Radiative and Photochemical Processes in Mesospheric Dynamics: Part II, Vertical Propagation of Long Period Disturbances at the Equator

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(Manuscript received 29 October 1965; in revised form 10 January 1966)

ABSTRACT

This paper considers the vertical propagation of a long-period, small-amplitude perturbation in a medium in which radiative transfer and photochemistry play important roles. The perturbation and the basic field are assumed to be axially symmetric and symmetric about the equator; the basic wind field is geostrophic and the basic temperature field is in radiative equilibrium.

It is found that long-period perturbations can only propagate by virtue of the physical effects of radiative transfer and photochemistry. The computed wave propagates downwards and, for a period of 2.2 years, the phase speed is close to the observed speed of 1.5 km month^{-1} for the "26-month" equatorial oscillation. The observed relative phases of velocity and temperature fields, and the sharp attenuation of the oscillation below 20 to 25 km are also found in the model wave.

There are discrepancies between the model and the observed "26-month" oscillation, which are to be expected in view of the nonlinearity of the observed phenomenon. However, it appears that, for complex reasons, the observed wave may satisfy equations similar to those occurring in the linear theory.

1. Introduction

In this paper we shall investigate the vertical propagation of a small amplitude, axially symmetric disturbance, also symmetric about the equator, on a symmetric mean field. This strongly contrived example has been selected because its conditions resemble those in the observed "26-month" equatorial wave (Reed and Rogers, 1962). However, the model differs in at least one significant respect from the observed phenomenon. We assume the wave to be a small perturbation on a mean field, while the observed shear, on the contrary, appears to be larger than that of the mean field. Thus we have, *a priori*, no reason to anticipate results in detailed agreement with observations. Nevertheless, it turns out that our results resemble qualitatively and even quantitatively many features of the observed wave. The reasons for this will be discussed in the last section of this Part.

In this paper we will be content to specify the horizontal structure of a temperature perturbation at a particular level, and to investigate the behavior of the disturbance away from that level. The extremely interesting question of a driving mechanism for the "26-month" wave is discussed elsewhere (Lindzen, 1965).

2. Equations

The models for photochemical and radiative processes are developed in detail in Part I (Lindzen and Goody, 1965). Part I constitutes a necessary preliminary to the work in this and subsequent Parts. As a result it will be assumed that the reader is familiar with the results and notation employed there. The effects of photochemistry and radiative transfer on the temperature perturbation are described by Eq. (36) of Part I, i.e.,

$$\left\{\frac{\partial^2}{\partial t^2}+(a+B)\frac{\partial}{\partial t}+(aB+\eta C)\right\}\theta = -\left(\frac{\partial}{\partial t}+B\right)\left[v\frac{\partial \bar{T}}{\partial y}+w\left(\frac{\partial \bar{T}}{\partial z}+\frac{g}{c_p}\right)\right] -\eta\left(v\frac{\partial \bar{\varphi}}{\partial y}+w\frac{\partial \bar{\varphi}}{\partial z}\right), \quad (1)$$

where θ (rather than T') is the temperature perturbation, and v and w are the meridional velocity perturbation components (primes have been omitted). $\frac{\partial \theta}{\partial x}=0$ by virtue of our assumption of axial symmetry.

Our basic fields (indicated by overbars) are assumed to be in photochemical-radiative equilibrium, and if we restrict ourselves to the region below 33 km, η is approximately constant (see Section 3 of Part I). Then $\eta\bar{\varphi}=a\bar{T}-b$, and

$$\left(\frac{\partial \bar{\varphi}}{\partial y}, \frac{\partial \bar{\varphi}}{\partial z}\right)=\frac{a}{\eta}\left(\frac{\partial \bar{T}}{\partial y}, \frac{\partial \bar{T}}{\partial z}\right), \quad (2)$$

where $\bar{\varphi}$ is the basic ozone mixing ratio distribution.

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Eq. (1) then becomes

$$\left\{\frac{\partial^2}{\partial t^2}+(a+B)\frac{\partial}{\partial t}+(aB+\eta C)\right\}\Theta = -\left(\frac{\partial}{\partial t}+a+B\right)\left(v\frac{\partial \bar{T}}{\partial y}+w\frac{\partial \bar{T}}{\partial z}\right) -\left(\frac{\partial}{\partial t}+B\right)\frac{g}{c_p}w. \quad (3)$$

The hydrodynamic equations for the disturbance are obtained by taking the Navier-Stokes equations for the motion of a compressible gas on a rotating sphere, and assuming: (a) axial symmetry (i.e., independence of longitude); (b) symmetry about the equator for θ and u; (c) the vertical components of the earth's rotation $\left(\Omega \sin\varphi=\Omega \sin\frac{y}{a}\right.$, where Ω=rotation rate, φ=latitude in radians, y=distance from the equator, and a=radius of the earth$\left.\right)$ can be approximated by $\Omega\frac{y}{a}$; (d) the parallel components ($\Omega \cos\varphi$) can be approximated by Ω; (e) all other effects of the earth's curvature can be neglected; (f) the zonal component of the disturbance wind to be geostrophic; (g) vertical accelerations are negligible; (h) the time derivative of density in the equation of continuity is negligible; (i) diffusion terms can be omitted; and (j) the motion consists of small perturbations about a basic field consisting of a steady zonal wind in geostrophic balance with a photochemical-radiative equilibrium temperature field.

Assumptions (a) and (b) correspond to the observed symmetry of the equatorial wave (Reed, 1964a, b). Item (i) is based on the uncertain estimates of Lettau (1951) for eddy coefficients, which indicate eddy effects considerably smaller than effects retained. However, Lettau considers only vertical eddy transports. Information from tungsten diffusion suggests that horizontal transports may be more important (Reed, 1964a). Given our symmetry assumptions, together with the restriction that we consider only waves with long periods ($\gtrsim$6 months), assumptions (c)–(h) may be justified by an order of magnitude analysis. Assumption (j) is made to reduce the problem to tractable proportions and it will be further discussed later in this paper.

