NOTES AND CORRESPONDENCE

Comments on the "Inversion by Slabs of Varying Thickness"

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15 June 1964

In a recent note King has described a method which he proposes to apply to the problem of inferring atmospheric temperature profiles from measurements of atmospheric radiance (e.g., from a satellite instrument).

The equation which is actually to be "inverted" in King's method is (his equation (9)):

$$I(1/\mu_i)-B(0)=\sum_j (\Delta B)_j e^{-\tau_j/\mu_i}, \qquad (1)$$

where I is to be observed at successive values of μ_i and unknowns $(\Delta B)_j$ and τ_j deduced therefrom. Since I is measured and since several approximations are required to reduce the original non-linear integral equation to the finite linear combination appearing in (1), the latter equation is necessarily subject to errors which certainly exceed 1 per cent.

The solution of (1) when it is exactly true is indeed given by the procedure described by King [which Whittaker and Robinson (1924), pg. 369, attribute to Prony]. The problem is unchanged but the symbolism becomes more familiar if one poses the question in the form of fitting a sum of exponentials to an observed $f(x)$, i.e., to make

$$f(x)=A_1e^{-a_1x}+A_2e^{-a_2x}+\cdots+A_ne^{-a_nx}$$

by a suitable selection of the A_i and a_i.

The algorithm is admittedly elegant but it is also extremely unstable. This is discussed in detail by Lanczos (1956); the following is a direct quote from that text: (page 275) "This simple and straightforward mathematical solution would hardly indicate what enormous practical difficulties arise if we try to apply it to physical problems. The difficulty is caused by the fact that the solution [of an equation equivalent to King's equation (13)] succeeds only if the data are given with *excessive* accuracy. If the separation of four or five exponentials is demanded, the associated linear system becomes so strongly skew-angular that an accuracy of 6 to 8 significant figures would be needed [in the data being fitted] for their successful solution."

Lanczos goes on to show that the Prony algorithm, applied to data constructed from the combination

$$0.0951e^{-x}+0.8607e^{-3x}+M1.5576e^{-5x}$$

gave as a "solution" the combination

$$2.202e^{-4.45x}+0.305e^{-1.58x}.$$

Yet the latter expression gave a close agreement—the rms error being 2 per cent—when computed at the 24 values of x (0, 0.05, 0.10, . . . 1.15) at which the data were tabulated. It is therefore a satisfactory solution, in the numerical sense—but the distortion of the individual amplitudes and exponent constants is very severe.

At least in the writer's experience of this algorithm, Lanczos' conclusions are not at all over-pessimistic. Since he states that 6 or 8 significant figures are needed to fit four or five exponentials successfully—and surely any coarser a stratification of the atmosphere would be virtually valueless—and since such accuracy is quite out of the question, the usefulness of King's method must be questioned.

A further criticism of the procedure followed by King is called for by his choice of 0, 1, 2. . . $2n-1$ for the parameter $1/\mu$. Since μ is the cosine of the angle of view θ, measured from the nadir, these values imply for $\cos\theta$ the values ∞ (!); 1, $\frac{1}{2}$, $\frac{1}{3}$, . . . $\frac{1}{9}$, or angles (?), 0°, 60°, 70.5°, 78.5°, 80.4°, 81.8°, 82.8°, 83.6°—a range of values which could not be realized or approached in practice. To restrict the data to a practical range of angles would produce a considerably greater degree of interdependence among the $2n$ measurements and make the instability emphasized by Lanczos still worse.

REFERENCES

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