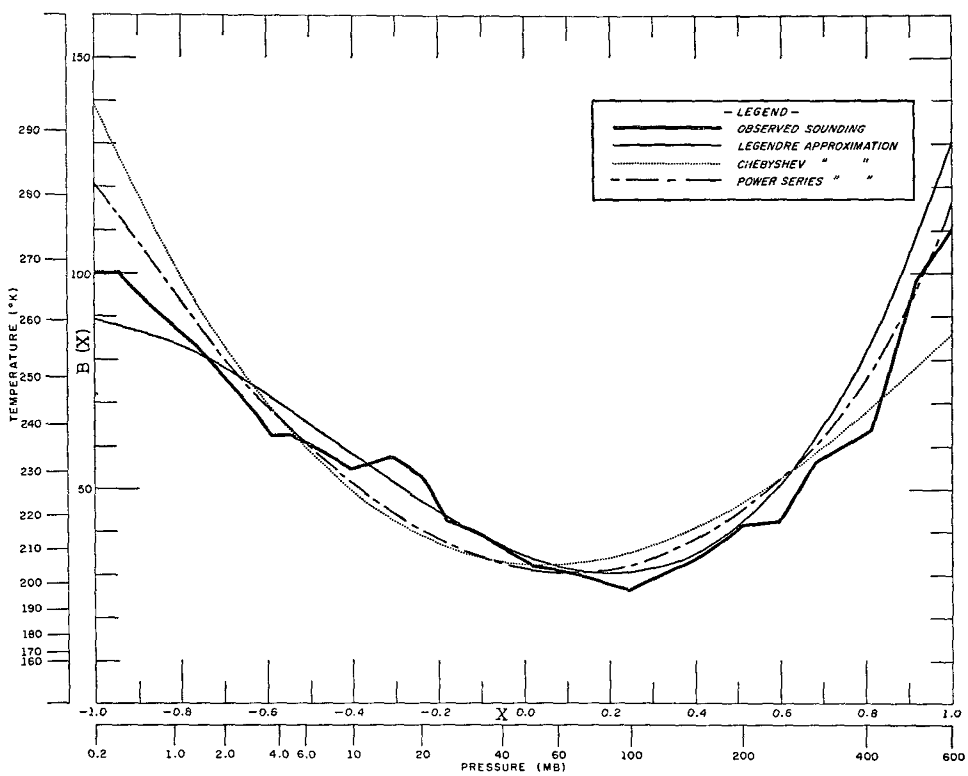

FIG. 2b. Estimated and observed temperature profiles for Balboa (8.9N). The observed sounding was taken at 00 GCT on 14 December 1958. The profile above 4 mb is extrapolated.

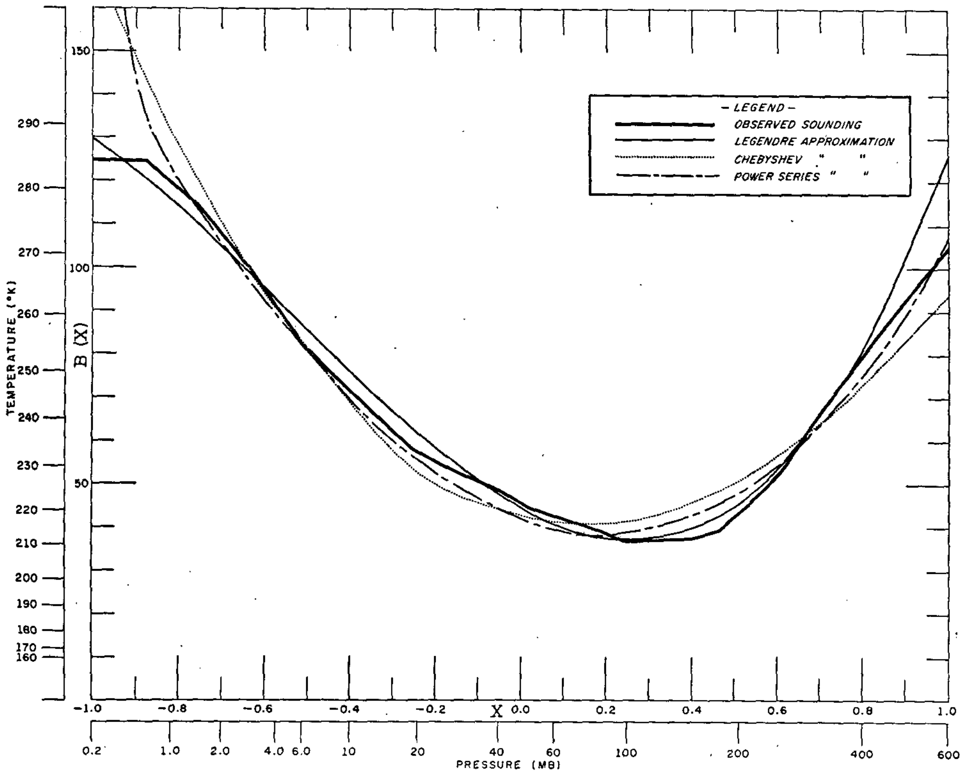

FIG. 2c. Estimated and observed temperature profiles for St. Cloud (45.6N). The observed sounding was taken at 12 GCT on 4 July 1958. The profile above 19 mb is extrapolated.

