Instability Associated with the Continuous Spectrum in a Baroclinic Flow[1]

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ABSTRACT

A general initial value problem is solved for wave perturbations of small amplitude on a zonal atmospheric flow with constant vertical shear and vanishing temperature lapse rate, using frictionless, hydrostatic, adiabatic and geostrophic theory and beta-plane dynamics.

For every west-east wavelength, the solution is seen to be constituted of a finite number of discrete spectrum modes and a continuous spectrum contribution. (As in many other hydrodynamical problems, the relevant operator of the form $L-\lambda M=0$ is singular and the differential operators L and M are of equal order.)

As was previously demonstrated, for almost all wavelengths one of the eigenvalue modes is unstable. The present analysis shows that, for the exceptional wavelengths for which no such unstable normal mode exists, the continuous spectrum contribution is unstable.

1. Introduction

In examining the stability of a fluid flow with respect to prescribed disturbances, the existence of one exponentially growing normal mode associated with any given perturbation wavelength is sufficient to establish instability. The converse does not hold, *viz.*, if no exponentially unstable mode exists for a given wavelength, this does not establish stability of the flow with respect to disturbances of this wavelength (even in a linear theory).

The baroclinic stability problem of a west wind with vertical shear on the rotating, spherical earth has been the subject of various investigations (Charney, 1947; Kuo, 1952; Burger, 1962; Miles, 1964). For a linear wind profile, a constant temperature profile, with hydrostatic and geostrophic conditions, and using β-plane dynamics, the existence of unstable normal modes for all but a finite number of wavelengths has been proven (Burger, 1962). To complete the picture it is of some interest to establish whether all perturbations corresponding to these exceptional wavelengths are in fact stable.

In the following analysis no harmonic time dependence is assumed. Instead, as has been done by Case (1960) for other problems, an initial perturbation of very general vertical structure is imposed on the basic flow and the solution on this initial value problem found. The general solution is constituted of normal modes plus continuous spectrum modes and it is shown that, for the exceptional wavelengths with only neutral normal modes, the continuous spectrum modes are unstable, growing linearly with time.

We may remark in passing that the extension of instability across the "neutral" value $a=0$ [see Eq. (5)] as previously proved by the author, has been verified by Miles (1964) for small values of the parameters B and m defined in the next section, by perturbing the formal perturbation solution $v=\bar{u}-c$ which holds for $B=m=0$.

2. Formulation of the problem

For atmospheric motion on the synoptic scale the following approximate vorticity and adiabatic equations may be used (see, e.g., Burger, 1958)

$$\frac{d_h\zeta}{dt}+\beta v-\frac{f}{\rho}(\rho w)_z=0,$$

$$\frac{1}{\theta}\frac{d_h\theta}{dt}+sw=0.$$

(The notation is fairly standard, but we note that spherical coordinates are implied, β is the northward derivative of the coriolis parameter f, θ is the potential temperature and s, the static stability, is its logarithmic vertical derivative.) We further assume hydrostatic and geostrophic relations, implying the thermal wind relation, by which horizontal derivatives of the potential temperature may be eliminated from the adiabatic equation, and yielding

$$\frac{d_hv_z}{dt}-v_xu_z+v_z\left(u_x+v\frac{\beta}{f}-\frac{v}{r}\tan\phi\right)+\frac{g}{f}(sw)_x=0.$$

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(Here the subscript x denotes eastward differentiation, although x is not a coordinate.) As boundary conditions we require the vertical velocity to vanish at the ground and some condition like finiteness of energy per unit volume at infinity. Also initial values satisfying certain regularity conditions may be specified.

Now consider a basic stationary west wind $\bar{u}(\phi,z)$ independent of longitude. Perturbing this and linearizing, we have

$$\frac{D\zeta}{dt}+(\beta+\bar{\zeta}_y)v-\frac{f}{\bar{\rho}}(\bar{\rho}w)_z=0, \tag{1}$$

$$\frac{Dv_z}{dt}-\bar{u}_z v_x+\frac{g\bar{s}}{f}w_x=0, \tag{2}$$

with $\dfrac{D}{dt}=\dfrac{\partial}{\partial t}+\bar{u}\dfrac{\partial}{\partial x}$. The basic and total flows are geostrophic, and therefore ζ_x can be expressed in terms of v only. Elimination of w from the two perturbation equations, therefore, yields an equation essentially in v as the only perturbation quantity. From (2) we have