As a result of the above assumptions and approximations we are left with the following equations [in addition to Eq. (3)];

$$\frac{\partial u}{\partial t}+v\left(\frac{\partial \bar{u}}{\partial y}-2\Omega\frac{y}{a}\right)+w\left(\frac{\partial \bar{u}}{\partial z}+2\Omega\right)=0, \quad (4)$$

$$2\Omega\frac{y}{a}u=-\frac{1}{\bar{\rho}}\frac{\partial p}{\partial y}, \quad (5)$$

$$\frac{\partial p}{\partial z}=-g\rho+2\Omega\bar{\rho}u, \quad (6)$$

$$\rho=-\frac{\bar{\rho}}{\bar{T}}\theta+\frac{\bar{\rho}}{\bar{p}}p, \quad (7)$$

and

$$\frac{\partial y}{\partial y}+\frac{\partial w}{\partial z}+\frac{1}{\bar{\rho}}\left(v\frac{\partial \bar{\rho}}{\partial y}+w\frac{\partial \bar{\rho}}{\partial z}\right)=0. \quad (8)$$

We will introduce some further approximations, which are neither necessary for our procedure nor significant in their effects, but which simplify the manipulations appreciably. We will approximate $\partial\bar{u}/\partial y$ by $\lambda_1 y$, $\partial\bar{T}/\partial y$ by $\lambda_2 y$, and take λ_1, λ_2, $\partial\bar{T}/\partial z$, B, and C to be independent of y. We will also neglect the terms $2\Omega\bar{\rho}u$ in Eq. (6). (The effect of this will be to predict an infinite length scale for the dissipation of infinitely long period waves, otherwise the length scale would be 400 km; for a region less than 40 km deep, this is of no consequence.) Details on this matter may be found in Lindzen (1964). Eqs. (5), (6) and (7) may be combined to yield

$$\left(\frac{\partial}{\partial z}+\frac{1}{H}\right)(\bar{\rho}u)=-\frac{ag}{2\Omega}\frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\bar{\rho}}{\bar{T}}\theta\right), \quad (9)$$

where

$$H=\frac{R\bar{T}}{g},$$

and (9) may be approximated by

$$\left(\frac{\partial}{\partial z}+\frac{1}{H}\right)(\bar{\rho}u)=-\frac{ag}{2\Omega\bar{T}}\frac{1}{y}\frac{\partial}{\partial y}(\bar{\rho}\theta). \quad (10)$$

It now proves convenient to adopt the following variables,

$$\left.\begin{aligned} U&=\bar{\rho}u,\\ V&=\bar{\rho}v,\\ W&=\bar{\rho}w,\\ \Theta&=\bar{\rho}\theta. \end{aligned}\right\}. \quad (11)$$

We will also assume for all disturbance fields a time dependence of the form $e^{i\omega t}$, yielding;

$$\Theta=-L(z)W-M(z)yV, \quad (12)$$

$$i\omega U+RyV+SW=0, \quad (13)$$

$$\left(\frac{\partial}{\partial z}+\frac{1}{H}\right)U=-\frac{ag}{2\Omega}\frac{1}{\bar{T}}\frac{1}{y}\frac{\partial\Theta}{\partial y}, \quad (14)$$

and

$$\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0, \quad (15)$$

where

$$L=\frac{\left\{(i\omega+a+B)\frac{\partial\bar{T}}{\partial z}+(i\omega+B)\frac{g}{c_p}\right\}}{\{(i\omega+B)(i\omega+a)+\eta C\}}, \quad (16)$$

$$M=\frac{\lambda_2(i\omega+a+B)}{\{(i\omega+B)(i\omega+a)+\eta C\}}, \quad (17)$$

$$R=\lambda_1-\frac{2\Omega}{a}, \quad (18)$$

and

$$S=\frac{\partial \bar{u}}{\partial z}+2\Omega. \tag{19}$$

3. Expansion about the equator

The complete solution of Eqs. (12)–(15) remains a formidable task, and this paper will restrict itself to investigating the behavior of the solution in the neighborhood of the equator. As a first step, we will expand U, V, W and Θ in power series in y. Using our assumed symmetry properties we obtain

$$\begin{aligned} U&=U_0+U_1y^2+U_2y^4+\cdots,\\ V&=V_0y+V_1y^3+V_2y^5+\cdots,\\ W&=W_0+W_1y^2+W_2y^4+\cdots,\\ \Theta&=\Theta_0(1-\alpha_1y^2-\alpha_2y^4\cdots),\end{aligned} \tag{20}$$

where U_i, V_i, W_i, Θ_0 and α_i are complex functions of z. We may now substitute the above expansions into Eqs. (12)–(15) and order terms according to powers of y. To zeroth order in y we obtain

$$\Theta_0=-LW_0, \tag{21}$$

$$i\omega U_0+SW_0=0, \tag{22}$$

$$\left(\frac{\partial}{\partial z}+\frac{1}{H}\right)=\frac{ag}{\Omega\bar{T}}\alpha_1\Theta_0, \tag{23}$$

and

$$V_0=-\frac{\partial W_0}{\partial z}. \tag{24}$$

Eqs. (21)–(24) are four equations in five unknowns. Let us for the moment, treat $\alpha_1(z)$ as though it were known, and solve for the remaining fields in terms of α_1. We obtain

$$\frac{i\omega L(z)}{S\Theta_0}\frac{d}{dz}\left(\frac{S\Theta_0}{i\omega L(z)}\right)=\alpha_1(z)\nu(z)-\frac{1}{H}, \tag{25}$$

or

$$\Theta_0=\Theta_0|_{z=z_0}\frac{\bar{T}}{\bar{T}(z_0)}\left(\frac{\nu}{\nu(z_0)}\right)\times\exp\left\{\int_{z_0}^{z}\left(-\frac{1}{H(z')}+\alpha_1(z')\nu(z')\right)dz'\right\}, \tag{26}$$

where

$$\nu=\frac{ag}{\Omega\bar{T}}\frac{i\omega L}{S}, \tag{27}$$

and z_0 is some level at which we will impose a temperature boundary condition. Also, for future reference,

$$V_0=\frac{1}{L}\left\{-\nu(z)\alpha_1(z)+\frac{1}{H}+\frac{1}{S}\frac{dS}{dz}\right\}\Theta_0. \tag{28}$$