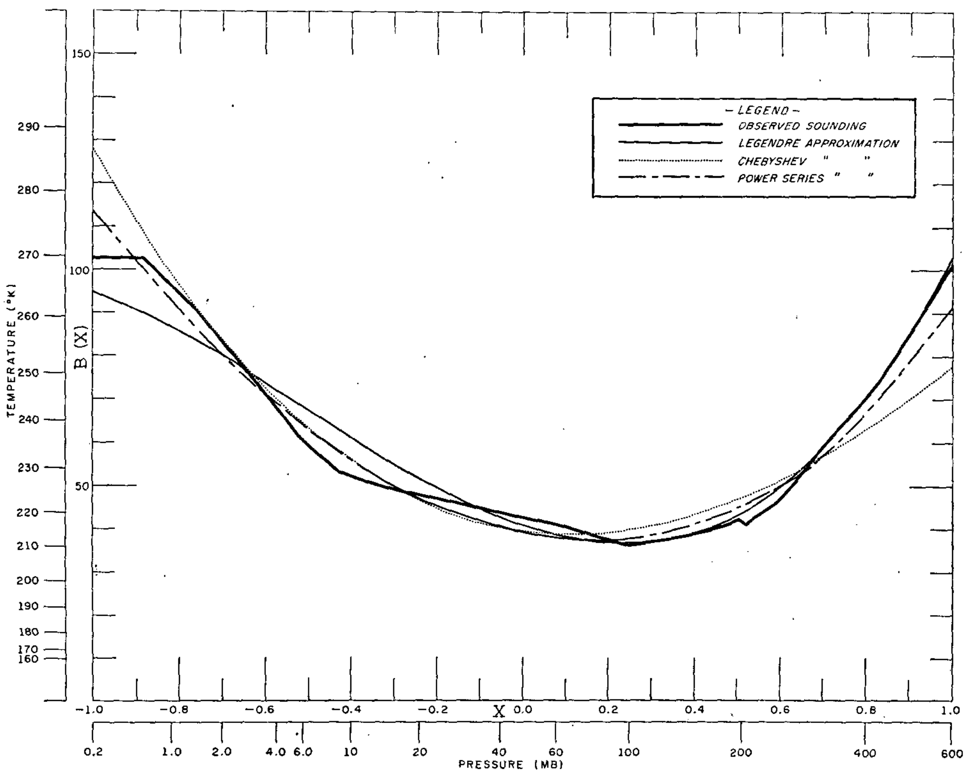

FIG. 2d. Estimated and observed temperature profiles for Green Bay (44.5N). The observed sounding was taken at 00 GCT on 7 October 1958. The profile above 11 mb is extrapolated.

