Statistical Effects in the Evolution of a Distribution of Cloud Droplets by Coalescence

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ABSTRACT

Several years ago Telford pointed out that the simplest coalescence model, in which a small group of droplets grew with unit collection efficiency by collecting droplets of half their volume, did not remain bimodal but that the statistical fluctuations in the discrete coalescence events caused many sizes to be evolved from the original two. A few droplets per million grew much more rapidly than the average; Telford was thereby able to shorten considerably the time necessary for rain formation in warm clouds.

Results have been obtained by numerical solution of the integro-differential equation which describes the time variation of a droplet distribution. Hocking's collection efficiencies were used. The computations show that Telford's mechanism is equally, if not more, effective when the initial distribution is continuous.

1. Introduction

In an important paper Telford (1955) pointed out that the usual method of computing the rate of growth of droplets in a cloud by coalescence, whereby a rate of increase of droplet radius dr/dt is computed as a function of drop size, collection efficiency and water content—and in which one ignores the fact that drops actually grow in finite increments as they coalesce with other cloud droplets—seriously underestimates the possible rate of growth by coalescence. This procedure gives only the average (and smoothed) rate of growth of all droplets of a certain radius; there will be some droplets which grow faster and some slower. Telford showed that a small but meteorologically significant ($100\ m^{-3}$) number could grow more than six times faster than the average, essentially as a result of experiencing better than average luck. Telford's computations employed an analytic method and envisaged a simplified cloud model in which there were originally present just two discrete groups of cloud droplets, one group being collector droplets with twice the mass of the collectee droplets in the other, main, group. He assumed unit collection efficiency at all sizes and considered only collisions with this main group.

The present paper will describe some preliminary results from numerical computations in which the statistical evolution of a cloud droplet *spectrum* was investigated, taking into account the variation of collection efficiency with size and including all possible collisions.

2. Statistical versus continuous growth models

In the "continuous" picture of droplet growth by coalescence, from the early calculations of Langmuir (1948) and Bowen (1950) to the complex computations of more recent investigations, *all* droplets with the same initial radius r_0 grow equally as time proceeds. Thus the time for growth to radius r_2 is taken to be precisely the same for all droplets of the same initial radius r_1; differences in the growth rate (dr/dt) for different radii are then the only influences broadening the droplet size spectrum.

The point emphasized by Telford was that the statistical, discrete nature of coalescence (i.e., growth by a finite number of finite steps) causes a spread in the time taken for originally identical drops to achieve a certain size, and that this statistical spread produces in time a distribution with a long "tail" towards larger sizes.

General considerations: Telford's original results are diagrammatically shown in Fig. 1a. The continuous growth process would envisage a steady rightward progress of the line representing the large group of droplets. This however is not physically realistic; if, for example, a computation gives $\frac{1}{2}V$ for the average growth of the larger droplets in a certain time interval, $2V$, V being the volumes of larger and smaller droplets, respectively, it does not of course mean that each large droplet has combined with half a small droplet. It is closer to the truth to say that one half of the large droplets have coalesced with one small droplet and attained volume $3V$. This is still a simplification, however, for a few $4V$ droplets will have been produced from $3V$ droplets, $5V$ droplets both from $4V$ and $3V$ droplets, and so on. So after *any* finite time all sizes, V, $2V$, $3V, \cdots NV$, are occupied, in the sense that there is a non-zero probability of a droplet of any size, however large, being found. If the rate of increase of the largest of all droplets is considered to be a "rate of growth," then it is in fact infinitely large. (From a meteorological