To first order in y^2 we get

$$\Theta_0\alpha_1=LW_1+MV_0, \tag{29}$$

$$i\omega U_1+RV_0+SW_1=0, \tag{30}$$

$$\left(\frac{\partial}{\partial z}+\frac{1}{H}\right)U_1=2\frac{ag}{\Omega\bar{T}}\alpha_2\Theta_0, \tag{31}$$

and

$$3V_1+\frac{\partial W_1}{\partial z}=0. \tag{32}$$

Solving for α_2 in terms of α_1 we obtain, using (25), (27) and (28),

$$\begin{aligned}2\nu\alpha_2=-\frac{1}{H}&\left\{\left(\frac{R}{S}-\frac{M}{L}\right)\left(\frac{1}{S}\frac{dS}{dz}+\frac{1}{H}-\nu\alpha_1\right)+\alpha_1\right\}\\ &-\left\{\left(\frac{1}{S}\frac{dR}{dz}-\frac{R}{S^2}\frac{dS}{dz}-\frac{1}{L}\frac{dM}{dz}+\frac{M}{L^2}\frac{dL}{dz}\right)\left(\frac{1}{S}\frac{dS}{dz}+\frac{1}{H}-\nu\alpha_1\right)\right.\\ &+\left(\frac{R}{S}-\frac{M}{L}\right)\left(-\frac{1}{S^2}\left(\frac{dS}{dz}\right)^2+\frac{1}{S}\frac{d^2S}{dz^2}-\frac{1}{H^2}\frac{dH}{dz}-\alpha_1\frac{d\nu}{dz}-\nu\frac{d\alpha_1}{dz}\right)\\ &\left.+\frac{d\alpha_1}{dz}+\left(\frac{R}{S}-\frac{M}{L}\right)\left(\frac{1}{S}\frac{dS}{dz}+\frac{1}{H}-\nu\alpha_1\right)\left(-\frac{1}{H}+\alpha_1\nu\right)+\alpha_1\left(-\frac{1}{H}+\alpha_1\nu\right)\right\}.\end{aligned} \tag{33}$$

Now, L, ν, R, S, H and M are all known function of z. Hence

$$\alpha_2=\frac{1}{2\nu}F_1\left(\alpha_1,\frac{d\alpha_1}{dz},z\right). \tag{34}$$

In a similar manner it can be shown that our solution for α_n will be of the form

$$\alpha_n=\frac{1}{n\nu}F_{n-1}\left(\alpha_1,\frac{d\alpha_1}{dz},\cdots,\frac{d^{n-1}\alpha_1}{dz^{n-1}},z\right), \tag{35}$$

and our solution for Θ will be

$$\Theta=\Theta_0\left(1-\alpha_1(z)y^2-\frac{1}{2\nu}F_1\left(\alpha_1,\frac{d\alpha_1}{dz},z\right)y^4-\cdots\right.$$
$$\left.-\frac{1}{n\nu}F_{n-1}\left(\alpha_1,\frac{d\alpha_1}{dz},\cdots,\frac{d^{n-1}\alpha_1}{dz^{n-1}},z\right)y^{2n}-\cdots\right. \quad (36)$$

Now let us introduce as a boundary condition

$$\Theta=\Theta(1-\beta_1y^2-\beta_2y^4-\cdots), \quad (37)$$

at $z=z_0$. Applying this boundary bondition to (36), we obtain

$$\Theta_0(z_0)=\Theta,$$
$$\alpha_1(z_0)=\beta_1,$$
$$\frac{1}{2\nu}F_1\left[\alpha_1(z_0),\left.\frac{d\alpha_1}{dz}\right|_{z=z_0},z_0\right]=\beta_2, \quad (38)$$
$$\frac{1}{n\nu}F_{n-1}\left[\alpha_1(z_0),\left.\frac{d\alpha_1}{dz}\right|_{z=z_0},\cdots,\left.\frac{d^{n-1}\alpha_1}{dz^{n-1}}\right|_{z=z_0},z_0\right]=\beta_n.$$

We can solve Eqs. (38) for

$$\alpha_1(z_0),\left.\frac{d\alpha_1}{dz}\right|_{z=z_0},\cdots,\left.\frac{d^n\alpha_1}{dz^n}\right|_{z=z_0},\cdots,$$

which in turn give us $\alpha_1(z)$, expressed as a Taylor series about $z=z_0$, i.e.,

$$\alpha_1(z)=\alpha_1(z_0)+\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}(z-z_0)+\frac{1}{2}\left.\frac{d^2\alpha_1}{dz^2}\right|_{z=z_0}(z-z_0)^2+\cdots. \quad (39)$$

Thus we see that the solution at the equator ($y=0$) depends on our knowing the complete horizontal structure at a single level.

4. Solution at the equator

The dependence of our solution at the equator on the complete horizontal structure at a particular level (for which information is not usually available) poses, in general, a substantial obstacle to the use of Eq. (26). However, if we can assume α_1 not to be a function of z, the problem is greatly simplified.

α_1 is a measure of the horizontal curvature of the temperature disturbance at the equator and $\alpha_1^{-\frac{1}{2}}$ is a measure of the horizontal scale of the disturbance. Conditions may well exist under which the disturbance preserves its horizontal form (at least to the extent of preserving its curvature at the equator) as it propagates vertically. Without wishing to overestimate the significance of the observation, it should be pointed out that the latest data of the "26-month" wave (Reed, 1964a) are consistent with preservation of form at the equator.

In what follows we will assert that the distribution of $\Theta(y^2)$ at $z=z_0$ is such that $\alpha_1\neq\alpha_1(z)$. Since this is an ad hoc assumption, a brief analysis of the meaning of this restriction in terms of the relation between β_2 and $\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}$ is instructive.

For the sake of the following discussion we will presume properties of $\nu(\omega,z)$ and R, S and L, which will not be derived until Section 5.

We can show that for equilibrium basic fields,

$$\left|\frac{R}{S}\nu\right|\gg\left|M\frac{\omega ag}{\Omega S\bar{T}}\right| \quad\text{and}\quad \left|\frac{R}{S}\frac{d\nu}{dz}\right|\gg\left|\frac{dM}{dz}\frac{\omega ag}{\Omega S\bar{T}}\right|;$$

also $\frac{2\Omega}{a}\gg|\lambda_1|$ and hence $R=-\frac{2\Omega}{a}$. Finally, it will suffice for this limited investigation to treat R, S, H and $1/\bar{T}S$ as though they are independent[2] of z. Then (33) becomes

$$2\nu\alpha_2\cong\left(\frac{R}{S}\nu-1\right)\left(\frac{d\alpha_1}{dz}+\alpha_1^2\nu\right)+\frac{R}{S}\alpha_1\left(\frac{d\nu}{dz}-\frac{\nu}{H}\right), \quad (40)$$

or, in view of (38),

$$\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}\cong\nu(z_0)\beta_1^2\left\{\frac{\frac{R}{S}\frac{1}{\beta_1}\left(\frac{1}{\nu}\frac{d\nu}{dz}-\frac{1}{H}\right)-\frac{2\beta_2}{\beta_1^2}}{1-\frac{R}{S}\nu}-1\right\}\Bigg|_{z=z_0}. \quad (41)$$