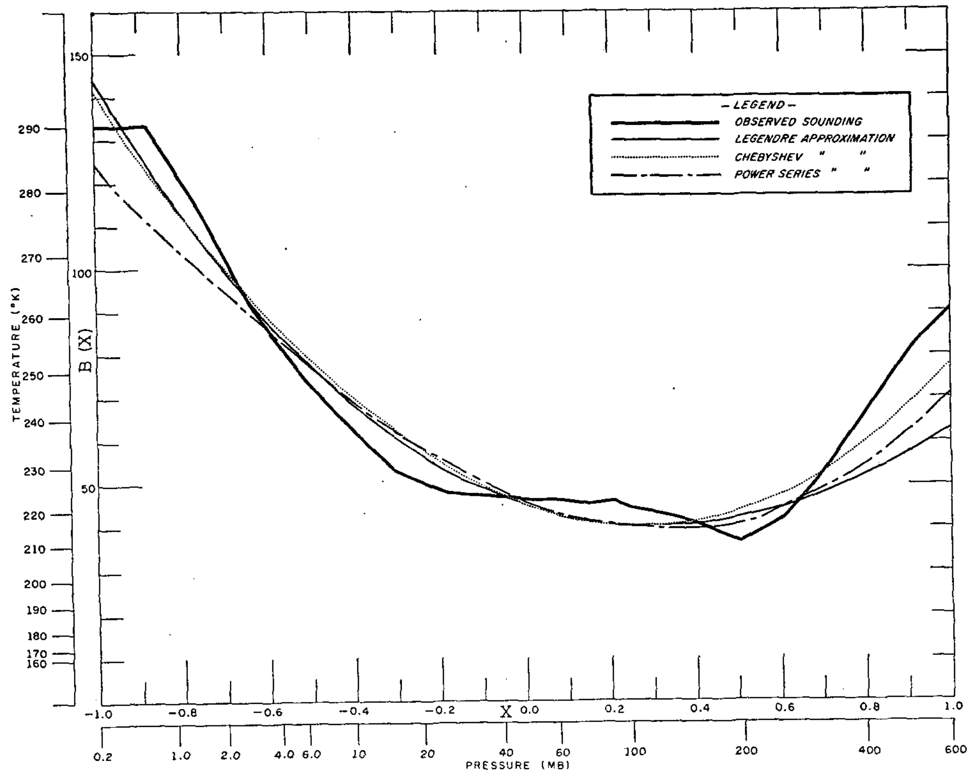

Fig. 2e. Estimated and observed temperature profiles for Argentia (47.2N). The observed sounding was taken at 12 GCT on 18 May 1958. The profile above 14 mb is extrapolated.

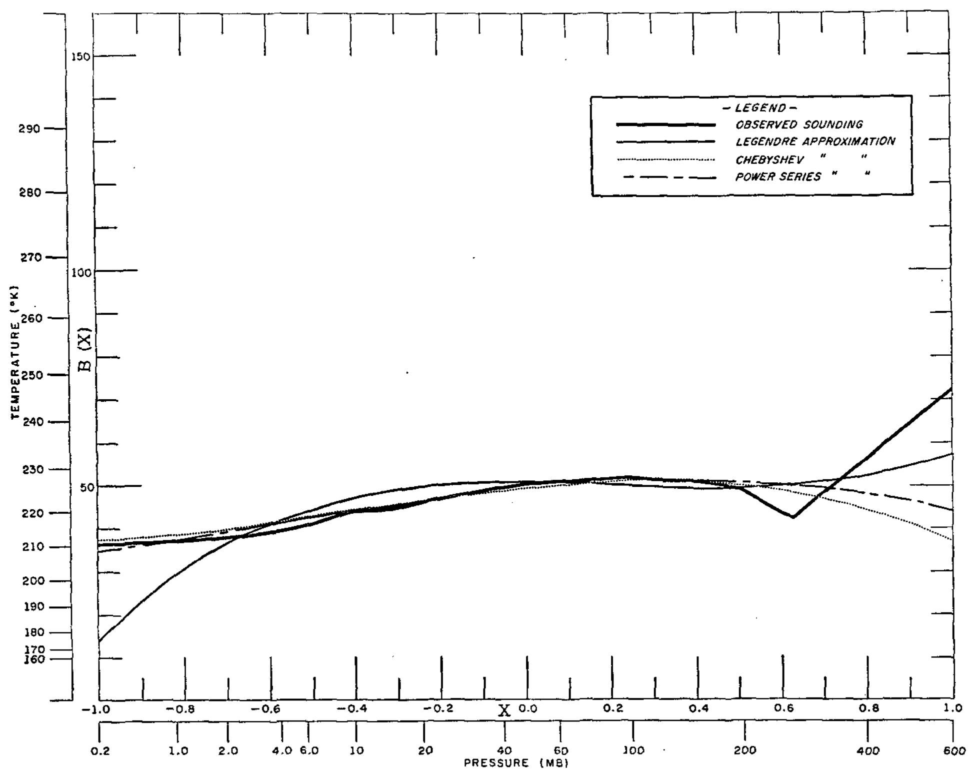

Fig. 2f. Estimated and observed temperature profile for Port Harrison (58.4N). The observed sounding was taken at 12 GCT on 4 December 1958. The profile above 10 mb is extrapolated.

of increasing the number of band regions is to decrease the interval assigned to each band region, which was assumed to be 5 cm^{-1} in this investigation. Quite recently Houghton [7] proposed an interesting idea of using a Fabry-Perot interferometer which, in principle, makes it possible to decrease the interval.

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REFERENCES

1. Kaplan, L. D., 1959: Inference of atmospheric structure from remote radiation measurements. *J. opt. Soc. Amer.*, **49**, 1004–1007.
2. Kaplan, L. D., 1960: The spectroscope as a tool for atmospheric sounding by satellites. *Instrument Society of America Conference Preprint* (*New York City*). 9-NY60.
3. King, J. I. F., 1959: Deduction of vertical thermal structure of a planetary atmosphere from a satellite. *General Electric, Tech. Info. series* No. R59SD477.
4. Wark, D. Q., 1960: On indirect temperature soundings of the stratosphere from satellites, *J. geophys. Res.*, **66**, 77–82.
5. Yamamoto, G. and T. Sasamori, 1960: Further studies on the absorption by the 15 micron carbon dioxide bands. Submitted to *Sci. Rep. Tohoku University, Series 5, Geophysics*.
6. Lowan, A. D., N. Davids and A. Levenson, 1942: Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula. *Bull. Amer. math. Soc.*, **48**, 739–743.
7. Houghton, J. T., 1960: The meteorological significance of remote measurements of infrared emission from atmospheric carbon-dioxide. Presented at IUGG meeting at Helsinki.