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point of view a probability of 10^{-20} is insignificant, since there are not as many as 10^{20} droplets in an entire average cloud; probabilities of 10^{-6} to 10^{-8} and perhaps smaller are, however, quite significant, for there may be 10^8–10^9 droplets in a cubic meter, and heavy rain requires only a few hundred raindrops m^{-3}.) It is evident that droplets which grow by increasing their volume in discrete steps cannot grow to a given size steadily and equally. When a droplet of radius r is formed from smaller droplets it immediately becomes statistically indistinguishable from all other droplets of radius r—and how or when it achieved this size is of no consequence. A fully satisfactory description of the process can only be given by an analysis treating the variation in the population of the various size intervals. If for a certain time interval there are non-zero probabilities that droplets of radius r_1 will grow to r_2, that droplets of radius r_2 will grow to r_3, and so on, then there is also a non-zero probability that many such steps will be taken during the specified interval. Hence there is no *maximum* radius, unless the probability of growth were to drop to zero at some point; once the process has commenced, there is a definite if small probability that any sized droplet will be found. There is no cutoff in the distribution, which will therefore show only an asymptotic falling off with increasing size. The importance of the statistical effect is greatly dependent on the fact that upon coalescing a droplet acquires a higher probability of further coalescence—since it becomes larger and falls faster. The effect of collection efficiency (not included by Telford) would be expected to add to the relative speed of the statistical growth in those circumstances where collection efficiency increases with size of collector droplet. Since collisions between larger droplets can only accelerate the process it would seem safe to conclude that consideration of these additional factors will in most cases tend to make statistical growth to be even more effective in comparison to "continuous" growth.

3. Computations of statistical growth in a continuous distribution

It is desirable to introduce the probability, per unit time and per unit concentration of droplets of radius r_2, that a droplet of radius r_1 will coalesce with a droplet of radius r_2. This quantity (the coagulation coefficient) will be denoted by $K(r_1,r_2)$. It is related to the collection efficiency $E(r_1,r_2)$ by the obvious relationship

$$K(r_1,r_2)=\pi r_1^2 E(r_1,r_2)[U(r_1)-U(r_2)] \quad (r_1 \geq r_2)$$
$$K(r_1,r_2)=K(r_2,r_1) \quad (r_2>r_1)$$

if $U(r_1)$ signifies the terminal velocity of a droplet of radius r_1. (For droplets small enough to be appreciably affected by diffusion, a diffusive-coagulation term should be included in $K(r_1,r_2)$; this is not, however, necessary in the present context).