It should be noted that had we neglected horizontal advection of momentum, then $R=0$, and the condition from (41) for $\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}=0$ becomes $\beta_2=-\frac{1}{2}\beta_1^2$. Since $\bar{\Theta}(1-\beta_1y^2+\frac{1}{2}\beta_1^2y^4)$ corresponds to the first three terms in the expansion of $\bar{\Theta}e^{-\beta_1y^2}$, solutions of this form are solutions of Eqs. (3)–(8) when horizontal advections are neglected. This neglect, however, is not justified. When $R\neq0$ the interpretation of (41) requires some further knowledge of $\nu(\omega,z)$. For ω corresponding to $\tau(=2\pi/\omega)=2$ years, there will be a height, between 20 and 25 km—depending upon the choice of reaction rate parameters (see Section 4, Part I; for our particular choice, 25 km is the height)—above which $\nu(z)$ is almost purely positive imaginary, corresponding to the fact that our model wave only propagates downward. Below this level, ν has a significant real part, resulting in severe attenuation (for details of the interpretation of ν in terms of propagation properties see Section 5). For

[2] One may choose to look at this as a model assumption wherein we assume (i) our basic velocity field has a constant vertical shear, and (ii) the variations of our basic temperature field with respect to position are small compared to $\bar{T}$.

$\tau=2$ years and z_0 in the neighborhood of 30 km, $\nu\sim 10^6 i$ km, $\frac{1}{\nu}\frac{d\nu}{dz}\sim -0.25$ km^{-1}, $H\sim 8$ km, $R=-0.25\times 10^{-7}$ km^{-1} sec^{-1} and $S\sim 2\times 10^{-3}$ sec^{-1}. Eq. (41) thus becomes

$$\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}\sim 10^6 i\beta_1^2\frac{\left(\dfrac{4.6\times 10^{-6}\text{ km}^{-2}}{\beta_1}-2\dfrac{\beta_2}{\beta_1^2}\right)}{(1+12.5i)}-1. \tag{42}$$

The condition that $\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}=0$ now becomes

$$\beta_2\sim -\tfrac{1}{2}\beta_1^2\left\{\left(1-\frac{4.6\times 10^{-6}\text{ km}^{-2}}{\beta_1}\right)+12.5i\right\}, \tag{43}$$

a considerable departure from the condition obtained when $R=0$. We can now show that (43) implies that the disturbance which leads to $\left.\frac{d\alpha_1}{dz}\right|_{z=z_0}=0$ must be such that its phase at latitudes away from the equator precedes its phase at the equator. Near the equator the phase lead in radians is approximately given by

$$\Phi\sim 6\beta_1^2 y^4\sim 6\beta_1^2 y^4\left(\frac{\tau}{2\pi}\right)\text{ months.} \tag{44}$$

For a 2 year wave, $\Phi\sim 1.5$ month at $y=1000$ km for $\beta_1=(2000\text{ km})^{-2}$. A phase lead of this magnitude has, in fact, been noted in the observed "26-month" wave (Angell and Korshover, 1962; Reed and Rogers, 1962). This is both suggestive and encouraging, but it should be pointed out that the observations on this matter are somewhat uncertain (Reed, 1964b).

5. Dispersion relation for $\alpha_1\neq\alpha_1(z)$

We will now restrict our discussion to those fields for which $\alpha_1\neq\alpha_1(z)$. The considerations of Sections 3 and 4 were framed in terms of the Θ field because the boundary conditions would most probably be given in terms of a temperature field. The dispersion properties of our system are, however, most clearly shown in terms of the U field. We find from Eqs. (21), (22) and (25)

$$\frac{1}{U_0}\frac{dU_0}{dz}=\alpha_1\nu(z)-\frac{1}{H}, \tag{45}$$

or, for constant α_1,

$$U_0=U_0|_{z=z_0}|\exp\left\{-\int_{z_0}^{z}\frac{dz'}{H(z')}+\alpha_1\int_{z_0}^{z}\nu(\omega,z')dz'\right\}.$$

Also, since $U_0=\bar{\rho}u_0$,

$$\bar{\rho}u_0=\bar{\rho}(z_0)u_0(z_0)\exp\left\{-\int_{z_0}^{z}\frac{dz'}{H(z')}+\alpha_1\int_{z_0}^{z}\nu(\omega,z')dz'\right\}.$$

Furthermore, over vertical distances on the order of 10 km

$$\bar{\rho}\cong\bar{\rho}(z_0)\exp\left\{-\int_{z_0}^{z}\frac{dz'}{H}\right\},$$

and

$$u_0\cong u_0(z_0)\exp\left\{\alpha_1\int_{z_0}^{z}\nu(\omega,z')dz'\right\}. \tag{46}$$

u_0 is unique in that its propagation (or z-dependence) is entirely determined by $\nu(\omega,z)$. Let $\mu(\omega,z)=\alpha_1\nu(\omega,z)$. For $|(z-z_0)|$ small

$$U_0=U_0|_{z=z_0}|\exp\{\mu(z_0)(z-z_0)\}.$$

We see that if μ_i and μ_r are the imaginary and real components of μ, then $2\pi/\mu_i$ may be interpreted as a local wavelength (μ_i/ω is a phase lag) and μ_r^{-1} as a scale length for decay.

From Eqs. (16), (19) and (27)

$$\mu=\alpha_1\frac{ag}{\Omega S\bar{T}}(i\omega)\frac{\left\{(i\omega+a+B)\dfrac{\partial\bar{T}}{\partial z}+(i\omega+B)\dfrac{g}{c_p}\right\}}{\{(i\omega+B)(i\omega+a)+\eta C\}}, \tag{47}$$

or

$$\mu_r=\frac{\alpha_1 ag}{\Omega S\bar{T}}\{(aB+\eta C-\omega^2)^2+\omega^2(a+B)^2\}^{-1}\times(\omega^2)\left\{-\sigma(aB+\eta C-\omega^2)+(a+B)\times\left[(a+B)\sigma-a\frac{g}{c_p}\right]\right\}, \tag{48}$$

and

$$\frac{\mu_i}{\omega}=\frac{\alpha_1 ag}{\Omega S\bar{T}}\{(aB+\eta C-\omega^2)^2+\omega^2(a+B)^2\}^{-1}\times\left\{(aB+\eta C)\left[(a+B)\sigma-a\frac{g}{c_p}\right]+\omega^2 a\frac{g}{c_p}\right\}, \tag{49}$$

where

$$\sigma\equiv\frac{\partial\bar{T}}{\partial z}+\frac{g}{c_p}.$$

We will initially investigate the properties of μ_i/ω. The first feature to be noted is that

$$\text{Sgn}(\mu_i)=\text{Sgn}\left(\frac{\omega}{S}\right). \tag{50}$$

This implies that if $S>0$, the model wave can only propagate downward, and if $S<0$ it can only propagate upward. Our basic fields are a temperature field in radiative-photochemical equilibrium, and a zonal velocity field in geostrophic balance with this tem-