$$-\frac{f}{\bar{\rho}}(\bar{\rho}w_x)_z=\frac{f^2}{g\bar{\rho}}\left[\frac{D}{dt}\frac{\bar{\rho}v_z}{\bar{s}}-\frac{\bar{\rho}\bar{u}_z}{\bar{s}}v_x\right]_z$$

$$=\frac{f^2}{g\bar{\rho}}\left\{\frac{D}{dt}\left[\frac{\bar{\rho}v_z}{\bar{s}}\right]_z-\left[\frac{\bar{\rho}\bar{u}_z}{\bar{s}}\right]_z v_x\right\},$$

which, with (1) differentiated with respect to x, gives

$$\frac{D}{dt}\left\{\zeta_x+\frac{f^2}{g\bar{\rho}}\left[\frac{\bar{\rho}v_z}{\bar{s}}\right]_z\right\}+\left\{\beta+\bar{\zeta}_y-\frac{f^2}{g\bar{\rho}}\left[\frac{\bar{\rho}\bar{u}_z}{\bar{s}}\right]_z\right\}v_x=0.$$

If the equation is interpreted in a β-plane formalism (*viz.*, with both f and β treated as constant, and with x and y as cartesian coordinates) it becomes possible to ignore latitude dependence of the basic state, so that the coefficients involve the height only. The perturbation velocity may then likewise be taken as independent of latitude, and ζ_x simplifies to v_{xx}. If, finally, the simplest basic temperature and velocity profiles are chosen, i.e., constant temperature and shear, our equation becomes

$$\frac{D}{dt}\left[v_{zz}+\frac{\bar{\rho}_z}{\bar{\rho}}v_z+\frac{g\bar{s}}{f^2}v_{xx}\right]+\left[\frac{g\bar{s}\beta}{f^2}-\frac{\bar{\rho}_z}{\bar{\rho}}\bar{u}_z\right]v_x=0,$$

in which, since

$$\frac{\bar{\rho}_z}{\bar{\rho}}=\frac{\bar{p}_z}{\bar{p}}=-\frac{g\bar{\rho}}{\bar{p}}=-\frac{g}{R\bar{T}},$$

all coefficients are constant, except for the linear height-dependence of $\bar{u}$ contained in the operator D/dt. Note that the quasi-balance between the terms of the static stability, $\bar{\rho}_z/\bar{\rho}=\bar{p}_z/\gamma\bar{p}$, normally holds in the troposphere, introducing a factor γ in the formalism.

The boundary condition at the ground, $w=0$, implies $w_x=0$ for $z=0$, which, from (2), becomes

$$\frac{Dv_z}{dt}-\bar{u}_z v_x=0, \quad \text{for} \quad z=0.$$

At infinite height, the condition of bounded energy per unit volume gives

$$v=O(\bar{\rho}^{-\frac{1}{2}})=O\left(\exp\left[\frac{gz}{2R\bar{T}}\right]\right).$$

At the initial moment we prescribe $v(x,z,0)$.

For convenience we non-dimensionalize these equations by first putting

$$z=Hz',$$
$$\bar{u}=H\bar{u}_z\bar{u}'=H\bar{u}_z(\bar{u}_0'+z'),$$

where H is the scale height, defined as $H=(R\bar{T}/g)$. Further, for the horizontal scaling, we write

$$x=2\frac{(g\bar{s})^{\frac{1}{2}}}{f}Hx',$$

and a time scale defined by

$$t=2\frac{(g\bar{s})^{\frac{1}{2}}}{f}\frac{1}{\bar{u}_z}t',$$

now leaves the expression D/dt invariant up to a factor. (Note that, typically, $H\sim10^4$ m; $H(g\bar{s})^{\frac{1}{2}}/f\sim10^6$ m; $H\bar{u}_z\sim10$ m sec^{-1}; $(g\bar{s})^{\frac{1}{2}}/f\bar{u}_z\sim10^5$ sec.) On assuming harmonic x-dependence, $\exp(i\sqrt{B}x)$, and omitting primes in the dimensionless variables, we have

$$\left(\frac{1}{i\sqrt{B}}\frac{\partial}{\partial t}+\bar{u}\right)(v_{zz}-v_z-\tfrac{1}{4}Bv)+(m+1)v=0,$$

where $m=(g\bar{s}/f^2)(\beta H/\bar{u}_z)$, while the wave number k for the x-direction has led to the nondimensional form

$$\sqrt{B}=2\frac{(g\bar{s})^{\frac{1}{2}}}{f}Hk.$$

The boundary and initial conditions become

$$\left.\begin{aligned}&\left(\frac{1}{i\sqrt{B}}\frac{\partial}{\partial t}+\bar{u}\right)v_z-v=0 &&\text{at}\quad z=0\\ &v=O(\exp[\tfrac{1}{2}z]) &&\text{at}\quad z=\infty\\ &v=v^0(z)\exp(i\sqrt{B}x) &&\text{at}\quad t=0.\end{aligned}\right\} \tag{3}$$

This problem we now proceed to solve.