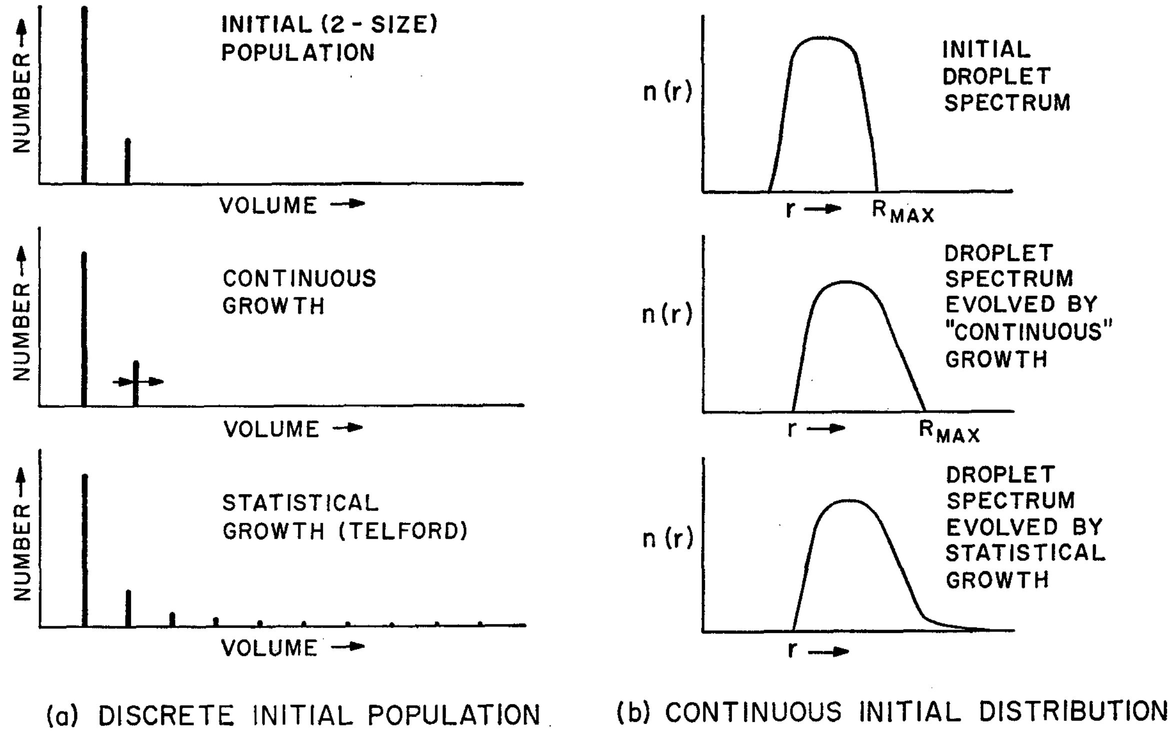

FIG. 1. Schematical diagram to illustrate the difference between statistical and "continuous" growth in (a) discrete droplet population, (b) continuously distributed droplet population.

If the droplet distribution is described by a distribution function $n(r)$ such that there are $n(r)\Delta r$ droplets per unit volume with radii between r and $r+\Delta r$, then evidently the total number of coalescences experienced by droplets in this size interval is, per unit volume,

$$n(r)\Delta r \int_0^\infty K(r,\rho)n(\rho)d\rho;$$

the total number of encounters per unit volume is equal to twice the decrease in the total droplet concentration N, (for each coalescence is experienced by two droplets), so that

$$\frac{dN}{dt}=\frac{1}{2}\int_0^\infty n(\sigma)\int_0^\infty K(\sigma,\rho)n(\rho)d\rho d\sigma. \quad (1)$$

Examining now the number of droplets greater than some size r, which will for convenience be called "big" droplets, one finds that this number

$$\int_r^\infty n(\rho)d\rho$$

is changed by: (a) a coalescence between two "big" droplets, which reduces the concentration of "big" droplets by one (b) a coalescence between two "little" droplets the sum of the volumes of which exceeds that of a droplet of radius r: such an encounter increases by one the number of "big" droplets. The rate of coalescence among "big" droplets is given by

$$\int_r^\infty n(\rho)\int_r^\infty K(\rho,\sigma)n(\sigma)d\sigma d\rho$$

the rate of those coalescences among "little" droplets which give new "big" droplets is given by

$$\int_0^r n(\rho)\int_{\sqrt[3]{r^3-\rho^3}}^r K(\rho,\sigma)n(\sigma)d\sigma d\rho.$$

Hence,

$$\frac{d}{dt}\int_r^\infty n(\rho)d\rho=-\int_r^\infty n(\rho)\int_r^\infty K(\rho,\sigma)n(\sigma)d\sigma d\rho+\int_0^r n(\rho)\int_{\sqrt[3]{r^3-\rho^3}}^r K(\rho,\sigma)n(\sigma)d\sigma d\rho.$$

When the distribution function is continuous, one may differentiate with respect to r, obtaining:

$$-\frac{d}{dt}n(r)=n(r)\int_r^\infty K(r,\sigma)n(\sigma)d\sigma+n(r)\int_r^\infty K(\rho,r)n(\rho)d\rho+n(r)\int_0^r K(r,\sigma)n(\sigma)d\sigma$$

$$+\int_0^r n(\rho)[K(\rho,r)n(r)-r^2(r^3-\rho^3)^{-\frac{2}{3}}K(\rho,\sqrt[3]{r^3-\rho^3})n(\sqrt[3]{r^3-\rho^3})]d\rho.$$