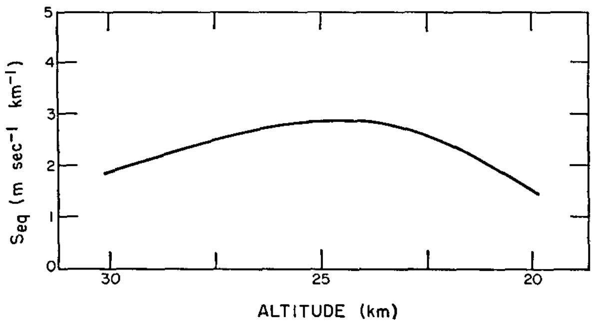

FIG. 1. The quantity $S\left(=\frac{\partial\bar{u}}{\partial z}+2\Omega\right)$ as a function of altitude. $\bar{u}$ is taken to be in geostrophic balance with a radiative-photochemical equilibrium temperature field symmetric about the equator.

perature field. Given our symmetry assumptions, this implies that

$$\frac{\partial\bar{u}}{\partial z}\simeq-\frac{ag}{2\Omega\bar{T}}\frac{\partial^2\bar{T}}{\partial y^2}. \tag{51}$$

Radiative-photochemical equilibrium theory requires that above 20 km there be a slight temperature maximum at the equator $[\{\bar{T}(y=0)-\bar{T}(y=1000\text{ km})\}\sim 1\text{K}]$. In consequence of this $\frac{\partial^2\bar{T}}{\partial y^2}<0$, and $\frac{\partial\bar{u}}{\partial z}$ (and hence S)>0. Fig. 1 shows S as a function of altitude for our basic field. The use of this S leads to a downward propagating wave whose properties we will describe shortly.

Before continuing, it should be remarked that there exist important differences between our basic state and the observed mean fields. In particular, according to the latest climatology of the equatorial stratosphere (Reed, 1964b), there is almost no horizontal variation in the mean temperature above 25 km, while below this level there is a weak temperature minimum at the equator. Thus the observed mean value of S is slightly negative—not positive. This matter is important only in connection with the value of S, which only appears in (13), the equation for the zonal momentum balance. We will return to this point in Section 7.

Continuing with our description of μ_i/ω, note that when $\omega\ll a$, B, the phase lag becomes independent of ω, i.e.,

$$\frac{\mu_i}{\omega}\xrightarrow[\omega\to 0]{}\frac{\alpha_1 ag}{\Omega S\bar{T}}\frac{\left\{(a+B)\sigma-a\frac{g}{c_p}\right\}}{aB+\eta C}. \tag{52}$$

The effect of omitting photochemistry and radiative transfer may be seen by taking the limit as $\omega\rightarrow\infty$, i.e.,

$$\frac{\mu_i}{\omega}\xrightarrow[\omega\to\infty]{}\frac{\alpha_1 ag}{\Omega S\bar{T}}a\frac{g}{c_p}\frac{1}{\omega^2}\sim 0. \tag{53}$$

The transition between these situations is shown in Fig. 2 where we show μ_i/ω as a function of $\tau\left(=\frac{2\pi}{\omega}\right)$ at $z=27.5$ km and for $\alpha_1=(2000\text{ km})^{-1}$. For $\tau=2$ years, $\mu_i/\omega=1.5$ months km^{-1} which is the estimated phase lag for the observed "26-month" wave at the equator. μ_i/ω also varies with altitude. In Fig. 3 we have plotted the phase $\Phi=-\int_{31.25\text{ km}}^{z}\mu_i(\omega,z')dz'$ for $S=S_{\text{equilibrium}}$ and also for $S=2$ m sec^{-1} km^{-1}, $S=5$ m sec^{-1} km^{-1}, $\mu_i/\omega=1.5$ month km^{-1}, and $\mu_i/\omega=1$ month km^{-1}. All the curves are for $\tau=2$ years and $\alpha_1=(2000\text{ km})^{-1}$. The observed "26-month" wave has a phase in the neighborhood of those for $\mu_i/\omega=1.5$ month km^{-1} and $\mu_i/\omega=1$ month km^{-1}. The phase for $S=S_{\text{equilibrium}}$ is in this neighborhood for $z>26$ km, but there is some discrepancy at lower altitudes.

No special importance ought to be attached to the level $z=26$ km: it results from our choice of reaction rate parameters (see Section 4 Part I). The experimental determinations of these parameters vary a great deal (Schiff, 1964), and had we chosen differently from among them we could have obtained good agreement to as low as 22 or 23 km.

We now come to the properties of μ_r, which determines the attenuation of the model wave. From Eq. (48) we see that there is no *a priori* statement which

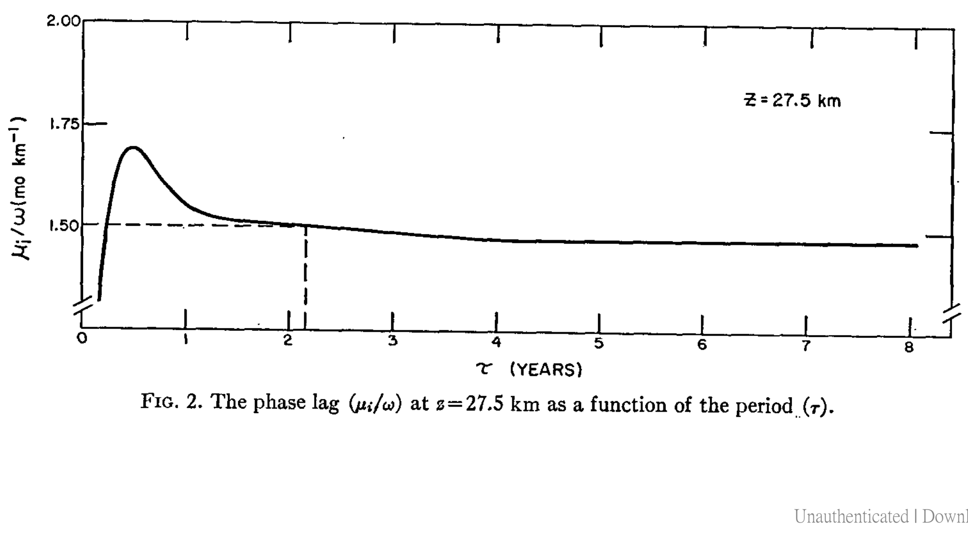

FIG. 2. The phase lag (μ_i/ω) at $z=27.5$ km as a function of the period (τ).