3. Solution of the initial value problem

To the above problem we apply a Laplace transform defined by

$$v^p(z)e^{i\sqrt{B}x}=\int_0^\infty e^{-pt}v(x,z,t)dt,$$

and by writing $\lambda=\bar{u}_0+\frac{p}{i\sqrt{B}}$, i.e., $\frac{p}{i\sqrt{B}}+\bar{u}=z+\lambda$, we obtain

$$(z+\lambda)(v_{zz}{}^p-v_z{}^p-\tfrac{1}{4}Bv^p)+(m+1)v^p =\frac{1}{i\sqrt{B}}(v_{zz}{}^0-v_z{}^0-\tfrac{1}{4}Bv^0),$$

$$\lambda v_z{}^p(0)-v^p(0)=\frac{1}{i\sqrt{B}}v_z{}^0(0),$$

together with the condition at infinite height.

Consider first the corresponding homogeneous equation

$$N''-N'-\tfrac{1}{4}BN+\frac{(m+1)}{X}N=0, \tag{4}$$

with independent solutions $N_1(X)$ and $N_2(X)$ and Wronskian, $W\equiv N_1N_2'-N_1'N_2$, satisfying $W'-W=0$, so that W is proportional to $\exp(X)$. The asymptotic behavior of the two solutions follows trivially from the differential equation, and the first solution is chosen to satisfy the stronger growth condition, so that for large X,

$$N_1\sim X^{1-a}e^{-bX}$$
$$N_2\sim X^{a-1}e^{(b+1)X}$$
$$W\sim(2b+1)e^X$$

in which a and b are defined by

$$\left.\begin{aligned} 2b&=(1+B)^{\frac{1}{2}}-1, \quad (\text{from } b^2+b-\tfrac{1}{4}B=0)\\ a&=\frac{2b-m}{2b+1}=1-(m+1)(1+B)^{-\frac{1}{2}}.\end{aligned}\right\} \tag{5}$$

For small X two independent solutions exist with behavior given by

$$\left.\begin{matrix} O(X) \\ 1-(m+1)X\ln X+O(X)\end{matrix}\right\}, \tag{6}$$

but the correspondence of N_1 and N_2 to either of these cannot be arbitrarily prescribed.

The solution for the transformed variable v^p can now be written as

$$\begin{aligned} i\sqrt{B}v^p(z)+v_z{}^0(0) &=\{h(\lambda)N_1(X)-N_2(X)\}\int_z^\infty \frac{N_1(z_0+\lambda)}{W(z_0+\lambda)}q(\lambda,z_0)dz_0\\ &\quad+N_1(X)\int_0^z \frac{q(\lambda,z_0)}{W(z_0+\lambda)} \times\{h(\lambda)N_1(z_0+\lambda)-N_2(z_0+\lambda)\}dz_0,\end{aligned}$$

where the following notation has been introduced:

$$X\equiv z+\lambda,$$

$$h(\lambda)\equiv\frac{\lambda N_2'(\lambda)-N_2(\lambda)}{\lambda N_1'(\lambda)-N_1(\lambda)},$$

$$q(\lambda,z)\equiv\frac{1}{X}(v_{zz}{}^0-v_z{}^0-\tfrac{1}{4}Bv^0)+\left(-\tfrac{1}{4}B+\frac{m+1}{X}\right)v_z{}^0(0).$$

The above expression for v^p is easily verified to satisfy all conditions of the problem. Note that account has been taken of the fact that N_2 does not satisfy the growth condition at infinity. A slight rearranging of the above solution gives

$$i\sqrt{B}v^p(z)+v_z{}^0(0) =N_1(X)\left\{h\int_0^\infty\frac{N_1q}{W}-\int_0^z\frac{N_2q}{W}\right\}-N_2(X)\int_0^\infty\frac{N_1q}{W}, \tag{7}$$

in which an abbreviated notation, which should be clear on comparison with the original form, is used.