Or

$$\frac{d}{dt}n(r)=-2n(r)\int_0^\infty K(r,\rho)n(\rho)d\rho+\int_0^r K(\rho,\sqrt[3]{r^3-\rho^3})(1-\rho^3/r^3)^{-\frac{2}{3}}n(\rho)n(\sqrt[3]{r^3-\rho^3})d\rho. \quad (2)$$

This integro-differential equation describes the time evolution of the droplet spectrum $n(r)$. It may be obtained in a slightly simpler form if the spectrum is described by a volume distribution function $\nu(v)$:

$$\frac{d}{dt}\nu(v)=-2\nu(v)\int_0^\infty K(v,u)\nu(u)du$$

$$+\int_0^v K(u,v-u)\nu(u)\nu(v-u)du.$$

The latter equation has no advantage over the equation (2) unless the function $K(v,u)$ can be expressed analytically. [This was the approach adopted recently in a series of papers by Golovin (1963)—who assumed $K(v,u)\propto(v+u)$.] In the computations to be described, numerical integration of (2) was carried out by finite difference methods on a high speed (IBM 7094) computer. The integrals were replaced by second-order quadratures (which when tested on analytic integrands were accurate to much better than 1 per cent), and the function $n(\rho)$ described approximately by a vector n containing the values of $n(\rho)$ at one hundred tabular points. The time evolution of this vector could then be obtained by iterated matrix operations, which, although time-consuming, were quite straightforward. The matrix $\|K_{ij}\|=\|K(\rho_i,\rho_j)\|$ was derived from Hocking's (1959) collection efficiencies, extended by logarithmic interpolation to cover the finer intervals used in the numerical computation.

For comparison, computations were also made by applying the familiar continuous growth equation, which in the present notation becomes:

$$r^2\frac{dr}{dt}=\frac{\pi}{3}\int_0^r \rho^3K(r,\rho)n(\rho)d\rho. \quad (3)$$

4. Results

The computations were applied to an initial distribution containing 135 droplets cm^{-3} with 2.56 g m^{-3} liquid water. However, the results can be converted to other, more realistic, values by scaling the ordinates and abscissae; for example a distribution with 1 g m^{-3} and 53 droplets cm^{-3} but with the same shape, would evolve according to the same curves if times were multiplied by a factor of 2.56 and the concentrations divided by 2.56.

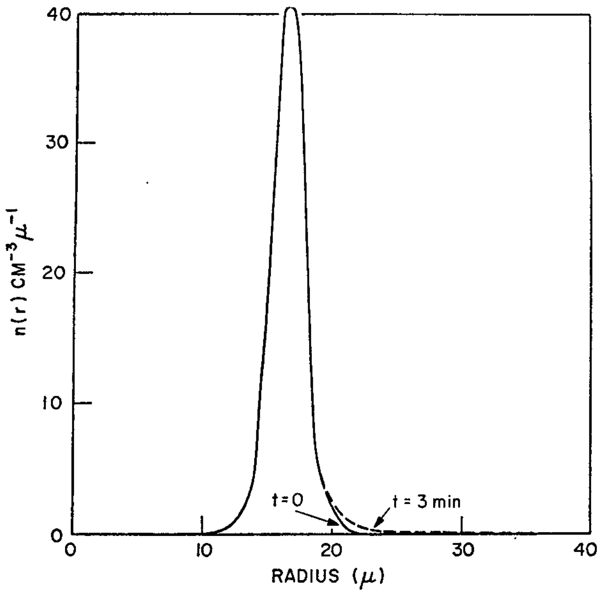

FIG. 2. Initial distribution and distribution after 3 minutes on a linear scale.

The initial distribution had changed so little after three minutes that the new distribution could not be plotted separately in Fig. 2 except at the larger radii. Computation by the continuous growth equation gave a curve differing so little from the curve for statistical growth that it could not be separated on the scale of Fig. 2. Yet when the spotlight is put on the tail of these curves by changing the scale suitably, the differences show up even after shorter periods. Figs. 3 and 4 show respectively spectra and cumulative curves after 20 seconds statistical growth and 20 seconds "continuous" growth, for the assumed initial population shown.