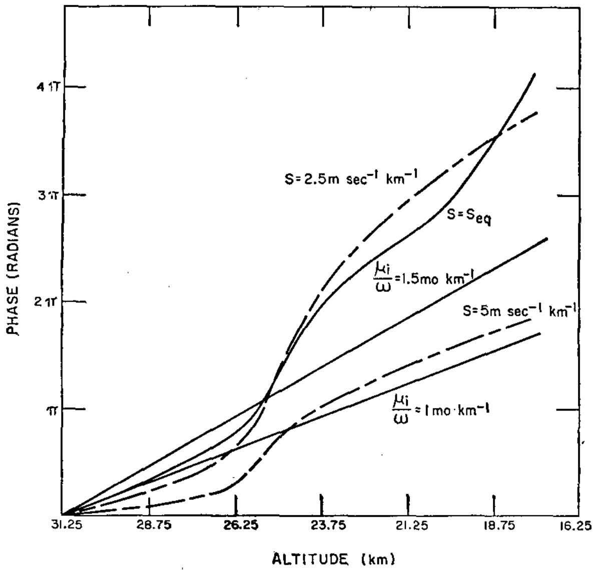

FIG. 3. The phase of a downward propagating wave as a function of z. τ is taken to be 2 years. Shown are the curves for $S=S_{\text{equilibrium}}$ (see Fig. 1) and for the arbitrary cases $S=2.5$ m sec^{-1} km^{-1} and $S=5$ m sec^{-1} km^{-1}. Also shown are the curves for the idealized situations where $\mu_i/\omega=1.5$ month km^{-1} and $\mu_i/\omega=1$ month km^{-1}. The phase of the observed "26-month" wave appears to lie in the neighborhood of these two curves. We have assumed zero phase at 31.25 km for all cases.

can be made concerning the relation of $\mathrm{Sgn}(\mu_r)$ to $\mathrm{Sgn}\left(\frac{\omega}{S}\right)$, and there exist choices of B, ηC, and ω for which $\mathrm{Sgn}(\mu_r)=-\mathrm{Sgn}\left(\frac{\omega}{S}\right)$, i.e., for which the wave will grow as it propagates. However, for values of B and ηC appropriate to the mesosphere, $\mathrm{Sgn}(\mu_i)=\mathrm{Sgn}\left(\frac{\omega}{S}\right)$ for all ω and the model wave is always attenuated as it propagates. This attenuation is small if $\omega \ll a$, B. Thus

$$\mu_r \xrightarrow[\omega\to 0]{} \frac{\alpha_1 a g}{\Omega S\bar{T}}\{aB+\eta C\}^{-2}\left\{-\sigma(aB+\eta C)+(a+B)\times\left[(a+B)\sigma-a\frac{g}{c_p}\right]\right\}\omega^2 \sim 0. \quad (54)$$

It is important to note that, in the absence of photochemistry and radiative transfer, our wave would hardly propagate at all. This may be seen by considering the limit as $\omega\to\infty$, i.e.,

$$\mu_r \xrightarrow[\omega\to\infty]{} \frac{\alpha_1 a g\sigma}{\Omega S\bar{T}} \sim 10\ \text{m}^{-1}, \quad (55)$$

and the wave would be largely extinguished after propagating a mere 10 cm.

Eqs. (54) and (55) show that our system contains a filtering property which favors the propagation of long period waves. As mentioned earlier, μ_r^{-1} may be interpreted as a local penetration depth. In Fig. 4 we see μ_r^{-1} as a function of τ at $z=25$ km. The filtering property is clearly displayed. In particular, for our choice of photochemical parameters, it would appear that the shortest period an oscillation may have and still be capable of penetrating much below 25 km is approximately 2 years. This filtering behavior depends on the relative magnitudes of a, B and ω. Since B has a strong altitude dependence we naturally expect these properties to vary from level to level. In Fig. 5 we have a plot of the normalized amplitude, $\exp\{\int_{z_0}^{z}\mu_r(\omega,z')dz'\}$, for $z_0=31.25$ km and $\tau=2$ years. We see that a 2 year oscillation undergoes very little

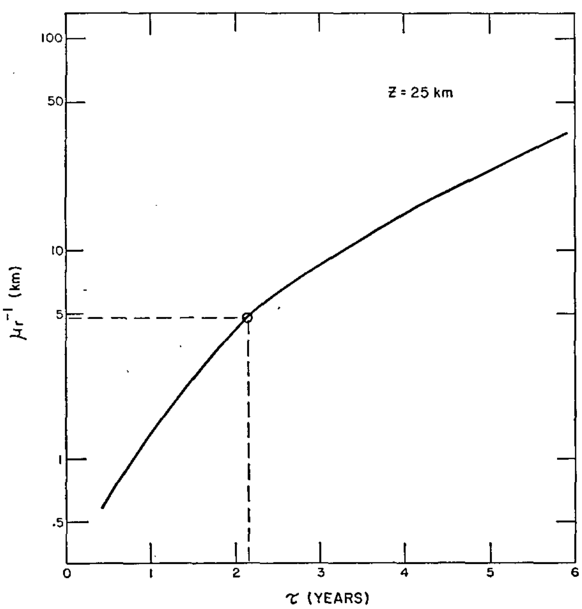

FIG. 4. Penetration depth (μ_r^{-1}) as a function of period (τ) at $z=25$ km.

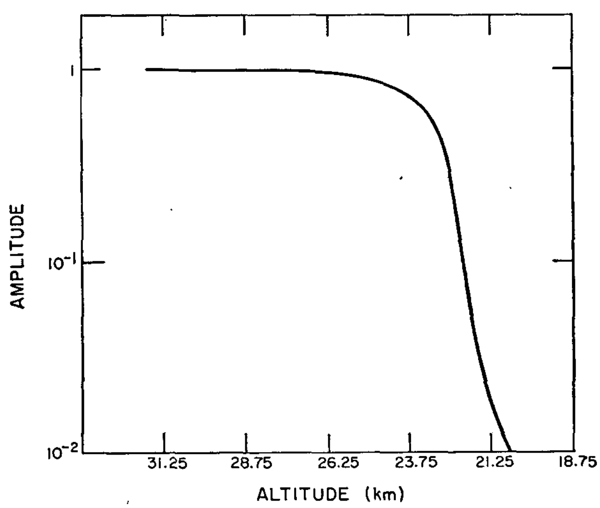

FIG. 5. Amplitude of u_0 as a function of altitude for $\tau=2$ years. Amplitude is normalized to unity at 31.25 km.

attenuation above 25 km but is sharply attenuated below this level.