For later use we note here that changing the second solution N_2 by the addition of a constant multiple of N_1 does not change the formalism, since $h(\lambda)N_1(X)-N_2(X)$ is invariant to such an adjustment. This means that we have the freedom of choosing N_2 to suit our convenience.

In order to find v from the above expression for v^p, the inversion formula

$$v(x,z,t)=\frac{e^{i\sqrt{B}x}}{2\pi i}\int_{A_1-i\infty}^{A_1+i\infty} e^{pt}v^p(z)dp$$

may be used where A_1 is a positive constant. Using the definition of λ, this may be transformed to

$$v(x,z,t)=\frac{e^{i\sqrt{B}(x-\bar{u}_0t)}}{2\pi i}\int_{-iA-\infty}^{-iA+\infty} e^{i\sqrt{B}t\lambda}i\sqrt{B}v^p(z)d\lambda.$$

The path of integration is now a line in the lower half-plane, parallel to the real axis.

Before proceeding, we consider the behavior of the integrand for large λ. Noting that

$$q(\lambda,z)=-\tfrac{1}{4}Bv_z{}^0(0)+O(\lambda^{-1}),$$

and

$$h(\lambda)=\frac{N_2'(\lambda)}{N_1'(\lambda)}[1+O(\lambda^{-1})]$$

$$=\frac{b+1}{-b}\lambda^{-2(1-a)}e^{(2b+1)\lambda}[1+O(\lambda^{-1})],$$

the terms in the expression for v^p may be evaluated for large λ as follows:

$$hN_1(X)\int_0^\infty \frac{N_1q}{W}=\frac{Be^{-bz}}{4b(2b+1)}v_z^0(0)+O(\lambda^{-1})$$

$$-N_1(X)\int_0^z \frac{N_2q}{W}=\frac{B(1-e^{-bz})}{4b(2b+1)}v_z^0(0)+O(\lambda^{-1})$$

$$-N_2(X)\int_z^\infty \frac{N_1q}{W}=\frac{B}{4(b+1)(2b+1)}v_z^0(0)+O(\lambda^{-1}).$$

Therefore, we have

$$i\sqrt{B}v^p(z)+v_z^0(0)$$

$$=\frac{B}{4(2b+1)}v_z^0(0)\left\{\frac{e^{-bz}}{b}+\frac{1-e^{-bz}}{b}+\frac{1}{b+1}\right\}+O(\lambda^{-1});$$

$$=v_z^0(0)+O(\lambda^{-1}),$$

since, by the definition of b, $4b(b+1)=B$. Thus we have finally

$$v^p(z)=O(\lambda^{-1}),$$

which illustrates that the expression for v^p is a well-behaved Laplace transform.

The integration contour in the inversion formula can now be deformed towards and into the upper half-plane. Due to the occurrence of the factor $(z_0+\lambda)^{-1}$ in the integrals defining v^p, the contour should not cross the negative λ-axis, and the inversion formula is ultimately constituted of contributions from the poles of v^p together with integration from minus infinity in the lower half-plane along the negative λ-axis, around the origin and to minus infinity in the upper half-plane. The solution to the initial value problem may thus be written in the form

$$v(x,z,t)e^{-i\sqrt{B}(x-\bar{u}_0t)}=\frac{1}{2\pi i}\int_{\rightarrow} e^{i\sqrt{B}t\lambda}v^p(z)d\lambda,$$

$$=\Sigma[\text{Residues of } e^{i\sqrt{B}t\lambda}v^p(z)]+\int_{\leftrightarrows\!\!\supset} e^{i\sqrt{B}t\lambda}v^p(z)d\lambda, \quad (8)$$

and is therefore given as the sum of normal modes and a continuous spectrum contribution.