The rapid growth of a few droplets as described by Telford, and the shape differences sketched in Fig. 1b are quite apparent in Figs. 3 and 4 even after so short a time. At a level of 100 droplets m^{-3} the "growth rate" was almost ten times faster than given by the "continuous growth" equation—which is in good agreement with the ratio of about $6\frac{1}{2}$ found by Telford, and the qualitative prediction that inclusion of the effect of collection efficiency and coalescences among all size groups would increase the advantage enjoyed by the statistically fortunate few.

The importance of this effect for precipitation formation stems from the disproportionate time which is consumed by the early part of the growth from cloud-to-rain-drop. For reference, Fig. 5 shows for the cloud

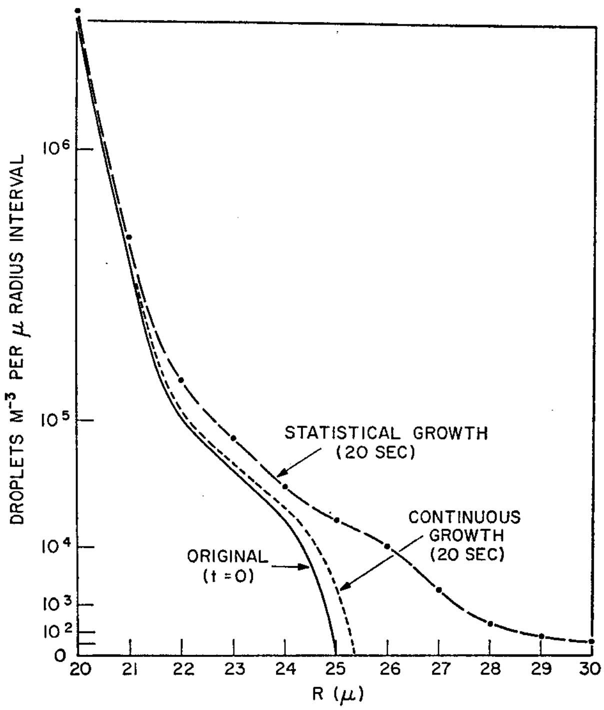

FIG. 3. The initial distribution and distribution after 20 seconds on a cube root scale.

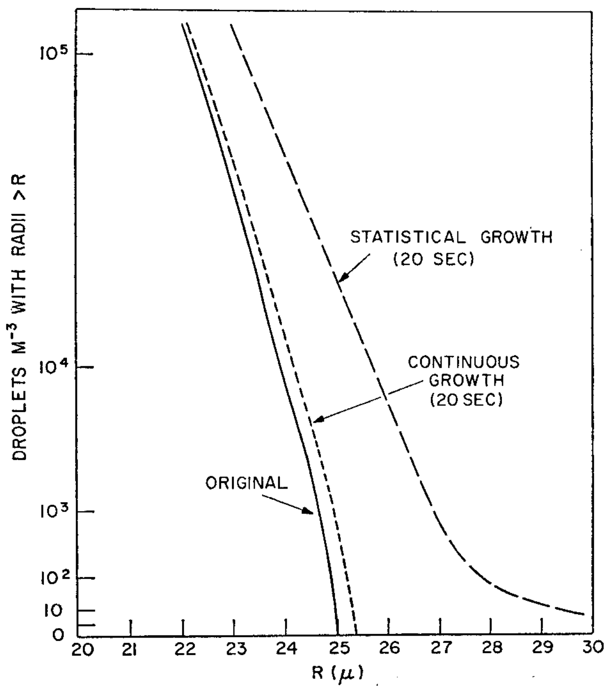

FIG. 4. The cumulative distribution corresponding to Fig. 3.