The reason for this attenuation is as follows. In order for the wave to propagate without attenuation, momentum must be advected [as given by Eq. (4)] and heat must be advected [as given by Eq. (3)] in such a manner as to maintain the thermal wind relation between the temperature and the zonal velocity [see Eq. (10)]. When $\omega < a, B$ then θ is approximately in phase with $-w$ at the equator and the advection maintains geostrophy. This is the case at higher altitudes; at lower altitudes where B (photochemical relaxation rate, see Section 5, Part I) becomes quite small, this phase relation between θ and $-w$ is no longer maintained with the consequence that the wave is attenuated. The general question of the phase relation between temperature and advection is dealt with in Section 5 of Part I. Again, it is important not to attach too much significance to the particular level $z=25$ km. Other choices of reaction rate could have led to a lower level for the onset of severe attenuation. However, as for the observed wave, there does exist a level, between 20 and 25 km, below which the wave is rapidly attenuated.

A consequence of the filtering property of the dispersion relation is that one might expect to find a prevalence of higher frequency components at higher altitudes. It is therefore interesting to note that Reed (personal communication) has found a strong 6-month oscillation in the 40–50 km region.

6. Relative behavior of fields

It is a straightforward matter, using Eqs. (11), (21), (22), (23), (24), (27) and (45), to show that

$$u \sim \frac{1}{\bar{\rho}} U_0 + O(y^2), \tag{56}$$

$$w \sim -\frac{i\omega}{\bar{\rho} S} U_0 + O(y^2), \tag{57}$$

$$v \sim \frac{i\omega}{\bar{\rho} S}\left\{\mu - \frac{1}{S}\frac{dS}{dz} - \frac{1}{H}\right\} U_0 y + O(y^3), \tag{58}$$

and

$$\theta \sim \frac{\mu \Omega \bar{T}}{a\alpha_1 g \bar{\rho}} U_0 + O(y^2), \tag{59}$$

where the common order of magnitude notation has been employed. Also, using Eq. (34b) from Part I, we get

$$\bar{\rho}\varphi' \sim \frac{\Omega}{\eta a \alpha_1}\left\{ i\omega\left[\frac{\bar{T}\mu}{g} - \sigma\frac{a\alpha_1}{\Omega S}\right] + a\frac{\bar{T}}{g}\mu\right\} U_0 + O(y^2), \tag{60}$$

for the ozone fluctuations. Since φ' is the ozone mixing ratio, $\bar{\rho}\varphi'$ is proportional to the perturbation of the ozone density.

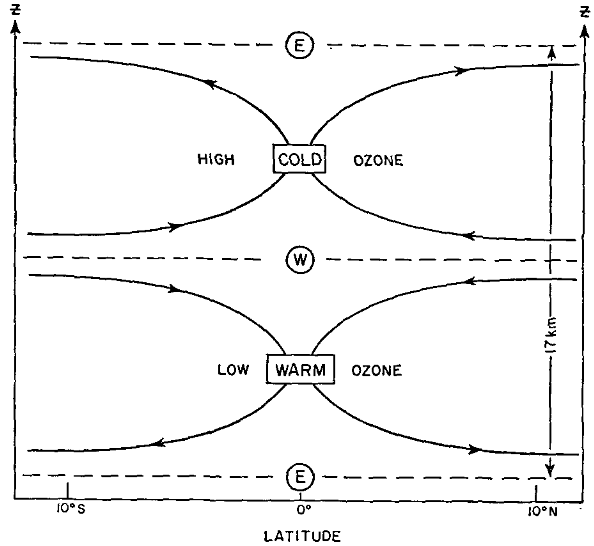

FIG. 6. Meridional cross section of perturbation fields above 25 km. 'E' refers to easterlies, and 'W' refers to westerlies. Arrowed curves refer to meridional circulation. The situation pictured is periodic in z and propagates downward at approximately 1 km month^{-1}.

Eq. (57) shows that w is always 90° behind u in phase. The phase relations between v, θ and φ', and u depend on μ, a complex function of z. At the higher altitudes, where there is a little attenuation, $\mu_i \gg \mu_r$. Also, $a \gg \omega$. If we also assume $\mu_i \gg \frac{1}{S}\frac{dS}{dz}, \frac{1}{H}$, then Eqs. (58)–(60) become

$$v \sim -\frac{\omega\mu_i}{\bar{\rho} S} U_0 y + O(y^3), \tag{61}$$

$$\theta \sim i\frac{\mu_i \Omega \bar{T}}{a\alpha_1 g\bar{\rho}} U_0 + O(y^2), \tag{62}$$

$$\bar{\rho}\varphi' \sim i\frac{\Omega}{\eta a\alpha_1}\left[-\omega\frac{\sigma a\alpha_1}{\Omega S} + a\frac{\bar{T}}{g}\mu_i\right] U_0 + O(y^2). \tag{63}$$

Under these conditions v is in phase or 180° out of phase with u depending on whether y is negative or positive and θ leads u by 90°. Using previously given values, it may be confirmed that (63) implies that φ' will be in phase or 180° out of phase with θ depending on whether S is greater than or less than 10^{-2} sec^{-1}. In general, it appears that S should be less than 10^{-2} sec^{-1}, and, as a result φ' is out of phase with θ in the model wave. A cross-sectional view of the various fields for the region of negligible attenuation is shown in Fig. 6. With respect to u, v, w, and θ fields, the picture is in good agreement with the semi-observational picture developed by Reed (1964a).

It can easily be shown that, as μ_r becomes larger (i.e., as we go to lower altitudes), θ leads u by less than 90°, that for positive y, v lags behind u by less than 180°, and that φ' lags beyond u by less than 90°, and behind θ by less than 180°. φ' is nowhere in phase with θ. This, however, says nothing about the variation of total ozone in a vertical column—the only quantity for which we have data.