4. The stability problem

For stability considerations, we turn first to the discrete spectrum. Evidently the only poles of v^p are situated at the poles of $h(\lambda)$. These occur precisely at all the zeros of its denominator, since none of these coincide with zeros of the numerator, as is easily proved by making use of the fact that N_1 and N_2 are independent solutions of the differential equation (4). Thus the poles of v^p are obtained by solving the equation

$$\lambda N_1'(\lambda)-N_1(\lambda)=0.$$

The residue of $v^p(z)$ at a root λ_1 is determined by the term containing $h(\lambda)$ and is therefore equal to $N_1(z+\lambda_1)$ times a proportionality factor which is independent of z. The normal modes occurring in the full solution for $v(x,z,t)$ are therefore proportional to

$$N_1(z+\lambda_1)e^{i\sqrt{B}[x+(\lambda_1-\bar{u}_0)t]}, \quad (9)$$

with

$$\lambda_1N_1'(\lambda_1)-N_1(\lambda_1)=0, \quad (10)$$

where complex values of λ_1 (with $\text{Im}\lambda_1<0$) evidently imply instability. The existence of this form of instability has been fully treated by Burger (1962).

It remains to examine the contour integral or continuous spectrum part of the solution.

Since the functions N_1 and N_2 are finite for all finite arguments and W does not vanish, the factor $(z_0+\lambda)^{-1}$ occurring in $q(z_0,\lambda)$ gives rise to at most a finite jump across the negative λ-axis in the value of the integrals defining $v^p(z)$, and thus to a contribution to $v(x,z,t)$ which decreases with time.

For stability considerations it is, therefore, sufficient to consider the contribution of the function $h(\lambda)$. Inasmuch as it does not have poles on the negative axis, (10) having no roots there (Burger, 1962), only the origin need be considered.

At $\lambda=0$, we note that the behavior of $h(\lambda)$ is determined by the behavior of the solutions N of Eq. (4) at the origin. This, as in (6), is either

$$N(\lambda)=O(\lambda),$$

or

$$N(\lambda)\propto 1-(m+1)\lambda\ln\lambda+O(\lambda),$$

corresponding, respectively, to

$$\lambda N'(\lambda)-N(\lambda)=O(\lambda^2),$$

and

$$\lambda N'(\lambda)-N(\lambda)=O(1).$$

In the general case where $N_1(\lambda)$ is finite at the origin, $N_2(\lambda)$ may be chosen to vanish there (without changing the solution for v, as has been remarked above), so that $h(\lambda)=O(\lambda^2)$. This yields a contribution to $v(x,z,t)$ which vanishes with time, and need, therefore, not be further considered.

In the special case where $N_1(\lambda)=O(\lambda)$, we have $N_2(\lambda)=O(1)$ and $h(\lambda)=O(\lambda^{-2})$. This, by virtue of the relation

$$\frac{1}{2\pi i}\oint\frac{e^{i\sqrt{B}t\lambda}d\lambda}{\lambda^2}=i\sqrt{B}t$$

(in which the path of integration circles the origin) yields a contribution to $v(x,z,t)$ proportional to

$$tN_1(z)e^{i\sqrt{B}(x-\bar{u}_0t)},$$

i.e., increasing linearly with t. *We have thus shown that*

$$N_1(\lambda)=O(\lambda)$$

at the origin leads to a form of instability associated with the continuous spectrum of the problem.

If the parameter a defined by (5) assumes the values zero or a negative integer $-n$, we have (Burger, 1962):

$$N_1(X)\propto Xe^{-\alpha X}L_n^1[(2\alpha+1)X],$$

in which

$$\alpha=\frac{b}{2b+1}=\tfrac{1}{2}[1-(1+B)^{-\frac{1}{2}}],$$

while $L_n^1(x)$ is the generalized Laguerre polynomial

$$L_n^1(x)=\sum_{r=0}^{n}\binom{n+1}{n-r}\frac{(-x)^r}{r!}.$$

Whereas these are the cases in which only real eigenvalues occur, i.e., where no normal mode leads to (exponential) instability, the above result shows that the vanishing of $N_1(\lambda)$ in the origin implies (algebraic) instability. These are, in fact, the only cases for which this holds.

We may remark here that for the very simple formulation of the problem obtained by formally putting $m+1=0$, the so-called Eady (1949) model, the purely exponential solution satisfying the growth condition at infinite height does not vanish at the origin, so that no continuous spectrum instability occurs in that case.

5. Pure normal modes

Finally, as a check on the formalism used, we will show that the special case of a pure normal mode, viewed as an initial value problem, does emerge from the analysis. In view of (3) and (9) we have to require

$$v^0(z)=N_1(z+\lambda_1)\equiv N_1(Y),$$

with $\lambda_1N_1'(\lambda_1)-N_1(\lambda_1)=0$ and $Y\equiv z+\lambda_1$.