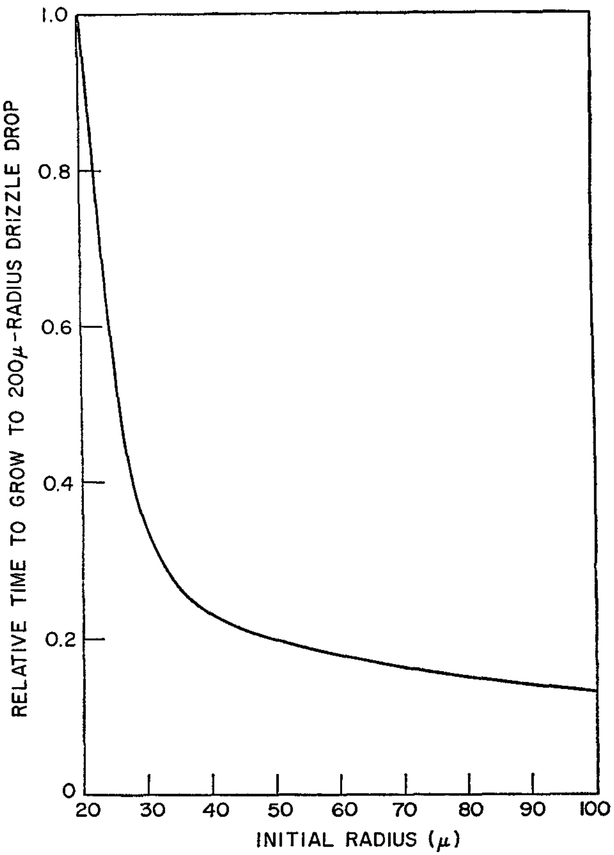

FIG. 5. Relative time spent at various sizes during the "continuous" growth of (average) droplets to 200 μ radius.

drop spectrum used in these calculations (but for constant collection efficiency) the relative time for growth to radius 200μ. The large proportion of the time which is spent in the 20–40μ region is cut down drastically for a statistically fortunate droplet; this large proportion of time, of course, would be even larger had collection efficiency been included, but since Hocking's values do not go beyond $r=30$ μ it was omitted.

It may be remarked that the growth of a few especially favored droplets into raindrops can give rain without drastically modifying the cloud droplet spectrum, whereas continuous growth would have the consequence that the entire spectrum would spread out and have become very different by the time rain had formed. The impression that rain is associated with cloud droplet spectra not dramatically different to those in similar clouds which are not raining seems to be general, and is supported by, e.g., Diem's (1948) curves for stratus and nimbostratus.

5. Conclusions

Computations with continuous size spectra fully accord with Telford's (1955) conclusion—from calculations employing discrete sizes—that statistical fluctuations in droplet coalesecnces can produce a meteorologically significant (100–1000 m^{-3}) number of large drops in a relatively short time. The production of these statistically fortunate drops took place about ten times faster than the average growth rate.

The results herein are the first results from a numerical study of the influence of cloud droplet concentration and size distribution on the evolution of precipitation. They are being reported separately at this early stage because they are apparently the first numerical integrations of the statistical equations for continuous size distributions, and as such confirm Telford's earlier conclusions.

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REFERENCES

Bowen, E. G., 1950: The formation of rain by coalescence. *Austral. J. Sci. Res.*, **A3**, 193–213.

Diem, M., 1948: Messungen der Grösse von Wolkenelementen, II. *Meteor. Rundschau*, **1**, 261–273.

Golovin, A. M., 1963: The solution of the coagulation equation for cloud droplets in a rising air current. *Izv. Ak. Nk. SSSR.* (Geophys. Ser.) **5**, 783–791.

Hocking, L. M., 1959: The collision efficiency of small droplets. *Quart. J. R. meteor. Soc.*, **85**, 44–50.

Langmuir, I., 1948: The production of rain by a chain reaction in cumulus clouds at temperatures above freezing. *J. Meteor.*, **5**, 175–192.

Telford, J. W., 1955: A new aspect of coalescence theory. *J. Meteor.*, **12**, 436–444.