The above equations tell us not only the relative phases of the various fields, but also their relative amplitudes (the equations, being linear and homogeneous, cannot, of course, tell us anything about absolute amplitudes). For the upper regions, where $\mu_i \gg \mu_r$, we have from Eqs. (56), (57), (61), (62), (63) and values from Part I and Section 5 of this Part

$$\left(\frac{w}{u}\right) \sim -\frac{i\omega}{S} \sim -4\times10^{-5}i, \tag{64}$$

$$\left(\frac{v}{u}\right) \sim -\frac{\omega\mu_i}{S}y \sim -10^{-5}\ \text{km}^{-1}\ y, \tag{65}$$

$$\left(\frac{\theta}{u}\right) \sim -\frac{\mu_i \Omega \bar{T}}{a\alpha_1 g} \sim 0.3i\ \text{sec deg m}^{-1}, \tag{66}$$

and

$$\left(\frac{\varphi'}{u}\right) \sim i\frac{\Omega}{\eta a\alpha_1}\left[-\omega\frac{\sigma a\alpha_1}{\Omega S} + a\frac{\bar{T}}{g}\mu_i\right] \sim -2.4\times10^{-7}\ i\ \text{sec m}^{-1}. \tag{67}$$

A typical value of u in the observed "26-month" oscillation is 15 m sec^{-1}. Inserting this value in (64)–(67) gives

$|w| \sim 6\times10^{-2}$ cm sec^{-1},

$|v| \sim 1.5\times10^{-2}$ cm km^{-1} sec^{-1} $y \sim 15$ cm sec^{-1} at $y = 1000$ km,

$|\theta| \sim 4.5$ deg,

and

$|\varphi'| \sim 4\times10^{-6}$.

The value of $|\varphi'|$ corresponds to about 30 per cent of $\bar{\varphi}$ above 25 km. It should be added that the choice $|u| \sim 15$ m sec^{-1} clearly violates the conditions postulated when linearizing the equations. Nevertheless, the results are in the same range as the estimates of Reed (1964a).

7. Relation of model wave to observed "26-month" oscillation

The model system described in the previous paragraphs differs from the observed system in at least two important respects, (a) the non-linear terms neglected in our equation for zonal momentum [Eq. (13)] are not small, and (b) our basic field is a geostrophic radiative-photochemical equilibrium field with a temperature maximum at the equator and consequently a positive shear, while the observed mean fields show a slight temperature minimum at the equator and a slightly negative shear.

Items (a) and (b) are not unrelated since the interaction of disturbance fields, neglected in a linearized treatment, can lead to discrepancies of the type indicated in (b).

The important feature of both effects is that they modify the parameter $S\left(=\frac{\partial\bar{u}}{\partial z}+2\Omega\right)$ which appears in Eq. (13) for the conservation of zonal momentum. Since S plays an important role in the dispersion relation for the model wave, close agreement between the model wave and the observed wave is unexpected. Surprisingly, however, the model wave not only illustrates qualitatively the importance of radiative and photochemical processes, but also resembles the observed wave quantitatively in a number of important respects:

i) Above a certain level (between 20 and 25 km depending upon the choice of reaction rates) the phase speed of the model wave is approximately 1 km month^{-1}, which is also the phase speed of the observed wave.

ii) Above this level the relative phases of the velocity and temperature fields are in agreement with observation. Observations of time variations of ozone distributions in the tropics are only now becoming available through the work of the North American Ozonesonde Network. They are, however, still insufficient for any meaningful comparison with theory.

iii) As the wave propagates below 20 to 25 km the rate of attenuation is sharply increased; the observed wave also undergoes its most marked attenuation in this region.

There are, however, also discrepancies to be noted. Above some level between 20 and 25 km, u_0 is approximately independent of altitude. The zonal velocity in the observed wave, on the other hand, appears to have a slight maximum in its amplitude at about 25 km. This disagreement does not, however, appear to result from the points (a) and (b) mentioned at the beginning of this section. In particularly, preliminary calculations (which are not presented here) suggest that a dependence of the form of (68) will hold whenever we have a drive at a particular, fixed level. Thus the discrepancy between the model wave and the observed wave may be due to our assumption of a drive at a fixed level.

This discussion suggests that the drive for the observed wave is distributed through the region in which the wave is observed; the peak at 25 km presumably indicates that the drive is primarily above this level.

The agreement between model and observation cited in (i), (ii), and (iii) above suggests that the observed wave, for reasons which are unclear, satisfies an equation analogous to (22) at the equator, with S positive and approximately time independent.

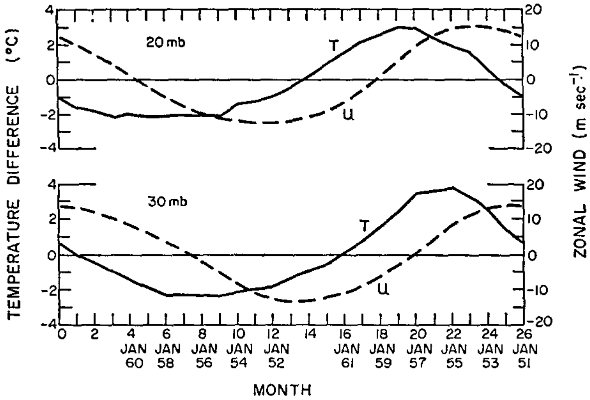

Fig. 7. Observed 26-month cycles. ΔT is the temperature difference between 3N and 27N (°K), and u the zonal wind component (m sec^{-1}) over Balboa (9N), at 20 mb and 30 mb. Annual cycles have been subtracted from monthly averages. After Read (1964a).

This last result also emerges from an independent semi-empirical argument. In Fig. 7 we see observed 26-month cycles at 20 mb and 30 mb of ΔT (temperature difference between 3N and 26N) and u near the equator. If we subtract out the mean value of ΔT, we may associate ΔT with θ. It is possible accurately to describe these cycles by a relation of the form

$$\frac{\partial u}{\partial t} \sim D\theta, \tag{68}$$

where D is positive, real and independent of time, for time scales of the order of 26 months. In any case, for time scales on the order of 26 months, it may be treated as approximately constant. Further, since $\frac{\partial \theta}{\partial z} \ll \frac{\partial \bar{T}}{\partial z}$ for the observed wave, Eq. (21) holds for both the model wave and the observed wave at the equator. Now, above some height between 20 and 25 km (depending on our choice of reaction rate parameters), B^{-1} becomes much smaller than 26 months; also a^{-1} is much smaller than 26 months. Under these circumstances, L becomes a positive real function of z, dependent upon the distribution with respect to z of the photochemical parameters B and C, and of $\partial\bar{T}/\partial z$ (see Section 5, Part I). Combining (68) and (21) we obtain

$$\frac{\partial u}{\partial t} = -DLw, \tag{69}$$

where DL may be identified with S in Eq. (22).

Acknowledgments. I would like to thank Prof. R. M. Goody, who suggested this area of research to me, for his continued interest and advice. I would further like to thank Prof. R. J. Reed for many stimulating discussions and much useful observational information. Thanks are also due to Prof. A. R. Robinson and Dr. P. Stone for their helpful criticism.

Finally, the support of the National Science Foundation at Harvard University (G 24903) and at the University of Washington (G 2282) is gratefully acknowledged.

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