Since v^0 now satisfies the differential equation (4), with X replaced by Y, the integrands in the expression for $v^p(z)$ can be written as

$$\frac{N(X)}{W(X)}q(\lambda,z)=-\frac{N(X)}{W(X)}\frac{m+1}{XY}N_1(Y)-N_1'(\lambda_1)\frac{N''(X)-N'(X)}{W(X)},$$

in which $N(X)$ stands for either $N_1(X)$ or $N_2(X)$.

The second of these two terms can be explicitly integrated by noting that $W'-W=0$, so that

$$\left(\frac{N'}{W}\right)'=\frac{N''-N'}{W}.$$

For the first term, we note that

$$\left\{\frac{N(X)N_1'(Y)-N'(X)N_1(Y)}{W(X)}\right\}'=\frac{N(X)[N_1''(Y)-N_1'(Y)]-N_1(Y)[N''(X)-N'(X)]}{W(X)},$$

$$=(m+1)\frac{N(X)N_1(Y)}{W(X)}\left(\frac{Y-X}{XY}\right).$$

Since $Y-X=\lambda_1-\lambda$, this yields the required explicit integration. We have, therefore,

$$\int\frac{Nq}{W}=-\frac{N(X)N_1'(Y)-N'(X)N_1(Y)}{(\lambda_1-\lambda)W(\lambda)}-N_1'(\lambda_1)\frac{N'(X)}{W(X)}.$$

Substituting limits of integration and making use of the relation $\lambda_1N_1'(\lambda_1)-N_1(\lambda_1)=0$, we get the following terms in the expression for v^p:

$$h\int_0^\infty\frac{N_1q}{W}=-\frac{N_1'(\lambda_1)}{(\lambda_1-\lambda)W(\lambda)}\{\lambda N_2'(\lambda)-N_2(\lambda)\},$$

$$\int_0^z\frac{N_2q}{W}=-\frac{N_2(X)N_1'(Y)-N_2'(X)N_1(Y)}{(\lambda_1-\lambda)W(X)}-N_1'(\lambda_1)\frac{N_2'(X)}{W(X)}-\frac{N_1'(\lambda_1)}{(\lambda_1-\lambda)W(\lambda)}\{\lambda N_2'(\lambda)-N_2(\lambda)\},$$

$$\int_z^\infty\frac{N_1q}{W}=\frac{N_1(X)N_1'(Y)-N_1'(X)N_1(Y)}{(\lambda_1-\lambda)W(X)}+N_1'(\lambda_1)\frac{N_1'(X)}{W(X)}.$$

Thus, from (7),

$$i\sqrt{B}v^p+N_1'(\lambda_1)$$

$$=\frac{N_1(X)N_2'(X)-N_1'(X)N_2(X)}{W(X)}\left\{-\frac{N_1(Y)}{\lambda_1-\lambda}+N_1'(\lambda_1)\right\},$$

and, therefore,

$$i\sqrt{B}v^p=\frac{N_1(Y)}{\lambda-\lambda_1}.$$

Evidently in this case the continuous spectrum falls away and the final solution is

$$v(x,z,t)e^{-i\sqrt{B}(x-u_0t)}=\text{Residue of } e^{i\sqrt{B}t\lambda}i\sqrt{B}v^p(z) \text{ at } \lambda=\lambda_1,$$

$$=e^{i\sqrt{B}\lambda_1 t}N_1(Y),$$

or

$$v(x,z,t)=e^{i\sqrt{B}(x-ct)}N_1(z+\lambda_1),$$

with $c=\bar{u}_0-\lambda_1$. This is precisely the normal mode (9) which was used in prescribing the initial value.

REFERENCES

Burger, A. P., 1958: Scale consideration of planetary motions of the atmosphere. *Tellus*, **10**, 195–205.

——, 1962: On the non-existence of critical wavelengths in a continuous baroclinic stability problem. *J. Atmos. Sci.*, **19**, 31–38.

Case, K. M., 1960: Stability of an idealized atmosphere I. *Phys. of Fluids*, **3**, 149–154.

Charney, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, **4**, 135–162.

Eady, E. T., 1949: Long waves and cyclone waves. *Tellus*, **1**, 33–52.

Kuo, H. L., 1952: Three-dimensional disturbances in a baroclinic zonal current. *J. Meteor.*, **9**, 260–278.

Miles, J. W., 1964: Baroclinic instability of the zonal wind, II. *J. Atmos. Sci.*, **21**, 500–506.